Filomat 28:6 (2014), 1225–1236
DOI 10.2298/FIL1406225S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Comparison of Strong and Statistical Convergences in Some Families of Summability Methods

Anna Šeletski[a], Anne Tali[b]

[a]*Institute of Cybernetics, Tallinn University of Technology, Akadeemia tee 21, 12618 Tallinn, Estonia*
[b]*Institute of Mathematics and Natural Sciences, Tallinn University, Narva mnt 25, 10120 Tallinn, Estonia*

Abstract. The paper deals with certain families $\{A_\alpha\}$ $(\alpha > \alpha_0)$ of summability methods. Strong and statistical convergences in Cesàro- and Euler–Knopp-type families $\{A_\alpha\}$ are investigated. Convergence of a sequence $x = (x_n)$ with respect to the different strong summability methods $[A_{\alpha+1}]_t$ (with positive exponents $t = (t_n)$) in a family $\{A_\alpha\}$ is compared, and characterized with the help of statistical convergence. A convexity theorem for comparison of three strong summability methods $[A_{\gamma+1}]_t$, $[A_{\delta+1}]_t$ and $[A_{\beta+1}]_t$ $(\beta > \delta > \gamma > \alpha_0)$ in a Cesàro-type family $\{A_\alpha\}$ is proved. This theorem can be seen as a generalization of some convexity theorems known earlier. Interrelations between strong convergence and certain statistical convergence are also studied and described with the help of theorems proved here. All the results can be applied to the families of generalized Nörlund methods (N, p_n^α, q_n).

1. Preliminaries and Introduction

1.1 We start with some basics of summability theory (see [1], [7]). Let us consider sequences $x = (x_n)$ with $x_n \in \mathbb{C}$ for every $n \in \mathbb{N}_0 = \{0, 1, 2, ...\}$. If a sequence x is bounded, we write $x_n = O(1)$. If $\lim_n x_n = s$ or $\lim_n x_n = 0$, we write also $x_n \to s$ or $x_n = o(1)$, respectively. Let A be a transformation which transforms a sequence x into the sequence $y = (y_n) = Ax = (A_n x)$. If the limit $\lim_n y_n = s$ exists, then we say that x is convergent with respect to the summability method A (in short, A-convergent) to s and write $x_n \to s(A)$. If $y_n = O(1)$, we say that x is bounded with respect to the method A and write $x_n = O(A)$. The most common summability method is a matrix method A defined with the help of the matrix $A = (a_{n,k})$, where $a_{n,k} \in \mathbb{C}$ $(n, k \in \mathbb{N}_0)$ and which transforms x into y with

$$y_n = \sum_{k=0}^{\infty} a_{n,k} x_k \quad (n \in \mathbb{N}_0). \tag{1}$$

If $x_n \to s \Longrightarrow x_n \to s(A)$ for any $x \in c$, then we say that the matrix method $A = (a_{n,k})$ is regular.

2010 *Mathematics Subject Classification*. Primary 40F05; Secondary 40G15, 40G05
Keywords. Summability method, strong summability method, generalized Nörlund methods, Cesàro methods, Euler-Knopp methods, convexity theorem, statistical convergence
Received: 28 May 2013; Revised: 20 August 2013; Accepted: 20 August 2013
Communicated by Eberhard Malkowsky
Research was partially supported by Estonian Target Funding Project SF140083s08
Email addresses: annatar@ioc.ee (Anna Šeletski), atali@tlu.ee (Anne Tali)

It is well-known that the method $A = (a_{n,k})$ is regular if and only if the following conditions are satisfied:

$$\lim_n a_{n,k} = 0 \ (k \in \mathbb{N}_0), \ \lim_n \sum_{k=0}^{\infty} a_{n,k} = 1, \ \sum_{k=0}^{\infty} |a_{n,k}| = O(1).$$

1.2 The notion of a A-statistically convergent sequence x (see [8], [4]) also belongs to the basics of this paper. We denote

$$\mathcal{K}_\varepsilon = \{k : |x_k - s| \geq \varepsilon\}, \tag{2}$$

where s and $\varepsilon > 0$ are some numbers.

Definition 1. *Let A be a non-negative regular matrix method defined by transformation* (1). *We say that a sequence $x = (x_n)$ is A-statistically convergent to s and write $x_n \to s(st_A)$, if for any $\varepsilon > 0$*

$$\lim_{n\to\infty} \sum_{\mathcal{K}_\varepsilon} a_{n,k} = 0,$$

where $\mathcal{K}_\varepsilon$ is the set defined by (2).

In particular, if $A = (C, 1)$ then A-statistical convergence of x turns into statistical convergence defined in [6], and we write $x_n \to s(st)$. This notion was generalized also in [11] where statistical $(C, 1)$-convergence was defined. About further developments of the notion of statistical convergence and appropriate references can be read, e.g., in [5], [9] and [2].

Let us have another summability method B, besides the non-negative matrix method A. Generalizing the notions of statistical $(C, 1)$-convergence and A-statistical convergence we define A-statistical B-convergence of x as A-statistical convergence of Bx.

Definition 2. *Let A be a non-negative regular matrix method and B be a summability method. We say that a sequence $x = (x_n)$ is A-statistically B-convergent to s if $B_n x \to s(st_A)$. In particular, if $A = (C, 1)$, i.e., if $B_n x \to s(st)$, we say that x is statistically B-convergent to s.*

In case of $A = B = (C, 1)$ Definition 2 defines the statistical $(C, 1)$-convergence (see [11]). In case of $B = I$ Definition 2 coincides with Definition 1.

We need also the following definition (see [10]).

Definition 3. *We say that a matrix method B is A-statistically regular if*

$$x_n = O(1), \ x_n \to s(st_A) \Longrightarrow B_n x = O(1), \ B_n x \to s(st_A).$$

In particular, if $A = (C, 1)$ then we say that a matrix method B is statistically regular if

$$x_n = O(1), \ x_n \to s(st) \Longrightarrow B_n x = O(1), \ B_n x \to s(st).$$

1.3 The main object of discussions in this paper is a family $\{A_\alpha\}$ of summability methods A_α, which transform sequences x into sequences $y_\alpha = (y_n^\alpha) = A_\alpha x$, and where α is a continuous parameter with values $\alpha > \alpha_0$ (α_0 is some fixed real number). Denote by ω_{A_α} the set of all x where $y_\alpha = A_\alpha x$ exists and suppose that $\omega_{A_\gamma} \subset \omega_{A_\beta}$ for any $\beta > \gamma > \alpha_0$.

The following definition is given in [15].

