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A Numerical Radius Version of the Arithmetic-Geometric Mean of Operators

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Abstract. In this paper, we obtain some numerical radius inequalities for operators, in particular for positive definite operators A, B a numerical radius and some operator norm versions for arithmetic-geometric mean inequality are obtained, respectively as

$$\omega^2(A\sharp B) \leqslant \omega\left(\frac{A^2+B^2}{2}\right) - \frac{1}{2}\inf_{\|x\|=1}\delta(x),$$

where $\delta(x) = \langle (A-B)x, x\rangle^2$, and

$$\|A\|\|B\| \leqslant \frac{1}{2}(\|A^2\| + \|B^2\|) - \frac{1}{2}\inf_{\|x\|=\|y\|=1}\delta(x,y),$$

where, $\delta(x,y) = (\langle Ay, y\rangle - \langle Bx, x\rangle)^2$.

1. Introduction

Let $\boldsymbol{H}$ be a complex Hilbert space with inner product $\langle .,. \rangle$ and let $\mathcal{B}(\boldsymbol{H})$ denote the algebra of all bounded linear operators on $\boldsymbol{H}$. Let $|||.|||$ denote any unitarily invariant norm, i.e., a norm with the property that $|||UAV||| = |||A|||$, for all $A \in \mathcal{B}(\boldsymbol{H})$ and for all unitary $U, V \in \mathcal{B}(\boldsymbol{H})$.
For $A \in \mathcal{B}(\boldsymbol{H})$, the spectral norm of A is defined by

$$\|A\| = \sup\{|\langle Ax, y\rangle| : \|x\| = \|y\| = 1,\ x, y \in \boldsymbol{H}\}.$$

It is evident that this norm is unitary invariant.
The numerical range of a $A \in \mathcal{B}(\boldsymbol{H})$ is defined as

$$W(A) = \sup\{\langle Ax, x\rangle : \|x\| = 1,\ x \in \boldsymbol{H}\}.$$

For any $A \in \mathcal{B}(\boldsymbol{H})$, $\overline{W(A)}$ is a convex subset of the complex plane containing the spectrum of A. See [5, Chapter 2] for this topic.
The numerical radius of $A \in \mathcal{B}(\boldsymbol{H})$ is defined by

$$\omega(A) = \sup\{|\lambda| : \lambda \in W(A)\}.$$

We recall the following results that were proved in [6].

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Lemma 1.1. *Let $A \in \mathcal{B}(\boldsymbol{H})$ and let $\omega(.)$ be the numerical radius. Then*
(i) $\omega(.)$ is a norm on $\mathcal{B}(\boldsymbol{H})$,
(ii) $\omega(UAU^*) = \omega(A)$, for all unitary operators U,
(iii) $\omega(A) = \|A\|$ if (but not only if) A is normal,
(iv) $\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|$.

Moreover, $\omega(.)$ is not a unitarily invariant norm and is not submultiplicative.
For positive real numbers a and b, the most familiar form of the Young inequality is the following:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \tag{1}$$

where $p, q > 1$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$, or equivalently

$$a^\nu b^{1-\nu} \leqslant \nu a + (1-\nu)b,$$

with $\nu \in [0, 1]$. Recently, Kittaneh and Manasrah [8] obtained a refinement of (1)

$$ab + r_0(a^{p/2} - b^{q/2})^2 \leq \frac{a^p}{p} + \frac{b^q}{q}, \tag{2}$$

where $r_0 = \min\{\frac{1}{p}, \frac{1}{q}\}$.
For positive definite operators $A, B \in \mathcal{B}(\boldsymbol{H})$, the operator geometric mean is defined by

$$A\sharp B \equiv A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

The operator geometric mean has the symmetric property ($A\sharp B = B\sharp A$). If $AB = BA$, then $A\sharp B = (AB)^{1/2}$. In this paper we obtain some inequalities (upper bound) for $\omega((A\sharp B)X)$, where $X \in \mathcal{B}(\boldsymbol{H})$ is arbitrary. Throughout the paper we use the notation $A > 0$ to mean that A is positive definite and $\mathbb{M}_n$ the space of all $n \times n$ matrices.

