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On Mixed C-Semigroups of Operators on Banach Spaces

Masoud Mosallanezhad[a], Mohammad Janfada[b]

[a]*Department of pure Mathematics, Ferdowsi University of Mashhad, International Campus, P.O. Box 1159-91775 Mashhad, Iran.*
[b]*Department of pure Mathematics, Ferdowsi University of Mashhad, P.O. Box 1159-91775 Mashhad, Iran.*

Abstract. In this paper an H-generalized Cauchy equation

$$S(t+s)C = H(S(s), S(t))$$

is considered, where $\{S(t)\}_{t\geq 0}$ is a one parameter family of bounded linear operators and $H : B(X)\times B(X) \to B(X)$ is a function. In the special case, when $H(S(s), S(t)) = S(s)S(t) + D(S(s) - T(s))(S(t) - T(t))$ with $D \in B(X)$, solutions of H-generalized Cauchy equation are studied, where $\{T(t)\}_{t\geq 0}$ is a C-semigroup of operators. Also a similar equations are studied on C-cosine families and integrated C-semigroups.

1. Introduction and Preliminaries

Suppose that X is a Banach space and A is a linear operator in X with domain $D(A)$ and range $R(A)$. For given $x \in D(A)$, the abstract Cauchy problem for A with the initial value x, consists of finding a solution $u(t)$ to the initial value problem

$$ACP(A;x)\begin{cases} \frac{du(t)}{dt} = Au(t), & t \in \mathbb{R}_+, \\ u(0) = x, \end{cases}$$

where by a solution we mean a function $u : \mathbb{R}_+ \to X$, which is continuous for $t \geq 0$, continuously differentiable for $t > 0$, $u(t) \in D(A)$ for $t \in \mathbb{R}_+$ and $ACP(A;x)$ is satisfied (see [15]).

If $C \in B(X)$, the space of all bounded linear operators on X, is injective, then a one-parameter C-semigroup (regularized semigroup) of operators is a family $\{T(t)\}_{t\in\mathbb{R}_+} \subset B(X)$ for which $T(0) = C$, and $T(s+t)C = T(s)T(t)$, $s,t \geq 0$, and also is strongly continuous, i.e. for each $x \in X$ the mapping $t \mapsto T(t)x$ is continuous. An operator $A : D(A) \to X$ with the domain

$$D(A) = \{x \in X : \lim_{t\to 0} \frac{T(t)x - Cx}{t} \text{ exists in the range of } C\}$$

define by $Ax := C^{-1}\lim_{t\to 0}\frac{T(t)x-Cx}{t}$ for $x \in D(A)$ is called the infinitesimal generator of $T(t)$.
With $C = I$, the identity operator on X, the C-semigroup $\{T(t)\}_{t\geq 0}$ is said to be a C_0-semigroup.
Regularized semigroups and their connection with the $ACP(A;x)$ have been studied, e.g., in [2, 11, 13, 16].

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Corresponding author: Mohammad Janfada
Email addresses: masvame63@gmail.com (Masoud Mosallanezhad), janfada@um.ac.ir (Mohammad Janfada)

Another important family of operators which is related to the following second order ACP

$$CP(A;x;y)\begin{cases} \frac{d^2u(t)}{dt^2} = Au(t), & t \in \mathbb{R}, \\ u(0) = x, u'(0) = y \end{cases}$$

is the C-cosine operator function. If $C \in B(X)$ and $\{C(t); t \in \mathbb{R}\} \subseteq B(X)$ is a strongly continuous family of operators, then $\{C(t)\}_{t\in\mathbb{R}}$ is a C-cosine operator function on X (see [8]) if it satisfy
(a) $C(0) = C$;
(b) $[C(t+s) + C(t-s)]C = 2C(t)C(s)$, $t, s \in \mathbb{R}$;
The associated sine operator function $\mathcal{S}(.)$ is defined by the formula $\mathcal{S}(t) = \int_0^t C(s)ds, t \in \mathbb{R}$. The second infinitesimal generator (or simply the generator) A of $S(.)$ is defined as $Ax = C^{-1}\lim_{t\to 0} \frac{2}{t^2}(S(t) - C)x$ with natural domain. A C-cosine operator family gives the solution of a well-posed Cauchy problem. For more details on the theory of cosine operator function we refer to [8, 10, 13, 17].