Definition 4. *A family $\{A_\alpha\}$ ($\alpha > \alpha_0$) is said to be $\mathcal{A}$) a Cesàro- or $\mathcal{B}$) an Euler–Knopp-type family, if for every $\beta > \gamma > \alpha_0$ the transformed sequences $y_\gamma = (y_n^\gamma)$ and $y_\beta = (y_n^\beta)$ of $x \in \omega_{A_\gamma}$ are related by the connection formula*

$$y_n^\beta = \frac{1}{r_n^\beta} \sum_{k=0}^{n} c_{n-k}^{\beta-\gamma} r_k^\gamma y_k^\gamma \quad (n \in \mathbb{N}_0), \tag{3}$$

where (r_n^α) $(\alpha > \alpha_0)$ are some positive sequences being related by

$$r_n^\beta = \sum_{k=0}^{n} c_{n-k}^{\beta-\gamma} r_k^\gamma \quad (n \in \mathbb{N}_0), \tag{4}$$

and

$$c_n^\alpha = A_n^{\alpha-1} = \binom{n+\alpha-1}{n} \quad (n \in \mathbb{N}_0) \tag{5}$$

in case $\mathcal{A}$) *and*

$$c_n^\alpha = \frac{\alpha^n}{n!} \quad (n \in \mathbb{N}_0) \tag{6}$$

in case $\mathcal{B}$).

Relations (3) and (4) give us the connection formula

$$A_\beta = D_{\gamma,\beta} \circ A_\gamma \quad (\beta > \gamma > \alpha_0)$$

where $D_{\gamma,\beta} = (d_{n,k}^{\gamma,\beta})$ with

$$d_{n,k}^{\gamma,\beta} = \begin{cases} c_{n-k}^{\beta-\gamma} r_k^\gamma / r_n^\beta & \text{if } 0 \le k \le n, \\ 0 & \text{if } k > n. \end{cases} \tag{7}$$

The connection methods $D_{\gamma,\beta}$ are regular (see [15], Lemma 1). The methods $D_{\gamma,\beta}$ can be seen as generalizations of Cesàro methods in case $\mathcal{A}$) and Euler-Knopp methods in case $\mathcal{B}$), that is why $\{A_\alpha\}$ is called a Cesàro-type family in case $\mathcal{A}$) and an Euler–Knopp-type family in case $\mathcal{B}$).

As examples of Cesàro-type (case $\mathcal{A}$)) and Euler–Knopp-type families (case $\mathcal{B}$)) can be seen the families of generalized Nörlund methods (see [15])

$$A_\alpha = N_\alpha = (N, p_n^\alpha, q_n) \ (\alpha > \alpha_0),$$

where

$$y_n^\alpha = \frac{1}{r_n^\alpha} \sum_{k=0}^{n} p_{n-k}^\alpha q_k x_k,$$

$r_n^\alpha = \sum_{k=0}^n p_{n-k}^\alpha q_k$, $p_n^\alpha = \sum_{k=0}^n c_{n-k}^\alpha p_k$ and c_n^α is defined by (5) in case $\mathcal{A}$) and by (6) in case $\mathcal{B}$), and (p_n) and (q_n) are two non-negative sequences with p_0, $q_0 > 0$. The number $\alpha_0 \in \mathbb{R}$ is chosen such that $r_n^\alpha > 0$ for all $n \in \mathbb{N}_0$ and $\alpha > \alpha_0$. Note that for $\beta > \gamma > \alpha_0$ the methods are related through (3). The particular cases of the methods (N, p_n^α, q_n) are the Cesàro methods $(C, \alpha)\,(\alpha > -1)$ in case $\mathcal{A}$) and the Euler-Knopp methods $E_{1/(\alpha+1)}\,(\alpha > 0)$ in case $\mathcal{B}$). More particular cases can be found in [15].

The inclusion relations in a family $\{A_\alpha\}$ are given by the following proposition (see Proposition 1 in [15]).

Proposition 1. *Let $\{A_\alpha\}$ $(\alpha > \alpha_0)$ be a Cesàro- or an Euler–Knopp-type family. Then for sequences $x = (x_n)$, and numbers s and $\beta > \gamma > \alpha_0$ we have:*

i) $x_n = O(A_\gamma) \Longrightarrow x_n = O(A_\beta)$,

ii) $x_n \to s(A_\gamma) \Longrightarrow x_n \to s(A_\beta)$.

The following convexity theorem is true (see Theorem 2.1 in [14]).

Proposition 2. *Let $\{A_\alpha\}$ $(\alpha > \alpha_0)$ be a Cesàro-type family satisfying for any $\beta > \gamma > \alpha_0$ the condition* [1)]

$$K_1 n^{\beta-\gamma} \leq \frac{r_n^\beta}{r_n^\gamma} \leq K_2 n^{\beta-\gamma} \quad (n \in \mathbb{N}) \tag{8}$$

with suitable positive constants K_1 and K_2. Then for sequences $x = (x_n)$, and numbers s and $\beta > \delta > \gamma > \alpha_0$ we have:

$$x_n = O(A_\gamma),\ x_n \to s(A_\beta) \Longrightarrow x_n \to s(A_\delta).$$

In case of Cesàro-type methods $A_\alpha = (N, p_n^\alpha, q_n)$ condition (8) holds for any $\alpha > 0$ if, for example, the conditions

$$np_n = O\left(\sum_{k=0}^n p_k\right) \tag{9}$$

and

$$nq_n = O\left(\sum_{k=0}^n q_k\right)$$

are satisfied (see [14], Lemma 2.1). Any nonincreasing sequence (p_n) satisfies (9). Also, (9) is satisfied if $(p_n) = n^\delta L(n)$, where $\delta > -1$ and $L(n)$ is a slowly varying function (see [14], p.45). In particular, if $A_\alpha = (C, \alpha)$, then (8) is satisfied for any $\alpha > -1$.

One of the main topics in our paper is the notion of strong convergence defined with the help of a given positive sequence $t = (t_n)$ (see [13]).

Definition 5. *Let $\{A_\alpha\}$ $(\alpha > \alpha_0)$ be a Cesàro- or an Euler–Knopp-type family and $t = (t_n)$ be a positive sequence. We say that a sequence $x = (x_n)$ is strongly convergent with respect to the method $A_{\alpha+1}$ with index $t = (t_n)$ (in short, $[A_{\alpha+1}]_t$-convergent) to s and write $x_n \to s[A_{\alpha+1}]_t$ if*

$$\mu_n^{\alpha+1}(t) = \frac{1}{r_n^{\alpha+1}} \sum_{k=0}^n c_{n-k}^1 r_k^\alpha \left|y_k^\alpha - s\right|^{t_k} = o(1). \tag{10}$$

We say that x is strongly bounded with respect to the method $A_{\alpha+1}$ with index $t = (t_n)$ (in short, $[A_{\alpha+1}]_t$-bounded) and write $x_n = O([A_{\alpha+1}]_t)$ if

$$\frac{1}{r_n^{\alpha+1}} \sum_{k=0}^n c_{n-k}^1 r_k^\alpha \left|y_k^\alpha\right|^{t_k} = O(1).$$

Recall that $c_{n-k}^1 = 1$ or $c_{n-k}^1 = 1/(n-k)!$ by (5) or (6), respectively. In particular case of constant exponent $t_n \equiv t$ the last definition was given in [16].