2. Main Results

Bhatia and Kittaneh in 1990 [3] established a matrix mean inequality as follows:

$$|||A^*B||| \leq \frac{1}{2}\,|||A^*A + B^*B|||, \tag{3}$$

for matrices $A, B \in \mathbb{M}_n$.
In [2] a generalization of (3) was proved, for all $X \in \mathbb{M}_n$,

$$|||A^*XB||| \leq \frac{1}{2}\,|||AA^*X + XBB^*|||.$$

Ando in 1995 [1] established a matrix Young inequality:

$$|||AB||| \leq \left|\left|\left|\frac{A^p}{p} + \frac{B^q}{q}\right|\right|\right| \tag{4}$$

for $p, q > 1$ with $1/p + 1/q = 1$ and positive matrices A, B. In [9] we considered the inequalities (3) and (4) with the numerical radius norm as follows:

Proposition 2.1. *[9, Proposition 1] If A, B are $n \times n$ matrices, then*

$$\omega(A^*B) \leq \frac{1}{2}\omega(A^*A + B^*B).$$

Also if A and B are positive matrices and $p, q > 1$ with $1/p + 1/q = 1$, then

$$\omega(AB) \leq \omega(\frac{A^p}{p} + \frac{B^q}{q}).$$

Moreover, the authors, in [9, Theorem 2] and [10, Theorem 2.3], showed that the inequality

$$|||AXB||| \leq \left|\left|\left|\frac{A^p}{p}X + X\frac{B^q}{q}\right|\right|\right|$$

does not holds for numerical radius and spectral norm for all $X \in \mathbb{M}_n$ and positive matrices A, B.
The following lemma is a consequence of the spectral theorem for positive operators and Jensen's inequality (see, e.g., [7]).

Lemma 2.2. *Let A be a positive semidefinite operator in $\mathcal{B}(\boldsymbol{H})$ and let $x \in \boldsymbol{H}$ be any unit vector. Then for all $r \geqslant 1$*

$$\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle, \tag{5}$$

and for all $0 \leqslant r \leqslant 1$

$$\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r.$$

Theorem 2.3. *Let $A, B, X \in \mathcal{B}(\boldsymbol{H})$, such that $A, B > 0$ and $p \geqslant q > 1$ where $1/p + 1/q = 1$. Then for all $r \geqslant \frac{2}{q}$*

$$\omega^r((A\sharp B)X) \leqslant \omega\left(\frac{A^{rp/2}}{p} + \frac{(X^*BX)^{rq/2}}{q}\right) - \frac{1}{p}\inf_{\|x\|=1}\delta(x), \tag{6}$$

where $\delta(x) = \left(\langle Ax, x\rangle^{rp/4} - \langle X^*BXx, x\rangle^{rq/4}\right)^2$.

Proof. Let $x \in \boldsymbol{H}$, with $\|x\| = 1$. By the Schwarz inequality in the Hilbert space $(\boldsymbol{H}; \langle ., .\rangle)$, we have

$$\begin{aligned}
\left|\langle (A\sharp B)Xx, x\rangle\right|^r &= \left|\left\langle A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}Xx, x\right\rangle\right|^r \\
&= \left|\left\langle (A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}Xx, A^{1/2}x\right\rangle\right|^r \\
&\leqslant \|(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}Xx\|^r.\|A^{1/2}x\|^r \\
&= \left\langle (A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}Xx, (A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}Xx\right\rangle^{r/2} \\
&\times \left\langle A^{1/2}x, A^{1/2}x\right\rangle^{r/2} \\
&= \langle Ax, x\rangle^{r/2} \langle X^*BXx, x\rangle^{r/2}.
\end{aligned}$$

Now, by Young's inequality and (2) we have

$$\begin{aligned}
&\langle Ax, x\rangle^{r/2} \langle X^*BXx, x\rangle^{r/2} \\
&\leqslant \frac{1}{p}\langle Ax, x\rangle^{rp/2} + \frac{1}{q}\langle X^*BXx, x\rangle^{rq/2} - \frac{1}{p}\left(\langle Ax, x\rangle^{rp/4} - \langle X^*BXx, x\rangle^{rq/4}\right)^2
\end{aligned}$$

and by (5) we have

$$\begin{aligned}
&\frac{1}{p}\langle Ax, x\rangle^{rp/2} + \frac{1}{q}\langle X^*BXx, x\rangle^{rq/2} - \frac{1}{p}\left(\langle Ax, x\rangle^{rp/4} - \langle X^*BXx, x\rangle^{rq/4}\right)^2 \\
&\leqslant \frac{1}{p}\left\langle A^{rp/2}x, x\right\rangle + \frac{1}{q}\left\langle (X^*BX)^{rq/2}x, x\right\rangle - \frac{1}{p}\left(\langle Ax, x\rangle^{rp/4} - \langle X^*BXx, x\rangle^{rq/4}\right)^2 \\
&= \left\langle\left(\frac{A^{rp/2}}{p} + \frac{(X^*BX)^{rq/2}}{q}\right)x, x\right\rangle - \frac{1}{p}\left(\langle Ax, x\rangle^{rp/4} - \langle X^*BXx, x\rangle^{rq/4}\right)^2.
\end{aligned}$$