Another useful tool to find solutions of the $ACP(A;x)$ is the notion of integrated C-semigroups. A strongly continuous family $\{S(t)\}_{t\geq 0}$ of bounded operators on X is called a integrated C-semigroup if $S(0) = 0$, $S(t)C = CS(t)$, $t > 0$, and

$$S(s)S(t)x = \int_0^s (S(r+t) - S(r))Cxdr = S(t)S(s)x, \quad (s, t > 0,\ x \in X)$$

An operator $A : D(A) \subseteq X \to X$ which is defined as follows

$$x \in D(A) \text{ and } Ax = y \Leftrightarrow S(t)x - Cx = \int_0^t S(s)yds$$

is called the infinitesimal generator of $\{S(t)\}_{t\geq 0}$. Indeed $\{S(t)\}_{t\geq 0}$ is uniquely determined by A (see Proposition 1.3 [9])
Integrated semigroups extend the theory of C_0-semigroups to abstract Cauchy problems with operators which do not satisfy the Hille-Yosida conditions. For more information on this subject one may see [1, 3, 4, 7, 9, 12, 14**?**].

A generalized Cauchy equation

$$S(t+s) = S(s)S(t) + \alpha(S(s) - T(s))(S(t) - T(t)), \quad (s, t > 0)$$

where $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup and $\alpha \in \mathbb{C}$, was first studied in [5, 6] which is called also a mixed semigroup. Trivially with $\alpha = 0$ this equation reduces to a C_0-semigroup.
In Section 2, we will consider a regularized type extension of this equation and study its solutions. Also its corresponding ACP will be introduced in a special case. In Section 3, a mixed type regularized cosine family is considered and its properties are studied. Next in Section 4, a similar regularized integrated mixed semigroup will be examined and properties of its solutions will be investigated.

2. Mixed Regularized Semigroups

Let X be a Banach space and C be an injective operator in $B(X)$. A family $\{S(t)\}_{t\geq 0} \subseteq B(X)$ is said to satisfy a H-generalized Cauchy equation if

$$S(t+s)C = H(S(s), S(t)), \quad (s, t > 0) \tag{1}$$

where $H : B(X) \times B(X) \to B(X)$ is a function. If $H(S(s), S(t)) = S(s)S(t)$, then $\{S(t)\}_{t\geq 0}$ satisfies in the first condition of C-semigroups of operators.

In this section, we consider a special case when

$$H(S(s), S(t)) = S(s)S(t) + D(S(s) - T(s))(S(t) - T(t)), \quad (s, t > 0) \tag{2}$$

where $\{T(t)\}_{t\geq 0}$ is a C-semigroup of operators with the infinitesimal generator A_0 and $D \in B(X)$. In this case we say that $\{S(t)\}$ is a H-C- semigroup. Trivially $D = 0$ is the C-semigroup condition.

Now consider the equation (1) with H as in (2). Define the operator $A : D(A) \subseteq X \to X$ by $A(x) = C^{-1} \lim_{s\to 0} \frac{S(s)x-Cx}{s}$, where $D(A) = \{x \in X : \lim_{s\to 0} \frac{S(s)x-Cx}{s}$ exists in the range of C$\}$. We shall think of A as the infinitesimal generator of $\{S(t)\}_{t\geq 0}$.

The following are some examples of H-C-semigroups.

Example 2.1. *Suppose that X is a Banach space, $A, B, C \in B(X)$, C is injective and $CA = AC$. Put*

$$S(t) = Ce^{tA} + t(B - A)Ce^{tA} \quad (t > 0).$$

Then one can see that with $D = -I$, $\{S(t)\}_{t\geq 0}$ is a H-C- semigroup where $T(t) = Ce^{tA}$.
In this case, one can see that $S(s)S(t) = S(t)S(s)$, $s, t \geq 0$ if and only if A commutes with B. It will be proved that with $D = -I$, every uniformly continuous H-C-semigroup is of this form.

Example 2.2. *Let $X = L^p(\Omega, \mu)$, for some σ-finite measureable space Ω. Suppose that $q_1, q_2, q_3 : \Omega \to \mathbb{C}$ are measurable functions for which q_1 is bounded and nonzero almost everywhere and q_2, q_3 satisfy*

$$\operatorname{ess\,sup}_{s\in\Omega} \operatorname{Re} q_i(s) < \infty, \quad i = 2, 3,$$

where $\operatorname{Re} q_i(s)$ *is the real part of $q_i(s)$. Then it is easy to verify that with $D = -I$, the family*

$$S(t)f := (1 - tq_2 + tq_3)q_1e^{tq_2} f, \quad (t \in [0, \infty))$$

of operators on X defines a H-C-semigroup where $T(t)f := q_1e^{tq_2} f$ and $Cf := q_1 f$.