Next proposition gives the inclusion relations (see Theorem 4 in [13]).

Proposition 3. *Let $\{A_\alpha\}$ $(\alpha > \alpha_0)$ be a Cesàro- or an Euler–Knopp-type family. Then for sequences $x = (x_n)$, and numbers s and $\beta > \gamma > \alpha_0$ we have:*

i) *$x_n \to s[A_{\gamma+1}]_t \Rightarrow x_n \to s[A_{\beta+1}]_t$ and $x_n = O([A_{\gamma+1}]_t) \Rightarrow x_n = O([A_{\beta+1}]_t)$, provided that $t = (t_n)$ is nonincreasing and $t_n \geq 1$;*

[1)] Constants K_1 and K_2 in (8) may depend on γ and β. Throughout our paper the coefficients in $O(1)$- and $o(1)$-conditions related to the family $\{A_\alpha\}$ may depend on values of the parameter α. Let us agree not to show this dependence explicitly with indices without special need.

ii) $x_n \to s(A_\gamma) \Rightarrow x_n \to s[A_{\gamma+1}]_t$ *and* $x_n = O(A_\gamma) \Rightarrow x_n = O([A_{\gamma+1}]_t)$, *provided that* $\inf_n t_n = m > 0$;

iii) $x_n \to s[A_{\gamma+1}]_t \Rightarrow x_n \to s[A_{\gamma+1}]_{t'}$ *and* $x_n = O([A_{\gamma+1}]_t) \Rightarrow x_n = O([A_{\gamma+1}]_{t'})$, *provided that* (t_n) *and* (t'_n) *satisfy the conditions* $0 < t'_n \le t_n \le Kt'_n$ *where K is some positive constant;*

iv) $x_n \to s[A_{\gamma+1}]_t \Rightarrow x_n \to s(A_{\gamma+1})$ *and* $x_n = O([A_{\gamma+1}]_t) \Rightarrow x_n = O(A_{\gamma+1})$, *provided that* $1 \le t_n \le M < \infty$.

1.4 The idea of the present paper is to continue the comparison of different strong summability methods $[A_{\alpha+1}]_t$ in Cesàro- and Euler–Knopp-type families started in [13]. A convexity theorem for comparison of three different strong summability methods $[A_{\gamma+1}]_t$, $[A_{\delta+1}]_t$ and $[A_{\beta+1}]_t$ ($\beta > \delta > \gamma > \alpha_0$) in a Cesàro-type family is proved. This convexity theorem can be seen as a generalization of convexity theorems published earlier in [16], [3] and [12] in case of constant exponent $t_n \equiv t$. Interrelations between $[A_{\alpha+1}]_t$-convergence and certain A-statistical A_α-convergence of $x = (x_n)$, i.e., A-statistical convergence of $A_\alpha x = (y_n^\alpha)$ for different values of the parameter α are also investigated and described with the help of theorems. All these results can be transferred to particular cases of the family $\{A_\alpha\}$, e.g., to the families of generalized Nörlund methods (N, p_n^α, q_n).

2. A Convexity Theorem

We prove the following convexity theorem.

Theorem 1. *Let* $\{A_\alpha\}$ $(\alpha > \alpha_0)$ *be a Cesàro-type family. Suppose that* $t = (t_n)$ *is nonincreasing and* $t_n \ge 1$. *Then the following statement is true for sequences* $x = (x_n)$, *and numbers s and* $\beta > \delta > \gamma > \alpha_0$:

$$x_n = O([A_{\gamma+1}]_t),\ x_n \to s[A_{\beta+1}]_t \Longrightarrow x_n \to s[A_{\delta+1}]_t, \tag{11}$$

provided that (8) *is satisfied for any* $\beta > \gamma > \alpha_0$.

For the proof of this theorem the next auxiliary result is needed.

Lemma 1. *Let* $r_\alpha = (r_n^\alpha)$ $(\alpha > \alpha_0)$ *be positive sequences satisfying* (4) *for any* $\beta > \gamma > \alpha_0$. *If* (8) *is satisfied for any* $\beta > \gamma > \alpha_0$, *then for non-negative sequences* $x = (x_n)$ *and numbers* $\gamma > \alpha_0$ *we have:*

i) $\frac{1}{r_n^{\gamma+2}} \sum_{k=0}^{n} r_k^{\gamma+1} x_k = o(1) \Longrightarrow \frac{1}{n} \sum_{k=0}^{n} x_k = o(1)$,

ii) $\frac{1}{n} \sum_{k=0}^{n} x_k = o(1) \Longrightarrow \frac{1}{r_n^{\gamma+1}} \sum_{k=0}^{n} r_k^{\gamma} x_k = o(1)$.

Proof. Statement i) is true due to Theorem 14 in [7].

ii) Fix γ and choose γ', such that $\alpha_0 < \gamma' < \gamma$. We denote $\delta = \gamma - \gamma'$ and get with the help of (8)

$$\begin{aligned} \frac{1}{r_n^{\gamma+1}} \sum_{k=0}^{n} r_k^{\gamma'+\delta} x_k &= O(1) \frac{1}{r_n^{\gamma+1}} \sum_{k=0}^{n} r_k^{\gamma'+1} k^{\delta-1} x_k = O(1) \frac{r_n^{\gamma'+1}}{r_n^{\gamma+1}} \sum_{k=0}^{n} k^{\delta-1} x_k = O(1) \frac{1}{n^\delta} \sum_{k=0}^{n} k^{\delta-1} x_k \\ &= O(1) \frac{1}{A_n^\delta} \sum_{k=0}^{n} A_k^{\delta-1} x_k. \end{aligned}$$

If $\frac{1}{n} \sum_{k=0}^{n} x_k = o(1)$, then $\frac{1}{A_n^\delta} \sum_{k=0}^{n} A_k^{\delta-1} x_k = o(1)$ due to Theorem 14 in [7], and thus ii) holds. □

***Proof of Theorem* 1.** Without loss of generality we may take $s = 0$ and by Proposition 3 i) also $\beta = \gamma + 1$. Suppose that x is strongly bounded with respect to the method $A_{\gamma+1}$ and x is strongly convergent to 0 with respect to the method $A_{\gamma+2}$, and show that x is strongly convergent to 0 with respect to the method $A_{\delta+1}$ for any $\gamma < \delta < \gamma + 1$.

In other words, suppose that $\mu_n^{\gamma+1}(t) = O(1)$ and $\mu_n^{\gamma+2}(t) = o(1)$, and show that $\mu_n^{\gamma+\rho+1}(t) = o(1)$ for any $\rho = \delta - \gamma$ such that $0 < \rho < 1$.