Now, the result follows by taking the supremum over all unit vectors in $\boldsymbol{H}$. □

Remark 2.4. *Let $r = p = q = 2$. Then $\delta(x) \equiv 0$ if and only if $A - X^*BX = 0$. In general, $\delta(x) = 0$ if and only if $\langle Ax, x\rangle^{rp/4} = \langle X^*BXx, x\rangle^{rq/4}$.*

The following example shows that, inequality (6) does not hold in general for spectral norm.

Example 2.5. *If we take $p = q = 2,\ r = 1,\ A = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}, B = I_2$ and $X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then*

$$1 = \|(A\sharp B)X\|^r > \left\| \frac{A^{rp/2}}{p} + \frac{(X^*BX)^{rq/2}}{q} \right\| = \frac{5}{8}.$$

Put $X = I$ in Theorem 2.3, we obtain the following corollary.

Corollary 2.6. *Let $A, B \in \mathcal{B}(\boldsymbol{H})$, be positive definite and $p \geqslant q > 1$ such that $1/p + 1/q = 1$. Then for all $r \geqslant \frac{2}{q}$*

$$\omega^r(A\sharp B) \leqslant \omega\left(\frac{A^{rp/2}}{p} + \frac{B^{rq/2}}{q}\right) - \frac{1}{p} \inf_{\|x\|=1} \delta(x),$$

where $\delta(x) = \left(\langle Ax, x\rangle^{rp/4} - \langle Bx, x\rangle^{rq/4}\right)^2$.

Note that it is enough to replace r with $2r$ and $X = I$ in the statement of Theorem 2.3 to obtain the following corollary.

Corollary 2.7. *Let $A, B \in \mathcal{B}(\boldsymbol{H})$ be positive definite operators and $p \geqslant q > 1$ such that $1/p + 1/q = 1$. Then for all $r \geqslant \frac{1}{q}$,*

$$\omega^{2r}(A\sharp B) \leqslant \omega(\frac{A^{rp}}{p} + \frac{B^{rq}}{q}) - \frac{1}{p} \inf_{\|x\|=1} \delta(x), \tag{7}$$

where $\delta(x) = \left(\langle Ax, x\rangle^{rp/2} - \langle Bx, x\rangle^{rq/2}\right)^2$.

Remark 2.8. *Note that, if we set $r = 1$ and $p = q = 2$ in (7), then we have*

$$\omega^2(A\sharp B) \leqslant \omega\left(\frac{A^2 + B^2}{2}\right) - \frac{1}{2} \inf_{\|x\|=1} \delta(x), \tag{8}$$

where $\delta(x) = \langle (A - B)x, x\rangle^2$. Notice that (8) is an operator numerical radius version for arithmetic-geometric mean and moreover if, $0 \notin \overline{W(A - B)}$, then $\inf_{\|x\|=1} \delta(x) > 0$.

In the proof of Theorem 2.3, if we put $r = 2$ and $X = I$, then we have the following corollary.

Corollary 2.9. *Let $A, B \in \mathcal{B}(\boldsymbol{H})$, be positive definite operators. Then*

$$\|A\sharp B\|^2 \leqslant \|A\|\|B\|.$$

Let $T, U \in \mathcal{B}(\boldsymbol{H})$. The Euclidean radius(see [4]) is defined by

$$\omega_e(T, U) = \sup_{\|x\|=1} \left(|\langle Tx, x\rangle|^2 + |\langle Ux, x\rangle|^2\right)^{1/2}.$$

Corollary 2.10. *Let $A, B \in \mathcal{B}(\boldsymbol{H})$, be positive definite operators. Then*

$$\sqrt{2}\|A\sharp B\| \leqslant \omega_e(A, B) \leqslant \|A^2 + B^2\|^{1/2},$$

inparticular,

$$\sqrt{2}\omega(A\sharp B) \leqslant \omega_e(A, B) \leqslant \omega^{1/2}(A^2 + B^2).$$