In the following lemma some elementary properties of H-C- semigroups is presented.

Lemma 2.3. *Let $\{S(t)\}_{t\geq 0} \subseteq B(X)$ be a strongly continuous family, which satisfies (1) with H as (2).*

1. *If $I + D$ is injective and for any $s, t \geq 0$, $T(s)S(t) = S(t)T(s)$, then $S(s)S(t) = S(t)S(s)$, for all $s, t \geq 0$ and in particular $S(s)C = CS(s)$.*
2. *If D is injective and $S(s)S(t) = S(t)S(s)$ for all $s, t \geq 0$, then for any $s, t \geq 0$, $T(s)S(t) = S(t)T(s)$.*
3. *If $S(s)S(t) = S(t)S(s)$ for all $s, t \geq 0$ and $x \in D(A)$, then for any $t \geq 0$, $S(t)x, T(t)x \in D(A)$ and $AS(t)x = S(t)Ax$, $AT(t)x = T(t)A(x)$. In addition, $S(t)x, T(t)x \in D(A_0)$ and $A_0S(t)x = S(t)A_0x$, $A_0T(t)x = T(t)A_0(x)$ for any $x \in D(A_0)$.*

Proof. Suppose that $I + D$ is injective and for any $s, t \geq 0$, $T(s)S(t) = S(t)T(s)$. For $s, t \geq 0$, we have

$$\begin{aligned} S(s)S(t) + D(S(s) - T(s))(S(t) - T(t)) &= S(t + s)C = S(s + t)C \\ &= S(t)S(s) + D(S(t) - T(t))(S(s) - T(s)), \end{aligned}$$

which implies that

$$(I + D)(S(s)S(t) - S(t)S(s)) = 0.$$

This proves 1.

Suppose that D is injective and for any $s, t \geq 0$, $S(s)S(t) = S(t)S(s)$. For $s, t \geq 0$, we have

$$\begin{aligned} S(s)S(t) + D(S(s) - T(s))(S(t) - T(t)) &= S(t + s)C = S(s + t)C \\ &= S(t)S(s) + D(S(t) - T(t))(S(s) - T(s)), \end{aligned}$$

which implies that

$$D(S(t)T(s) - T(s)S(t) + T(t)S(s) - S(s)T(t)) = 0.$$

Thus injectivity of D yields that 2.

To show 3, let $x \in D(A)$. Then $\lim_{s\to 0} \frac{S(s)x-Cx}{s}$ exists in the range of C. For given $t \geq 0$ we have

$$\frac{S(s)S(t)x - CS(t)x}{s} = S(t)\frac{S(s)x - Cx}{s}.$$

Now let $y := \lim_{s\to 0} \frac{S(s)x-Cx}{s}$ which is in the range of C. If $y = Cz$ for some $z \in X$, then

$$\lim_{s\to 0} \frac{S(s)S(t)x - CS(t)x}{s} = S(t)y = S(t)Cz = CS(t)z.$$

It follows that $\lim_{s\to 0} \frac{S(s)S(t)x-CS(t)x}{s}$ is in the range of C and

$$\begin{aligned} AS(t)x &= C^{-1} \lim_{s\to 0} \frac{S(s)S(t)x - CS(t)x}{s} \\ &= C^{-1}S(t)y = S(t)C^{-1}y = S(t)Ax. \end{aligned}$$

The last part of 3 can be proved similarly. □

Set $A_1 = (1 + D)A - DA_0$, where A_0 is the infinitesimal generator of the C-semigroup $\{T(t)\}_{t\geq 0}$ and A is the infinitesimal generator of $\{S(t)\}_{t\geq 0}$. The next result reads as follows.