Choose some $\theta \in (0; \frac{1}{2})$ and divide $y_n^{\gamma+\rho} r_n^{\gamma+\rho}$ into two parts:

$$y_n^{\gamma+\rho} r_n^{\gamma+\rho} = \sum_{k=0}^{n} A_{n-k}^{\rho-1} r_k^{\gamma} y_k^{\gamma} = \sum_{k=0}^{n-[\theta n]} A_{n-k}^{\rho-1} r_k^{\gamma} y_k^{\gamma} + \sum_{k=n-[\theta n]+1}^{n} A_{n-k}^{\rho-1} r_k^{\gamma} y_k^{\gamma}.$$

Using Abel transformation for the first sum of the right side of the last equation and substituting $v = n-k$ for the second sum we get the equality

$$y_n^{\gamma+\rho} r_n^{\gamma+\rho} = A_{[\theta n]}^{\rho-1} r_{n-[\theta n]}^{\gamma+1} y_{n-[\theta n]}^{\gamma+1} + \sum_{k=0}^{n-[\theta n]-1} A_{n-k}^{\rho-2} r_k^{\gamma+1} y_k^{\gamma+1} + \sum_{v=0}^{[\theta n]-1} A_v^{\rho-1} r_{n-v}^{\gamma} y_{n-v}^{\gamma} = U_n + V_n + W_n.$$

Therefore, $y_n^{\gamma+\rho} = (U_n + V_n + W_n)/r_n^{\gamma+\rho}$.

Now, using the inequality

$$|a+b+c|^r \leq |a|^r + |b|^r + |c|^r \quad (r \leq 1)$$

with $r = \frac{t_n}{M}$, where $M = \sup_n t_n$, we have

$$\left|y_k^{\gamma+\rho}\right|^{\frac{t_k}{M}} \leq \left|\frac{U_k}{r_k^{\gamma+\rho}}\right|^{\frac{t_k}{M}} + \left|\frac{V_k}{r_k^{\gamma+\rho}}\right|^{\frac{t_k}{M}} + \left|\frac{W_k}{r_k^{\gamma+\rho}}\right|^{\frac{t_k}{M}}.$$

Further we get with the help of the Minkowski inequality:

$$\begin{aligned}
\left[\mu_n^{\gamma+\rho+1}(t)\, r_n^{\gamma+\rho+1}\right]^{\frac{1}{M}} &= \left[\sum_{k=0}^{n} r_k^{\gamma+\rho} \left(\left|y_k^{\gamma+\rho}\right|^{\frac{t_k}{M}}\right)^M\right]^{\frac{1}{M}} \leq \left[\sum_{k=0}^{n} r_k^{\gamma+\rho} \left(\left|\frac{U_k}{r_k^{\gamma+\rho}}\right|^{\frac{t_k}{M}} + \left|\frac{V_k}{r_k^{\gamma+\rho}}\right|^{\frac{t_k}{M}} + \left|\frac{W_k}{r_k^{\gamma+\rho}}\right|^{\frac{t_k}{M}}\right)^M\right]^{\frac{1}{M}} \\
&\leq \left[\sum_{k=0}^{n} \frac{|U_k|^{t_k}}{\left(r_k^{\gamma+\rho}\right)^{t_k-1}}\right]^{\frac{1}{M}} + \left[\sum_{k=0}^{n} \frac{|V_k|^{t_k}}{\left(r_k^{\gamma+\rho}\right)^{t_k-1}}\right]^{\frac{1}{M}} + \left[\sum_{k=0}^{n} \frac{|W_k|^{t_k}}{\left(r_k^{\gamma+\rho}\right)^{t_k-1}}\right]^{\frac{1}{M}} \\
&= (U_n')^{\frac{1}{M}} + (V_n')^{\frac{1}{M}} + (W_n')^{\frac{1}{M}}.
\end{aligned}$$

It follows immediately from the last inequality that

$$\left[\mu_n^{\gamma+\rho+1}(t)\right]^{\frac{1}{M}} \leq \left(\frac{U_n'}{r_n^{\gamma+\rho+1}}\right)^{\frac{1}{M}} + \left(\frac{V_n'}{r_n^{\gamma+\rho+1}}\right)^{\frac{1}{M}} + \left(\frac{W_n'}{r_n^{\gamma+\rho+1}}\right)^{\frac{1}{M}}. \tag{12}$$

Next we evaluate the sums U_n', V_n' and W_n' separately, starting from W_n'.

Using Hölder inequality and the definition of a Cesàro-type family, we have

$$\begin{aligned}
|W_n|^{t_n} &= \left|\sum_{k=0}^{[\theta n]-1} A_k^{\rho-1} r_{n-k}^{\gamma} y_{n-k}^{\gamma}\right|^{t_n} \leq \left(\sum_{k=0}^{[\theta n]} A_k^{\rho-1} r_{n-k}^{\gamma} \left|y_{n-k}^{\gamma}\right|\right)^{t_n} \leq \left(\sum_{k=0}^{[\theta n]} A_k^{\rho-1} r_{n-k}^{\gamma}\right)^{t_n-1} \sum_{k=0}^{[\theta n]} A_k^{\rho-1} r_{n-k}^{\gamma} \left|y_{n-k}^{\gamma}\right|^{t_n} \\
&\leq \left(r_n^{\gamma+\rho}\right)^{t_n-1} \sum_{k=0}^{[\theta n]} A_k^{\rho-1} r_{n-k}^{\gamma} \left|y_{n-k}^{\gamma}\right|^{t_n}.
\end{aligned}$$

Further we get

$$\begin{aligned}
W_n' &= \sum_{k=0}^{n} \frac{|W_k|^{t_k}}{\left(r_k^{\gamma+\rho}\right)^{t_k-1}} \leq \sum_{k=0}^{n} \frac{\left(r_k^{\gamma+\rho}\right)^{t_k-1} \sum_{v=0}^{[\theta k]} A_v^{\rho-1} r_{k-v}^{\gamma} \left|y_{k-v}^{\gamma}\right|^{t_k}}{\left(r_k^{\gamma+\rho}\right)^{t_k-1}} = \sum_{k=0}^{n} \sum_{v=0}^{[\theta k]} A_v^{\rho-1} r_{k-v}^{\gamma} \left|y_{k-v}^{\gamma}\right|^{t_k} \\
&\leq \sum_{v=0}^{[\theta n]} A_v^{\rho-1} \sum_{k=[v/\theta]}^{n} r_{k-v}^{\gamma} \left|y_{k-v}^{\gamma}\right|^{t_k}.
\end{aligned}$$

So we have the inequality

$$W'_n \le \sum_{v=0}^{[\theta n]} A_v^{\rho-1} \sum_{k=0}^{n-v} r_k^{\gamma} \left|y_k^{\gamma}\right|^{t_{k+v}}. \tag{13}$$

Let us denote

$$u_{k,v}^{\gamma} = \begin{cases} \left|y_k^{\gamma}\right|^{t_{k+v}} & \text{if } \left|y_k^{\gamma}\right| \ge 1, \\ 0 & \text{if } \left|y_k^{\gamma}\right| < 1 \end{cases}$$

and

$$w_{k,v}^{\gamma} = \begin{cases} \left|y_k^{\gamma}\right|^{t_{k+v}} & \text{if } \left|y_k^{\gamma}\right| < 1, \\ 0 & \text{if } \left|y_k^{\gamma}\right| \ge 1\,. \end{cases}$$