Proof. Same as, in the proof of Theorem 2.3, if we set $r = p = q = 2$, then we have

$$\left|\langle (A\sharp B)x, x\rangle\right|^2 \leqslant \frac{1}{2}(\langle Ax, x\rangle^2 + \langle Bx, x\rangle^2) \tag{9}$$

and by Lemma 2.2,

$$\frac{1}{2}(\langle Ax, x\rangle^2 + \langle Bx, x\rangle^2) \leqslant \frac{1}{2}(\left\langle A^2x, x\right\rangle + \left\langle B^2x, x\right\rangle) = \frac{1}{2}\left\langle (A^2 + B^2)x, x\right\rangle. \tag{10}$$

Now, the result follows by taking the supremum in (9) and (10) over all unit vectors in $\boldsymbol{H}$. □

3. Additional Results

Proposition 3.1. *Let $A, B, X \in \mathcal{B}(\boldsymbol{H})$ such that $A, B > 0$ and $p \geqslant q > 1$ such that $1/p + 1/q = 1$. Then for all $r \geqslant \frac{2}{q}$*

$$\|(A\sharp B)X\|^r \leqslant \|\frac{A^{rp/2}}{p}\| + \|\frac{(X^*BX)^{rq/2}}{q}\| - \frac{1}{p}\inf_{\|x\|=\|y\|=1}\delta(x, y),$$

where $\delta(x, y) = \left(\langle Ay, y\rangle^{rp/4} - \langle X^*BXx, x\rangle^{rq/4}\right)^2$.

Proof. Let $x, y \in \boldsymbol{H}$, such that $\|x\| = \|y\| = 1$. By the Schwarz inequality in the Hilbert space $(\boldsymbol{H}; \langle ., .\rangle)$, we have

$$\begin{aligned}
\left|\langle (A\sharp B)Xx, y\rangle\right|^r &= \left|\left\langle A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}Xx, y\right\rangle\right|^r \\
&= \left|\left\langle (A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}Xx, A^{1/2}y\right\rangle\right|^r \\
&\leqslant \|(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}Xx\|^r.\|A^{1/2}y\|^r \\
&= \left\langle (A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}Xx, (A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}Xx\right\rangle^{r/2} \\
&\times \left\langle A^{1/2}y, A^{1/2}y\right\rangle^{r/2} \\
&= \langle Ay, y\rangle^{r/2}\langle X^*BXx, x\rangle^{r/2}.
\end{aligned}$$

Now, by Young's inequality and (2) we have

$$\begin{aligned}
&\langle Ay, y\rangle^{r/2}\langle X^*BXx, x\rangle^{r/2} \\
&\leqslant \frac{1}{p}\langle Ay, y\rangle^{rp/2} + \frac{1}{q}\langle X^*BXx, x\rangle^{rq/2} - \frac{1}{p}\left(\langle Ay, y\rangle^{rp/4} - \langle X^*BXx, x\rangle^{rq/4}\right)^2
\end{aligned}$$

and by (5) we have

$$\begin{aligned}
&\frac{1}{p}\langle Ay, y\rangle^{rp/2} + \frac{1}{q}\langle X^*BXx, x\rangle^{rq/2} - \frac{1}{p}\left(\langle Ay, y\rangle^{rp/4} - \langle X^*BXx, x\rangle^{rq/4}\right)^2 \\
&\leqslant \frac{1}{p}\left\langle A^{rp/2}y, y\right\rangle + \frac{1}{q}\left\langle (X^*BX)^{rq/2}x, x\right\rangle - \frac{1}{p}\left(\langle Ay, y\rangle^{rp/4} - \langle X^*BXx, x\rangle^{rq/4}\right)^2.
\end{aligned}$$

Now, the result follows by taking the supremum over all unit vectors $x, y \in \boldsymbol{H}$. □

Corollary 3.2. *Let $A, B, X \in \mathcal{B}(\boldsymbol{H})$ be such that $A, B > 0$. Then for all $r \geqslant 1$*

$$2\|(A\sharp B)X\|^r \leqslant \|A^r\| + \|(X^*BX)^r\| - \inf_{\|x\|=\|y\|=1}\delta(x, y), \tag{11}$$

where $\delta(x, y) = \left(\langle Ay, y\rangle^{r/2} - \langle X^*BXx, x\rangle^{r/2}\right)^2$.

If in relation (11) we set $X = I$ we obtain the following corollary.

Corollary 3.3. *Let $A, B \in \mathcal{B}(\boldsymbol{H})$ be positive definite operators. Then for all $r \geqslant 1$*

$$2\|A\sharp B\|^r \leqslant \|A^r\| + \|B^r\| - \inf_{\|x\|=\|y\|=1} \delta(x,y),$$

where $\delta(x,y) = \left(\langle Ay, y\rangle^{r/2} - \langle Bx, x\rangle^{r/2}\right)^2$.