Theorem 2.4. *Suppose that $\{S(t)\}_{t\geq 0} \subseteq B(X)$ is a family, which is strongly continuous with $S(0) = C$ and satisfies (1) with H as (2).*

1. *Let $T_1(t)(x) = (1 + D)S(t)x - DT(t)x$, $x \in X$ and $CD = DC$. Then $\{T_1(t)\}_{t\geq 0}$ is a C-semigroup of operators, whose infinitesimal generator is an extension of A_1.*
2. *If $D + I$ is invertible, then the solution of (1) with H as (2) in the strong operator topology is of the form*

$$S(t)x = D(D + I)^{-1}T(t)x + (1 + D)^{-1}T_1(t)x, \quad x \in X.$$

Proof. Trivially $T_1(0) = C$, since $T(0) = C = S(0)$. Also for $s, t \geq 0$, using (1) and (2), and a simple calculation one may see that $T_1(s + t)C = T_1(s)T_1(t)$.
Now we are going to show that an extension of $A_1 = (1 + D)A - DA_0$ is the infinitesimal generator of $\{T_1(t)\}_{t\geq 0}$. Let B be the infinitesimal generator of $\{T_1(t)\}_{t\geq 0}$. For given $x \in D(A_1) = D(A) \cap D(A_0)$, by definition of $D(A)$ and $D(A_0)$, $\lim_{t\to 0} \frac{T(t)x-Cx}{t}$ and $\lim_{t\to 0} \frac{S(t)x-Cx}{t}$ are in the range of C. Thus

$$\begin{aligned} \lim_{t\to 0} \frac{T_1(t)x - Cx}{t} &= \lim_{t\to 0} \frac{(1 + D)S(t)x - DT(t)x - Cx}{t} \\ &= (I + D) \lim_{t\to 0} \frac{S(t)x - Cx}{t} + D \lim_{t\to 0} \frac{T(t)x - Cx}{t} \end{aligned}$$

exists in the range of C. It follows that $x \in D(B)$ and $A_1(x) = B(x)$. Thus the infinitesimal generator of $\{T_1(t)\}_{t\geq 0}$ is an extension of A_1. This proves 1.
2 is evident. □

For $D = -I$, the equation (1) reduces to

$$S(s + t)C - S(s)S(t) = (T(s) - S(s))(S(t) - T(t)) \quad (s, t > 0). \tag{3}$$

or equivalently

$$S(s + t)C - T(s + t)C = T(s)S(t) - S(s)T(t) \quad (s, t > 0).$$

Also in this case, with A_0 and A as above we have the following theorem which characterize solutions of (3).

Theorem 2.5. *Let $\{S(t)\}_{t\geq 0}$ be a strongly continuous commuting family satisfying (3). Then*

1. *For any $x \in D(A) \cap D(A_0)$, $CS(t)x$ is a differentiable function of t and $\frac{d}{dt}CS(t)x = C[A_0S(t)x + (A - A_0)T(t)x]$.*
2. *For any $x \in D(A) \cap D(A_0)$, $S(t)x = T(t)x + t(A - A_0)T(t)x$.*

Proof. For $x \in D(A) \cap D(A_0)$, applying Lemma 2.3 we have

$$\begin{aligned}
\frac{d}{dt}S(t)Cx &= \lim_{h\to 0}\frac{S(h+t)Cx - S(t)Cx}{h} \\
&= \lim_{h\to 0}\frac{S(h)S(t)x + (T(h) - S(h))(S(t) - T(t))x - S(t)Cx}{h} \\
&= \lim_{h\to 0}\frac{T(h)S(t)x - CS(t)x}{h} + \lim_{h\to 0}\frac{S(h)T(t)x - CT(t)x}{h} - \lim_{h\to 0}\frac{T(h)T(t)x - CT(t)x}{h} \\
&= C[A_0S(t)x + AT(t)x - A_0T(t)x].
\end{aligned}$$

This complete the proof of 1.

For establishing 2, let $x \in D(A) \cap D(A_0)$. By the part 1 we have

$$\begin{aligned}
C(S(t)x - T(t)x) &= \int_0^t \frac{d}{d\tau}CT(t-\tau)S(\tau)d\tau \\
&= \int_0^t (-CT(t-\tau)A_0S(\tau)x) \\
&+ (C[A_0S(\tau)T(t-\tau)x + (A - A_0)T(\tau)T(t-\tau)x])d\tau \\
&= \int_0^t (A - A_0)T(t)Cxd\tau \\
&= t(A - A_0)T(t)Cx = tC(A - A_0)T(t)x.
\end{aligned}$$

Now injectivity of C completes the proof of 2. □

Theorem 2.5 show that for $D = -I$ if $\{S(t)\}_{t\geq 0}$ is the mixed semigroup with the generator A and $\{T(t)\}_{t\geq 0}$ is the C-semigroup generated by A_0 then $u(t) = S(t)x$ is a solution of the following inhomogeneous ACP

$$\begin{cases} \frac{du(t)}{dt} = A_0u(t) + (A - A_0)f(t), & t \in \mathbb{R}_+, \\ u(0) = Cx, \quad x_0 \in D(A) \cap D(A_0), \end{cases}$$

with $f(t) = T(t)x$.