Thus we have the relations:

$$\left|y_k^{\gamma}\right|^{t_{k+v}} = u_{k,v}^{\gamma} + w_{k,v}^{\gamma}, \qquad u_{k,v}^{\gamma} \le \left|y_k^{\gamma}\right|^{t_k}, \qquad w_{k,v}^{\gamma} \le \left|y_k^{\gamma}\right|.$$

Now we can develop inequality (13), denoting $\mathbf{1} = (1, 1, ...)$:

$$\begin{aligned} W'_n &\le \sum_{v=0}^{[\theta n]} A_v^{\rho-1} \sum_{k=0}^{n-v} r_k^{\gamma} \left|y_k^{\gamma}\right|^{t_{k+v}} = \sum_{v=0}^{[\theta n]} A_v^{\rho-1} \sum_{k=0}^{n-v} r_k^{\gamma} u_{k,v}^{\gamma} + \sum_{v=0}^{[\theta n]} A_v^{\rho-1} \sum_{k=0}^{n-v} r_k^{\gamma} w_{k,v}^{\gamma} \\ &\le \sum_{v=0}^{[\theta n]} A_v^{\rho-1} \sum_{k=0}^{n-v} r_k^{\gamma} \left|y_k^{\gamma}\right|^{t_k} + \sum_{v=0}^{[\theta n]} A_v^{\rho-1} \sum_{k=0}^{n-v} r_k^{\gamma} \left|y_k^{\gamma}\right| = \sum_{v=0}^{[\theta n]} A_v^{\rho-1} r_{n-v}^{\gamma+1} \mu_{n-v}^{\gamma+1}(t) + \sum_{v=0}^{[\theta n]} A_v^{\rho-1} r_{n-v}^{\gamma+1} \mu_{n-v}^{\gamma+1}(\mathbf{1}) \\ &\le \sum_{v=0}^{[\theta n]} A_v^{\rho-1} r_n^{\gamma+1} \mu_n^{\gamma+1}(t) + \sum_{v=0}^{[\theta n]} A_v^{\rho-1} r_n^{\gamma+1} \mu_n^{\gamma+1}(\mathbf{1}) = O(r_n^{\gamma+1}(\theta n)^{\rho}), \end{aligned}$$

because $\mu_n^{\gamma+1}(t) = O(1) \Longrightarrow \mu_n^{\gamma+1}(\mathbf{1}) = O(1)$ by Proposition 3 iii). Further we get with the help of (8) that

$$W'_n = \sum_{k=0}^{n} \frac{|W_k|^{t_k}}{\left(r_k^{\gamma+\rho}\right)^{t_k-1}} = O(1)\left(r_n^{\gamma+\rho+1}\theta^{\rho}\right). \tag{14}$$

Next we evaluate the sum U'_n.

Using characteristics of Cesàro numbers (see [1], [7]) and relation (8) we get:

$$\frac{|U_k|}{r_k^{\gamma+\rho}} = \frac{A_{[\theta k]}^{\rho-1} r_{k-[\theta k]}^{\gamma+1} \left|y_{k-[\theta k]}^{\gamma+1}\right|}{r_k^{\gamma+\rho}} \le \frac{M([\theta k]+1)^{\rho-1} r_k^{\gamma+1} \left|y_{k-[\theta k]}^{\gamma+1}\right|}{r_k^{\gamma+\rho}} \le M(\theta k)^{\rho-1} k^{1-\rho} \left|y_{k-[\theta k]}^{\gamma+1}\right| \le H_{\theta} \left|y_{k-[\theta k]}^{\gamma+1}\right|.$$

Thus we have

$$\sum_{k=0}^{n} \frac{|U_k|^{t_k}}{\left(r_k^{\gamma+\rho}\right)^{t_k}} = O_{\theta}(1) \sum_{k=0}^{n} \left|y_{k-[\theta k]}^{\gamma+1}\right|^{t_k} = O_{\theta}(1) \sum_{k=0}^{n-[\theta n]} \left|y_k^{\gamma+1}\right|^{t_{[k/(1-\theta)]}} = O_{\theta}(1) \sum_{k=0}^{n} \left|y_k^{\gamma+1}\right|^{t_{[k/(1-\theta)]}}. \tag{15}$$

Denoting

$$u_{k,\theta}^{\gamma+1} = \begin{cases} \left|y_k^{\gamma+1}\right|^{t_{[k/(1-\theta)]}} & \text{if } \left|y_k^{\gamma+1}\right| \ge 1, \\ 0 & \text{if } \left|y_k^{\gamma+1}\right| < 1 \end{cases}$$

and

$$w_{k,\theta}^{\gamma+1}=\begin{cases}\left|y_k^{\gamma+1}\right|^{t_{[k/(1-\theta)]}} & \text{if } \left|y_k^{\gamma+1}\right|<1,\\ 0 & \text{if } \left|y_k^{\gamma+1}\right|\geq 1\,,\end{cases}$$

we get the relations:

$$\left|y_k^{\gamma+1}\right|^{t_{[k/(1-\theta)]}}=u_{k,\theta}^{\gamma+1}+w_{k,\theta}^{\gamma+1},\qquad u_{k,\theta}^{\gamma+1}\leq\left|y_k^{\gamma+1}\right|^{t_k},\qquad w_{k,\theta}^{\gamma+1}\leq\left|y_k^{\gamma+1}\right|.$$

Developing relations (15) we get

$$\sum_{k=0}^{n}\frac{|U_k|^{t_k}}{\left(r_k^{\gamma+\rho}\right)^{t_k}} = O_\theta(1)\sum_{k=0}^{n}\left|y_k^{\gamma+1}\right|^{t_{[k/(1-\theta)]}}=O_\theta(1)\sum_{k=0}^{n}(u_{k,\theta}^{\gamma+1}+w_{k,\theta}^{\gamma+1})=O_\theta(1)\left(\sum_{k=0}^{n}\left|y_k^{\gamma+1}\right|^{t_k}+\sum_{k=0}^{n}\left|y_k^{\gamma+1}\right|\right).$$

As $\mu_n^{\gamma+2}(t)=o(1)$, then $\mu_n^{\gamma+2}(\mathbf{1})=o(1)$ and therefore

$$\sum_{k=0}^{n}\left|y_k^{\gamma+1}\right|^{t_k}+\sum_{k=0}^{n}\left|y_k^{\gamma+1}\right|=o(n)$$

by Lemma 1 i). That is why

$$\sum_{k=0}^{n}\frac{|U_k|^{t_k}}{\left(r_k^{\gamma+\rho}\right)^{t_k}}=o_\theta(n).$$