Proposition 3.4. *Let $A, B, X \in \mathcal{B}(\boldsymbol{H})$ such that $A, B > 0$ and $p \geqslant q > 1$ such that $1/p + 1/q = 1$. Then for all $r \geqslant 2/q$*

$$(\|A\|\|X^*BX\|)^{r/2} \leqslant \|\frac{A^{rp/2}}{p}\| + \|\frac{(X^*BX)^{rq/2}}{q}\| - \frac{1}{p} \inf_{\|x\|=\|y\|=1} \delta(x,y), \tag{12}$$

where $\delta(x,y) = \left(\langle Ay, y\rangle^{rp/4} - \langle X^*BXx, x\rangle^{rq/4}\right)^2$.

Proof. Let $x, y \in \boldsymbol{H}$ with $\|x\| = \|y\| = 1$. By the inequality (2), we have
$\langle Ay, y\rangle^{r/2} \langle X^*BXx, x\rangle^{r/2}$
$\leqslant \frac{1}{p} \langle Ay, y\rangle^{rp/2} + \frac{1}{q} \langle X^*BXx, x\rangle^{rq/2} - \frac{1}{p} \left(\langle Ay, y\rangle^{rp/4} - \langle X^*BXx, x\rangle^{rq/4}\right)^2$
and by (5) we have
$\frac{1}{p} \langle Ay, y\rangle^{rp/2} + \frac{1}{q} \langle X^*BXx, x\rangle^{rq/2} - \frac{1}{p} \left(\langle Ay, y\rangle^{rp/4} - \langle X^*BXx, x\rangle^{rq/4}\right)^2$
$\leqslant \frac{1}{p} \left\langle A^{rp/2}y, y\right\rangle + \frac{1}{q} \left\langle (X^*BX)^{rq/2}x, x\right\rangle - \frac{1}{p} \left(\langle Ay, y\rangle^{rp/4} - \langle X^*BXx, x\rangle^{rq/4}\right)^2$.
Now, the result follows by taking the supremum over all unit vectors $x, y \in \boldsymbol{H}$. □

If in relation (12), we set $X = I$ and $r = 2$, then we obtain the following corollary.

Corollary 3.5. *Let $A, B \in \mathcal{B}(\boldsymbol{H})$ be positive definite operators and $p \geqslant q > 1$ such that $1/p + 1/q = 1$. Then*

$$\|A\|\|B\| \leqslant \|\frac{A^p}{p}\| + \|\frac{B^q}{q}\| - \frac{1}{p} \inf_{\|x\|=\|y\|=1} \delta(x,y), \tag{13}$$

where $\delta(x,y) = \left(\langle Ay, y\rangle^{p/2} - \langle Bx, x\rangle^{q/2}\right)^2$.

Remark 3.6. *Note that, if we set $p = q = 2$ in (13), then we have*

$$\|A\|\|B\| \leqslant \frac{1}{2}(\|A^2\| + \|B^2\|) - \frac{1}{2} \inf_{\|x\|=\|y\|=1} \delta(x,y), \tag{14}$$

where $\delta(x,y) = (\langle Ay, y\rangle - \langle Bx, x\rangle)^2$. Notice that (14) is an operator norm version for arithmetic-geometric mean and moreover if, $W(A)$ and $W(B)$ are separated, then $\inf_{\|x\|=\|y\|=1} \delta(x,y) > 0$.

Example 3.7. *Let $p = q = 2$ and $A = diag(1,2), B = diag(5,6)$ in the inequality (13). Then $\inf_{\|x\|=\|y\|=1} \delta(x,y) = 9 > 0$ and hence,*

$$12 = \|A\|\|B\| \leqslant \frac{1}{2}(\|A^2\| + \|B^2\|) - \frac{1}{2} \inf_{\|x\|=\|y\|=1} \delta(x,y) = \frac{31}{2}.$$

Whereas, if we set this values in the inequality (4), with the spectral norm, then we obtain

$$12 = \|AB\| \leqslant \|\frac{A^p}{p} + \frac{B^q}{q}\| = 20.$$

Thus, in this case, we have

$$\|AB\| = \|A\|\|B\| < \|\frac{A^p}{p}\| + \|\frac{B^q}{q}\| - \frac{1}{p} \inf_{\|x\|=\|y\|=1} \delta(x,y) < \|\frac{A^p}{p} + \frac{B^q}{q}\|.$$

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