In the following theorem it will be proved that multiplication of a H-C-semigroup and a C-semigroup is a H-C-semigroup if these two families commute.

Theorem 2.6. *Let $\{V(t)\}_{t\geq 0}$ be a C-semigroup with the infinitesimal generator B which commute with $D \in B(H)$ and $\{S(t)\}_{t\geq 0}$ be a commuting strongly continuous H-C-semigroup with the C-semigroup $\{T(t)\}_{t\geq 0}$ which also commute with $\{V(t)\}_{t\geq 0}$. Then $W(t) := V(t)S(t)$ is a H-C^2-semigroup with the infinitesimal generator $A + B$ where A and A_0 are the infinitesimal generators of $\{S(t)\}_{t\geq 0}$ and $\{T(t)\}_{t\geq 0}$, respectively.*

Proof. Trivially $T(0) = C^2$. Also for any $s, t \geq 0$,

$$\begin{aligned}
W(s+t)C^2 &= V(s+t)CS(s+t)C \\
&= V(s)V(t)[S(s)S(t) + D(S(s) - T(s))(S(t) - T(t))] \\
&= W(s)W(t) + D(W(s) - V(s)T(s))(W(t) - V(t)T(t)).
\end{aligned}$$

Thus $\{W(t)\}_{t\geq 0}$ is a H-C^2-semigroup which is obviously strongly continuous. Also for any $x \in D(A) \cap D(B)$,

$$\begin{aligned}
\lim_{t\to 0}\frac{W(t)x - C^2x}{h} &= \lim_{t\to 0}\frac{V(t)S(t)x - CS(t)x}{t} + \frac{CS(t)x - C^2x}{t} \\
&C^2Bx + C^2Ax.
\end{aligned}$$

Thus

$$C^{-2} \lim_{t\to 0} \frac{W(t)x - C^2x}{h} = (B + A)x.$$

□

3. Mixed *C*-Cosine Family

Let X be a Banach space and C be an injective operator in $B(X)$. A family $\{S(t)\}_{t\in\mathbb{R}} \subseteq B(X)$ is said to satisfy a H-C-cosine Cauchy equation if

$$[S(s + t) + S(s - t)]C = H(S(s), S(t)), \tag{4}$$

where $H : B(X) \times B(X) \to B(X)$ is a function. If $H(S(s), S(t)) = 2S(s)S(t)$, then $\{S(t)\}_{t\in\mathbb{R}}$ satisfy in the first condition of C-cosine family of operators.

In this section we consider a special case when

$$H(S(s), S(t)) = 2S(s)S(t) + 2D(S(s) - T(s))(S(t) - T(t)) \quad (s, t \in \mathbb{R}), \tag{5}$$

where $\{T(t)\}_{t\in\mathbb{R}}$ is a C-cosine family of operators and $D \in B(X)$. Trivially $D = 0$ is a the C-cosine condition.

Now consider the equation (4) with H as in (5). Let $A : D(A) \subseteq X \to X$ be defined as $A(x) = C^{-1} \lim_{h\to 0} \frac{2}{h^2}[S(h)x - Cx]$, where $D(A) = \{x \in X : \lim_{h\to 0} \frac{2}{h^2}[S(s)x - Cx]$ exists in the range of C$\}$. We shall think of A as the infinitesimal generator of $\{S(t)\}_{t\in\mathbb{R}}$.