Further we get with the help of Lemma 1 ii) that

$$U_n'=\sum_{k=0}^{n}\frac{|U_k|^{t_k}\,r_k^{\gamma+\rho}}{\left(r_k^{\gamma+\rho}\right)^{t_k}}=o_\theta(r_n^{\gamma+\rho+1}).\tag{16}$$

Finally we evaluate the sum V_n'.
As $\mu_n^{\gamma+2}(t)=o(1)$, then $\mu_n^{\gamma+2}(\mathbf{1})=o(1)$ and therefore

$$\begin{aligned}|V_k| &= \left|\sum_{v=0}^{k-[\theta k]}A_{k-v}^{\rho-2}r_v^{\gamma+1}y_v^{\gamma+1}\right|\leq\sum_{v=0}^{k-[\theta k]}\left|A_{k-v}^{\rho-2}\right|r_v^{\gamma+1}\left|y_v^{\gamma+1}\right|\leq H\sum_{v=0}^{k-[\theta k]}(k-v+1)^{\rho-2}r_v^{\gamma+1}\left|y_v^{\gamma+1}\right|\\ &\leq H(\theta k)^{\rho-2}\sum_{v=0}^{k}r_v^{\gamma+1}\left|y_v^{\gamma+1}\right|=o\left((\theta k)^{\rho-2}r_k^{\gamma+2}\right).\end{aligned}$$

With the help of condition (8) we get

$$|V_k|=o\left(\frac{r_k^{\gamma+\rho}k^{2-\rho}}{\theta^{2-\rho}k^{2-\rho}}\right)=o_\theta(r_k^{\gamma+\rho}).$$

As the method $D_{\gamma+\rho,\gamma+\rho+1}$ is regular we have

$$\sum_{k=0}^{n}\frac{|V_k|^{t_k}}{\left(r_k^{\gamma+\rho}\right)^{t_k-1}}=\sum_{k=0}^{n}\frac{|V_k|^{t_k}\,r_k^{\gamma+\rho}}{\left(r_k^{\gamma+\rho}\right)^{t_k}}=o_\theta(r_n^{\gamma+\rho+1}),$$

and thus

$$V_n'=o_\theta(r_n^{\gamma+\rho+1}).\tag{17}$$

Now we are able to complete our proof.

Let $\varepsilon > 0$ be a given number. Due to (14) we can choose $\theta_\varepsilon \in (0;1/2)$, so that $\left[W'_n/r_n^{\gamma+\rho+1}\right]^{\frac{1}{M}} < \varepsilon/3$ for any n. Due to (16) and (17) we can choose now n_0, so that $\left[U'_n/r_n^{\gamma+\rho+1}\right]^{\frac{1}{M}} < \varepsilon/3$ and $\left[V'_n/r_n^{\gamma+\rho+1}\right]^{\frac{1}{M}} < \varepsilon/3$ for all $n > n_0$.

It follows now from (12) that

$$\left[\mu_n^{\gamma+\rho+1}(t)\right]^{\frac{1}{M}} = \left[\frac{1}{r_n^{\gamma+\rho+1}} \sum_{k=0}^{n} r_k^{\gamma+\rho} \left|y_v^{\gamma+\rho}\right|^{t_k}\right]^{\frac{1}{M}} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

for all $n > n_0$.

Thus we have shown that $\mu_n^{\gamma+\rho+1}(t) = \mu_n^{\delta+1}(t) = o(1)$, and therefore implication (11) holds for any $\beta > \delta > \gamma > \alpha_0$. □

In particular case of constant exponent $t_n \equiv t$ Theorem 1 turns into theorem which was formulated without proof as Theorem 3 in [16]. Moreover, for methods $A_\alpha = (N, p_n^\alpha, q_n)$ an analogous theorem was proved in [12] and, in particular, for $A_\alpha = (N, p_n^\alpha, 1)$ in [3].

The following remark bases on Lemma 1.

Remark 1. Let $\{A_\alpha\}$ $(\alpha > \alpha_0)$ be a Cesàro-type family. If (8) is satisfied for any $\beta > \alpha > \alpha_0$, then $[A_{\alpha+1}]_t$-convergence of $x = (x_n)$ to s can be defined as

$$\frac{1}{n} \sum_{k=0}^{n} \left|y_k^\alpha - s\right|^{t_k} = o(1) \tag{18}$$

for any $\alpha > \alpha_0 + 1$ due to Lemma 1. In particular, if $A_\alpha = (C, \alpha)$, then (18) defines $[A_{\alpha+1}]_t$-convergence for any $\alpha > -1$ due to Theorem 14 in [7].

3. Comparison of A_α- and $[A_{\alpha+1}]_t$-convergences with Some Statistical Convergence

We compare A_α- and $[A_{\alpha+1}]_t$-convergences of x with its A-statistical A_α-convergence (and, in particular, with its statistical A_α-convergence) for different values of parameter α, where $A = D_{\alpha,\alpha+1}$ is the matrix method defined by (7).

Denote $(A_n^\alpha x) = (y_n^\alpha) = A_\alpha x$, and recall that x is $D_{\alpha,\alpha+1}$-statistically A_α-convergent by Definition 2 if $A_n^\alpha x \to s(st_{D_{\alpha,\alpha+1}})$, and x is statistically A_α-convergent if $A_n^\alpha x \to s(st)$. Recall also that $t = (t_n)$ is a positive sequence.

The following auxiliary result will be used.

Lemma 2. *Let A be a regular non-negative matrix method defined by transformation* (1)*. Suppose that* $\sup_n t_n = M < \infty$*. Then the following statements are true for sequences $x = (x_n)$ and numbers s:*

i) *if*

$$\sum_{k=0}^{\infty} a_{n,k} \left|x_k - s\right|^{t_k} = o(1), \tag{19}$$

then $x_n \to s\,(st_A)$,

ii) *if $x_n = O(1)$ and $x_n \to s(st_A)$, then* (19) *is satisfied, provided that* $\inf_n t_n = m > 0$.