Lemma 3.1. *Let $\{S(t)\}_{t\in\mathbb{R}} \subseteq B(X)$ be a strongly continuous family, with $S(0) = C$ and it satisfies* (4) *with H as* (5)*. Then*

1. *$S(s) = S(-s)$ and $S(s)C = CS(s)$ for all s.*
2. *If $I + D$ is injective and for any s, t, $T(s)S(t) = S(t)T(s)$, then $S(s)S(t) = S(t)S(s)$ for all s, t.*
3. *If D is injective and $S(s)S(t) = S(t)S(s)$ for all s, t, then for any s, t, $T(s)S(t) = S(t)T(s)$.*
4. *If $\{S(t)\}_{t\in\mathbb{R}}$ is a commuting family and $x \in D(A)$ then for any t, $S(t)x \in D(A)$ and $AS(t)x = S(t)Ax$.*

Proof. Letting $s = 0$ in (4) we get

$$[S(t) + S(-t)]C = 2CS(t) + 2D(0) = 2CS(t) \quad (t \in \mathbb{R}) \tag{6}$$

so

$$S(t)C + S(-t)C = 2CS(t) \text{ and } S(-t)C + S(t)C = 2CS(-t) \quad (t \in \mathbb{R}).$$

Subtracting these equalities and using injectivity of C we find

$$S(t) = S(-t) \quad (t \in \mathbb{R}).$$

So by (6) we obtain $2S(t)C = 2CS(t)$ or $S(t)C = CS(t)$ for all $t \in \mathbb{R}$. This prove 1.
For proving 2, let $I + D$ be injective and for any s, t, $T(s)S(t) = S(t)T(s)$. From part 1 we have

$$\begin{aligned} 2S(s)S(t) + 2D(S(s) - T(s))(S(t) - T(t)) &= [S(s + t) + S(s - t)]C \\ &= [S(t + s) + S(t - s)]C \\ &= 2S(t)S(s) + 2D(S(t) - T(t))(S(s) - T(s)), \end{aligned} \tag{7}$$

On the other hand $T(s)T(t) = T(t)T(s)$ (see [8]) and by hypothesis $S(t)T(s) = T(s)S(t)$. Thus (7) implies that

$$(I + D)(S(s)S(t) - S(t)S(s)) = 0, \quad (s, t \in \mathbb{R}).$$

Now injectivity of $I + D$ implies 2.
Suppose that D is injective and for any s, t, $S(s)S(t) = S(t)S(s)$. Applying this condition on (7) we get

$$D(S(t)T(s) - T(s)S(t) + T(t)S(s) - S(s)T(t)) = 0 \quad (s, t \in \mathbb{R}).$$

which implies 3.
For proving 4, let $x \in D(A)$. So $y := \lim_{h\to 0} \frac{2}{h^2}[S(h)x - Cx]$ exists in the range of C. Put $y = Cz$ for some $z \in X$. Now for a given t by part 2 we have

$$\frac{2}{h^2}[S(h)S(t)x - CS(t)x] = S(t)\frac{2}{h^2}[S(h)x - Cx] \quad (t \in \mathbb{R}).$$

Thus

$$\lim_{h\to 0} \frac{2}{h^2}[S(h)S(t)x - CS(t)x] = S(t)y = S(t)Cz = CS(t)z \quad (t \in \mathbb{R}).$$

This implies that $\lim_{h\to 0} \frac{2}{h^2}[S(h)S(t)x - CS(t)x]$ is in the range of C and also

$$\begin{aligned} AS(t)x &= C^{-1}\lim_{h\to 0} \frac{2}{h^2}[S(h)S(t)x - CS(t)x] \\ &= C^{-1}S(t)y = S(t)C^{-1}y = S(t)Ax. \end{aligned}$$

□

Theorem 3.2. *Suppose that $\{S(t)\}_{t\in\mathbb{R}} \subseteq B(X)$ is a commuting strongly continuous family with $S(0) = C$ and it satisfies (4) with H as (5). Let $T_1(t)x := (1+D)S(t)x - DT(t)x$, $x \in X$, then $\{T_1(t)\}_{t\in\mathbb{R}}$ is a C-cosine family. Furthermore if A_0 is the infinitesimal generator of the C-cosine family $\{T(t)\}_{t\in\mathbb{R}}$, then an extension of $A_1 := (1 + D)A - DA_0$ is the generator of $\{T_1(t)\}_{t\in\mathbb{R}}$.*

Proof. Applying (5) and Lemma 3.1 we get

$$\begin{aligned} [T_1(t+s) + T_1(t-s)]Cx &= [((1+D)S(t+s) - DT(t+s)) + ((1+D)S(t-s) - DT(t-s))]Cx \\ &= [(1+D)(S(t+s) + S(t-s)) - D(T(t+s) + T(t-s))]Cx \\ &= [(1+D)(2S(t)S(s) + 2D(S(t) - T(t))(S(s) - T(s))) - D(2T(t)T(s))]x \\ &= 2[(1+D)S(t)S(s) + D(1+D)S(t)S(s) - D(1+D)S(t)T(s) \\ &- D(1+D)T(t)S(s) + D(1+D)T(t)T(s) - DT(t)T(s)]x \\ &= 2[(1+D)^2S(t)S(s) - D(1+D)S(t)T(s) \\ &- D(1+D)T(t)S(s) + D^2T(t)T(s)]x \\ &= 2[(1+D)S(t) - DT(t)][(1+D)S(s) - DT(s)]x \\ &= 2T_1(t)T_1(s)x. \end{aligned}$$

Moreover $T_1(0)x = (1 + D)S(0)x - DT(0)x = (1 + D)Cx - DCx = Cx$. These establish the C-cosine properties of $\{T_1(t)\}_{t\in\mathbb{R}}$.
We are going to show that an extension of $A_1 = (1 + D)A - DA_0$ is the infinitesimal generator of $\{T_1(t)\}$. Let B be the infinitesimal generator of $\{T_1(t)\}_{t\in\mathbb{R}}$. For a given $x \in D(A_1) = D(A) \cap D(A_0)$, by definition of $D(A)$ and $D(A_0)$, $\lim_{h\to 0} \frac{2}{h^2}[T(h)x - Cx]$ and $\lim_{h\to 0} \frac{2}{h^2}[S(h)x - Cx]$ are in the range of C. Thus

$$\begin{aligned} \lim_{h\to 0} \frac{2}{h^2}[T_1(h)x - Cx] &= \lim_{h\to 0} \frac{2}{h^2}[(1+D)S(h)x - DT(h)x - Cx] \\ &= (I+D)\lim_{h\to 0} \frac{2}{h^2}[S(h)x - Cx] + D\lim_{h\to 0} \frac{2}{h^2}[T(h)x - Cx] \end{aligned}$$

exists in the range of C. This implies that $x \in D(B)$ and $A_1(x) = B(x)$ and the proof is complete. □

The previous theorem implies that if $(I + D)$ is invertible then the solution $S(t)$ of (4) with H as (5) is of the form

$$S(t)x = D(D + I)^{-1}T(t)x + (1 + D)^{-1}T_1(t)x, \quad (x \in X).$$

4. Mixed Integrated Semigroups

In this section we consider the following equation for C-integrated case

$$V(s)V(t) - \int_0^s (V(t+\tau) - V(\tau))Cd\tau = D(V(s) - W(s))(W(t) - V(t)) \quad (s, t > 0) \tag{8}$$

where $\{W(t)\}_{t\geq 0}$ is a integrated C-semigroup and $D \in B(X)$. This equation is called a mixed integrated C-semigroup.

Proposition 4.1. *Let $\{V(t)\}_{t\geq 0}$ be a mixed integrated C-semigroup.*

1. *If $I + D$ is injective and for any $s, t \geq 0$, $V(s)W(t) = W(t)V(s)$, then $V(s)V(t) = V(t)V(s)$ for all $s, t \geq 0$.*
2. *If D is injective and $V(s)V(t) = V(t)V(s)$ for all $s, t \geq 0$, then for any $s, t \geq 0$, $V(s)W(t) = W(t)V(s)$.*

Proof. For any $s, t \geq 0$, we have

$$\int_0^s (V(t+\tau) - V(\tau))Cd\tau = \int_0^t (V(s+\tau) - V(\tau))Cd\tau, \tag{9}$$

since

$$\begin{aligned}
\int_0^s (V(t+\tau) - V(\tau))Cd\tau &= \int_0^s V(t+\tau)Cd\tau - \int_0^s V(\tau)Cd\tau \\
&= \int_t^{s+t} V(\tau)Cd\tau - \int_0^s V(\tau)Cd\tau \\
&= \int_0^{s+t} V(\tau)Cd\tau - \int_0^t V(\tau)Cd\tau - \int_0^s V(\tau)Cd\tau \\
&= \int_0^s V(\tau)Cd\tau + \int_s^{s+t} V(\tau)Cd\tau - \int_0^t V(\tau)Cd\tau - \int_0^s V(\tau)Cd\tau \\
&= \int_0^t (V(s+\tau) - V(\tau))Cd\tau
\end{aligned}$$