Proof. Choose an arbitrary $\varepsilon > 0$ and consider the set $\mathcal{K}_\varepsilon$ defined by (2).

i) We have the inequalities

$$\sum_{k=0}^{\infty} a_{n,k} |x_k - s|^{t_k} \geq \sum_{\mathcal{K}_\varepsilon} a_{n,k} |x_k - s|^{t_k} \geq h(\varepsilon) \sum_{\mathcal{K}_\varepsilon} a_{n,k},$$

where $h(\varepsilon) = \min\{1, \varepsilon^M\}$. If (19) holds, also the sum in the right side of the last inequalities tends to zero, i.e., $x_n \to s(st_A)$. That proves i).

ii) Denoting $\mathcal{K}_\varepsilon^* = \{k : |x_k - s| < \varepsilon\}$, we get:

$$\sum_{k=0}^{\infty} a_{n,k} |x_k - s|^{t_k} = \sum_{\mathcal{K}_\varepsilon} a_{n,k} |x_k - s|^{t_k} + \sum_{\mathcal{K}_\varepsilon^*} a_{n,k} |x_k - s|^{t_k} \leq (L + |s|)^M \sum_{\mathcal{K}_\varepsilon} a_{n,k} + H(\varepsilon) \sum_{k=0}^{\infty} a_{n,k},$$

where $|x_n| \leq L$ and $H(\varepsilon) = \max\{\varepsilon^m, \varepsilon^M\}$. Further, if $x_n \to s(st_A)$, i.e., if $\lim_{n\to\infty} \sum_{\mathcal{K}_\varepsilon} a_{n,k} = 0$, then

$$\limsup_n \sum_{k=0}^{\infty} a_{n,k} |x_k - s|^{t_k} \leq H(\varepsilon)$$

because $\lim_n \sum_{k=0}^{\infty} a_{n,k} = 1$ by regularity of A. As $\varepsilon > 0$ is arbitrarily chosen, the last inequality implies (19). □

We note that statements i) and ii) of Lemma 2 can be proved also as direct applications of Corollaries 3.3 and 3.4 in [9], respectively. References on developments of these statements can be also found in [9].

Theorem 2. *Let $\{A_\alpha\}$ ($\alpha > \alpha_0$) be a Cesàro- or an Euler–Knopp-type family. Then for sequences $x = (x_n)$, and numbers s and $\beta \geq \gamma > \alpha_0$ we have:*

i) $x_n \to s(A_\gamma) \Longrightarrow A_n^\beta x \to s(st_{D_{\beta,\beta+1}})$,

ii) $x_n = O(A_\gamma)$, $A_n^\gamma x \to s(st_{D_{\gamma,\gamma+1}}) \Longrightarrow x_n \to s(A_{\beta+1})$.

Proof. The implications

$$x_n \to s(A_\gamma) \Longrightarrow y_n^\gamma \to s[A_{\gamma+1}]_\mathbf{1} \Longrightarrow y_n^\gamma \to (st_{D_{\gamma,\gamma+1}}) \tag{20}$$

are true by Proposition 3 ii) and Lemma 2 i), if we apply Lemma 2 i) to $A = D_{\gamma,\gamma+1}$ and (y_n^γ) (instead of (x_n)) and remember that $A_{\gamma+1} = D_{\gamma,\gamma+1} \circ A_\gamma$. The implications

$$y_n^\gamma = O(1),\ y_n^\gamma \to s(st_{D_{\gamma,\gamma+1}}) \Longrightarrow y_n^\gamma \to s[A_{\gamma+1}]_\mathbf{1} \Longrightarrow x_n \to s(A_{\gamma+1}) \tag{21}$$

are true due to Lemma 2 ii) (with $A = D_{\gamma,\gamma+1}$) and Proposition 3 iv). Statements i) and ii) follow from (20) and (21), respectively, because $x_n \to s(A_\gamma) \Longrightarrow x_n \to s(A_\beta)$ by Proposition 1 ii). □

Theorem 3. *Let $\{A_\alpha\}$ ($\alpha > \alpha_0$) be a Cesàro- or an Euler–Knopp-type family. Suppose that $\sup_n t_n = M < \infty$. Then we have for any $\beta \geq \gamma > \alpha_0$:*

i) $x_n \to s[A_{\gamma+1}]_t \Longrightarrow A_n^\gamma x \to s(st_{D_{\gamma,\gamma+1}})$;

ii) $x_n = O(A_\gamma)$, $A_n^\gamma x \to s(st_{D_{\gamma,\gamma+1}}) \Longrightarrow x_n \to s[A_{\gamma+1}]_t$, *provided that* $\inf_n t_n = m > 0$;

iii) $x_n = O(A_\gamma)$, $x_n \to s[A_{\gamma+1}]_t \Longrightarrow x_n \to s(A_{\beta+1})$.

Moreover, if $t = (t_n)$ is nonincreasing and $\inf_n t_n = m \geq 1$, *then:*

iv) $x_n \to s[A_{\gamma+1}]_t \Longrightarrow A_n^\beta x \to s(st_{D_{\beta,\beta+1}})$;

v) $x_n = O(A_\gamma)$, $A_n^\gamma x \to s(st_{D_{\gamma,\gamma+1}}) \Longrightarrow x_n \to s[A_{\beta+1}]_t$.

Proof. i) and ii) are true by Lemma 2, if we take $A = D_{\gamma,\gamma+1}$ in it.

iii) follows immediately from statement i) and Theorem 2 ii).

iv) and v) follow from statements i) and ii), respectively, because $x_n \to s[A_{\gamma+1}]_t \Longrightarrow x_n \to s[A_{\beta+1}]_t$ by Proposition 3 i). □

Remark 2. i) Theorem 3 i) and Theorem 3 ii) imply the statement: if $x_n = O(A_\gamma)$ then $[A_{\gamma+1}]_t$- and $[A_{\gamma+1}]_{\mathbf{1}}$- convergences of x and $D_{\gamma,\gamma+1}$-statistical convergence of $A_\gamma x$ to s are equivalent, provided that $\sup_n t_n = M < \infty$ and $\inf_n t_n = m > 0$ (compare with Proposition 3 iii)).

ii) If $t_n \geq 1$ then the condition $x_n = O(A_\gamma)$ can be dropped in Theorem 3 iii) (compare with Proposition 3 iv) and Proposition 1 i)).

Further we consider only Cesàro-type families.

Theorem 4. *Let $\{A_\alpha\}$ $(\alpha > \alpha_0)$ be a Cesàro-type family. Suppose that $\sup_n t_n = M < \infty$ and (8) is satisfied for any $\beta > \gamma > \alpha_0$. Then we have:*

i) $x_n \to s[A_{\gamma+1}]_t \Longrightarrow A_n^\gamma x \to s(st)$ *for any* $\gamma > \alpha_0 + 1$;

ii) $x_n = O(A_\gamma)$, $A_n^\gamma x \to s(st) \Longrightarrow x_n \to s[A_{\gamma+1}]_t$ *for any* $\gamma > \alpha_0$, *provided that* $\inf_n t_n = m > 0$;

iii) $x_n = O(A_\gamma)$, $A_n^\gamma x \to s(st) \Longrightarrow A_n^\beta x \to s(st)$ *for any* $\gamma > \alpha_0$ *and* $\beta > \max\{\gamma, \alpha_0 + 1\}$.

Moreover, if $t = (t_n)$ is nonincreasing and $\inf_n t_n = m \geq 1$, then we have:

iv) $x_n \to s[A_{\gamma+1}]_t \Longrightarrow A_n^\beta x \to s(st)$ *for any* $\gamma > \alpha_0$ *and* $\beta > \max\{\gamma, \alpha_0 + 1\}$;

v) $x_n = O(A_\gamma)$, $A_n^\gamma x \to s(st) \Longrightarrow x_n \to s[A_{\beta+1}]_t$ *for any* $\beta \geq \gamma > \alpha_0$.