First suppose that $V(s)W(t) = W(t)V(s)$ for any $s, t \geq 0$. It follows from the definition of $\{V(t)\}_{t\geq 0}$ that

$$\begin{aligned}
V(s)V(t) &= \int_0^s (V(t+\tau) - V(\tau))Cd\tau + D(V(s) - W(s))(W(t) - V(t)) \\
&= \int_0^s (V(t+\tau) - V(\tau))Cd\tau + D(V(s)W(t) - V(s)V(t) - W(s)W(t) + W(s)V(t)).
\end{aligned}$$

On the other hand

$$\begin{aligned}
V(t)V(s) &= \int_0^t (V(s+\tau) - V(\tau))Cd\tau + D(V(t) - W(t))(W(s) - V(s)) \\
&= \int_0^t (V(s+\tau) - V(\tau))Cd\tau \\
&+ D(V(t)W(s) - V(t)V(s) - W(t)W(s) + W(t)V(s)).
\end{aligned}$$

Hence by (9) we have

$$I + D(V(s)V(t)) = I + D(V(t)V(s)).$$

Now injectivity of $I + D$ implies that $V(s)V(t) = V(t)V(s)$.
The proof of (2) is similar and so we omit it. $\square$

Now suppose that A_0 is the infinitesimal generator of the integrated C-semigroup $\{W(t)\}_{t\geq 0}$. With $A_1 = (1 + D)A - DA_0$, we have the following result.

Theorem 4.2. *Suppose that $\{V(t)\}_{t\geq 0} \subseteq B(X)$ is a family with $V(0) = 0$ and it satisfies (8).*

1. *Let $T_1(t)(x) = (1 + D)V(t)x - DW(t)x$, $x \in X$. Then $\{T_1(t)\}_{t\geq 0}$ is a integrated C-semigroup of operators whose infinitesimal generator is an extension of A_1.*
2. *If $D + I$ is invertible then the solution of (8) in the strong operator topology is of the form*

$$V(t)x = D(D + I)^{-1}W(t)x + (1 + D)^{-1}T_1(t)x, \quad x \in X.$$

Proof. Trivially $T_1(0) = 0$, since $W(0) = 0 = V(0)$. Also for $s, t \geq 0$, using (8), and a simple calculation one may observe that $T_1(t)C = CT_1(t)$ as well as

$$\begin{aligned}
\int_0^s (T_1(t+\tau) - T_1(\tau))Cd\tau &= \int_0^s ((1 + D)V(t+\tau) - DW(t+\tau) - (1 + D)V(\tau) + DW(\tau))Cd\tau \\
&= (1 + D)\int_0^s (V(t+\tau) - V(\tau))Cd\tau - D\int_0^s (W(t+\tau) - W(\tau))Cd\tau \\
&= (1 + D)[V(s)V(t) - D(V(s) - W(s))(W(t) - V(t))] - D[W(s)W(t)] \\
&= (1 + D)^2V(s)V(t) - (1 + D)DV(s)W(t) - (1 + D)DW(s)V(t) + D^2W(s)W(t) \\
&= [(1 + D)V(s) - DW(s)][(1 + D)V(t) - DW(t)] \\
&= T_1(s)T_1(t)x.
\end{aligned}$$

Now we are going to show that an extension of $A_1 = (1+D)A - DA_0$ is the infinitesimal generator of $\{T_1(t)\}_{t\geq 0}$. Let B be the infinitesimal generator of $\{T_1(t)\}_{t\geq 0}$. For a given $x \in D(A_1) = D(A) \cap D(A_0)$, by definition of $D(A)$ and $D(A_0)$, $\int_0^t V(s)yds = V(t)x - Cx \Leftrightarrow Ax = y$ and $\int_0^t W(s)yds = W(t)x - Cx \Leftrightarrow A_0x = y$. Thus

$$\begin{aligned}
&\int_0^t T_1(s)yds = \int_0^t [(1 + D)V(s) - DW(s)]yds \\
&= (1 + D)(V(t)x - Cx) - D(W(t)x - Cx) = T_1(t)x - Cx
\end{aligned}$$

if and only if $A_1x = y$. This implies that $x \in D(B)$ and $A_1(x) = B(x)$. Thus the infinitesimal generator of $\{T_1(t)\}_{t\geq 0}$ is an extension of A_1. This proves (1).
(2) is trivial. $\square$

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