Proof. i) If $x_n \to s[A_{\gamma+1}]_t$ for some $\gamma > \alpha_0 + 1$ then (18) holds by Lemma 1 i); (18), in its turn, implies $y_n^\gamma \to s(st)$ by Lemma 2 i) (take $A = (C, 1)$ in it).

ii) If $y_n^\gamma = O(1)$ and $y_n^\gamma \to s(st)$, then (18) holds by Lemma 2 ii), and $x_n \to s[A_{\gamma+1}]_t$ for any $\gamma > \alpha_0$ by Lemma 1 ii).

iii) If $y_n^\gamma = O(1)$ and $y_n^\gamma \to s(st)$, then $x_n \to s[A_{\gamma+1}]_{\mathbf{1}}$ by ii) and $x_n \to s[A_{\beta+1}]_{\mathbf{1}}$ by Proposition 3 i), and therefore $y_n^\beta \to s(st)$ by i).

iv) and v) follow from i) and ii) with the help of Proposition 3 i). □

Theorem 5. *Let $\{A_\alpha\}$ $(\alpha > \alpha_0)$ be a Cesàro-type family. Suppose that (8) holds for any $\beta > \gamma > \alpha_0$ and t is a nonincreasing sequence with $t_n \geq 1$. If $x_n = O([A_{\gamma+1}]_t)$ and $x_n \to s[A_{\beta+1}]_t$ for some $\beta > \gamma > \alpha_0$, then $A_n^\delta x \to s(st)$ for any $\delta > \max\{\gamma, \alpha_0 + 1\}$.*

Proof. Due to Theorem 1 the conditions $x_n = O([A_{\gamma+1}]_t)$ and $x_n \to s[A_{\beta+1}]_t$ imply $x_n \to s[A_{\delta+1}]_t$ for any $\beta > \delta > \gamma$. Also, $x_n \to s[A_{\beta+1}]_t$ implies $x_n \to s[A_{\delta+1}]_t$ for any $\delta > \beta$ due to Proposition 3 i) which, in its turn, implies $A_n^\delta x \to s(st)$ for any $\delta > \max\{\gamma, \alpha_0 + 1\}$ due to Theorem 4 i). □

Theorem 6. *Let $\{A_\alpha\}$ $(\alpha > \alpha_0)$ be a Cesàro-type family. Suppose that (8) is satisfied for any $\beta > \gamma > \alpha_0$. If $x_n = O(A_\gamma)$ and $A_n^\beta x \to s(st)$ for some $\beta > \gamma > \alpha_0$, then $A_n^\delta x \to s(st)$ for any $\delta > \max\{\gamma, \alpha_0 + 1\}$.*

Proof. If $A_n^\gamma x = O(1)$ then $A_n^\beta x = O(1)$ and $x_n = O([A_{\gamma+1}]_1)$ due to Proposition 1 i) and Proposition 3 ii), respectively. By Theorem 4 ii) we have

$$A_n^\beta x = O(1),\ A_n^\beta x \to s(st) \Longrightarrow x_n \to s[A_{\beta+1}]_1.$$

Now we can use Theorem 5 to finish our proof. □

Remark 3. Theorem 4 iii) remains valid if we replace $y_\gamma = (A_n^\gamma x)$ with any sequence $y = (y_n)$ in it. Thus the implication

$$y_n = O(1),\ y_n \to s(st) \Longrightarrow z_n^\beta = O(1),\ z_n^\beta \to s(st),$$

where $(z_n^\beta) = D_{\gamma,\beta}y$, is true for any $\gamma > \alpha_0$ and $\beta > \max\{\gamma, \alpha_0 + 1\}$, which shows that $D_{\gamma,\beta}$ is statistically regular.

Theorem 7. *Let $\{A_\alpha\}$ $(\alpha > \alpha_0)$ be a Cesàro-type family satisfying* (8) *for any $\beta > \gamma > \alpha_0$. Then the matrix methods $D_{\gamma,\beta}$ defined by this family are statistically regular for any $\gamma > \alpha_0$ and $\beta > \max\{\gamma, \alpha_0 + 1\}$.*

Remark 4. In particular, if $A_\alpha = (C, \alpha)$ $(\alpha > -1)$, the inequalities $\gamma > \alpha_0 + 1$ and $\beta > \max\{\gamma, \alpha_0 + 1\}$ can be replaced by $\gamma > -1$ and $\beta > \gamma > -1$, respectively, everywhere in Theorems 4 and 7. Also, the inequality $\delta > \max\{\gamma, \alpha_0 + 1\}$ can be replaced by $\delta > \gamma > -1$ everywhere in Theorems 5 and 6.

References

[1] J. Boos, Classical and Modern Methods in Summability, Oxford University Press (2000) 283–298.
[2] M. Burgin and O. Duman, Statistical fuzzy convergence, International Journal of Uncertaintly, fuzziness and knowledge-based systems 16 (2008) 879–902.
[3] F. P. Cass, Convexity theorems for Nörlund and strong Nörlund summability, Math. Z. 112 (1968) 357–363.
[4] J. Connor, On strong matrix summability with respect to a modulus and statistical convergence, Canad. Math. Bull. 32 (1989) 194–198.
[5] J. Connor, A topological and functional analytic approach to statistical convergence, In Analysis of divergence, Appl. Numer. Harmon. Anal., Birkhauser Boston, Boston, MA (1999) 403–413.
[6] H. Fast, Sur la convergence statistique, Colloq. Math. 2 (1951) 241–244.
[7] G. H. Hardy, Divergent series, Oxford Univ. Press, London, U.K., 1949.
[8] E. Kolk, Matrix summability of statistically convergent sequences, Analysis 13 (1993) 77–83.
[9] E. Kolk, Inclusion relations between the statistical convergence and strong summability, Acta Comment. Univ. Tartu. Math. (1998) 39–54.
[10] E. Kolk, Mann iterations via statistically regular matrices, Fixed Point Theory 11 (2010) 59–65.
[11] F. Móricz, Tauberian conditions under which statistical convergence follows from statistical summability $(C, 1)$, J. Math. Anal. Appl. 275 (2002) 277–287.
[12] R. Sinha, Convexity theorem for $[N, p, q]$ summability, Kyungpook Math. J. 15 (1975) 65–79.
[13] V. Soomer and A. Tali, On strong summability of sequences, Acta Comment. Univ. Tartu. Math. 11 (2007) 57–68.
[14] U. Stadtmüller and A. Tali, On some families of certain Nörlund methods and power series methods, J. Math. Anal. Appl. 238 (1999) 44–66.
[15] U. Stadtmüller and A. Tali, Comparison of certain summability methods by speeds of convergence, Analysis Mathematica 29 (2003) 227–242.
[16] U. Stadtmüller and A. Tali, Strong summability in certain families of summability methods, Acta Sci. Math. (Szeged) 70 (2004) 639–657.