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Weakly Convex Domination Subdivision Number of a Graph

Magda Dettlaff[a], Saeed Kosary[b], Magdalena Lemańska[a], Seyed Mahmoud Sheikholeslami[b]

[a]*Faculty of Applied Physics and Mathematics*
Gdańsk University of Technology,
ul. Narutowicza 11/12 80-233 Gdańsk, Poland
[b]*Department of Mathematics*
Azarbaijan Shahid Madani University
Tabriz, I.R. Iran

Abstract. A set X is *weakly convex* in G if for any two vertices $a, b \in X$ there exists an ab–geodesic such that all of its vertices belong to X. A set $X \subseteq V$ is a *weakly convex dominating set* if X is weakly convex and dominating. The *weakly convex domination number* $\gamma_{\rm wcon}(G)$ of a graph G equals the minimum cardinality of a weakly convex dominating set in G. *The weakly convex domination subdivision number* $\mathrm{sd}_{\gamma_{\rm wcon}}(G)$ is the minimum number of edges that must be subdivided (each edge in G can be subdivided at most once) in order to increase the weakly convex domination number. In this paper we initiate the study of weakly convex domination subdivision number and establish upper bounds for it.

1. Introduction

Throughout this paper, G is a simple connected graph with vertex set $V(G)$ and edge set $E(G)$ (briefly V and E). For every vertex $v \in V(G)$, the *open neighborhood* of v, $N_G(v) = N(v)$, is the set $\{u \in V(G) \mid uv \in E(G)\}$ and its *closed neighborhood* is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. The *open neighborhood* of a set $S \subseteq V$ is the set $N_G(S) = N(S) = \cup_{v \in S} N(v)$, and the *closed neighborhood* of S is the set $N_G[S] = N[S] = N(S) \cup S$. The *degree* of a vertex v is $d_G(v) = |N_G(v)|$. A *leaf* is a vertex of degree one and a *universal vertex* is a vertex of degree $|V(G)| - 1$. We denote the number of leaves in a graph G by $\ell(G)$. The minimum and maximum degrees of G are respectively denoted by $\delta(G)$ and $\Delta(G)$. The private neighborhood of a vertex $u \in D$ with respect to a set $D \subseteq V$, is the set $PN_G[u, D] = N_G[u] - N_G[D - \{u\}]$. If $v \in PN_G[u, D]$, then we say that v is a private neighbor of u with respect to the set D. For a set S of vertices of G we denote by $G[S]$ the subgraph induced by S in G. The *distance* $d_G(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest uv–path in G. A uv–path of length $d_G(u, v)$ is called a *uv–geodesic*. The largest distance between any pair of vertices u, v in G is the *diameter* of G, denoted by $\mathrm{diam}(G)$. The girth $g(G)$ of a graph G is the length of a shortest cycle in G. The *edge connectivity number* $\kappa'(G)$ of G is the minimum number of edges whose removal results in a disconnected graph. For every connected graph, $\kappa'(G) \leq \delta(G)$.

A set $A \subset V$ is a *dominating set* of G if $N_G[A] = V$, and is a *connected dominating set* if $N_G[A] = V$ and the induced subgraph $G[A]$ is connected. The (*connected*) *domination number* $\gamma(G)$ ($\gamma_c(G)$) is the minimum

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Email addresses: mdettlaff@mif.pg.gda.pl (Magda Dettlaff), saeedkosari38@yahoo.com (Saeed Kosary), magda@mif.pg.gda.pl (Magdalena Lemańska), s.m.sheikholeslami@azaruniv.edu (Seyed Mahmoud Sheikholeslami)

cardinality of a (connected) dominating set of G, and a (connected) dominating set of minimum cardinality is called a $\gamma(G)$–set ($\gamma_c(G)$–set).

A set X is *weakly convex* in G if for any two vertices $a, b \in X$ there exists an ab–geodesic such that all of its vertices belong to X. A set $X \subseteq V$ is a *weakly convex dominating set* if X is weakly convex and dominating. The *weakly convex domination number of a graph G*, denoted by $\gamma_{wcon}(G)$, equals to the minimum cardinality of a weakly convex dominating set in G. Weakly convex domination number was first introduced by Jerzy Topp, Gdańsk University of Technology, 2002.

In application, network design for example, if a parameter $\mu(G)$ is important to study, then it is important to know the effect that modifications of G have on $\mu(G)$. For example, vertices can be deleted and edges can be deleted or added. In network design, deleting a vertex or an edge may represent component's failure. From the other perspective, networks can be made fault-tolerant by providing redundant communication link (adding edges). The effects on the domination number of a graph, when G is modified by deleting a vertex or deleting or adding an edge, have been investigated extensively (see chapter 7 of [16]). In particular, the effects on the weakly convex domination number of a graph, when G is modified by deleting a vertex or deleting or adding an edge, have been investigated in [19].

Alternatively, one can consider how many modifications must take place before a parameter changes. Along these lines, Fink et al. [12], defined the *bondage number* of a graph to equal the minimum number of edges whose removal increases the domination number. On the other hand, Kok and Mynhardt [17] defined the *reinforcement number* of a graph to equal the minimum number of edges which must be added to a graph in order to decrease the domination number. Considering a different type of graph modification, Velammal [20] defined the domination subdivision number $\mathrm{sd}_\gamma(G)$ to be the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the domination number. The domination subdivision number has been studied by several authors (see for instance [1, 11, 14, 15]). A similar concepts related to connected domination were studied in [10], to total domination in [14], to Roman domination in [2], to rainbow domination in [5, 13], and to 2-domination in [3]. It is known that the domination subdivision parameters can take arbitrarily large values [2, 5, 9, 10, 14] and an interesting problem is to find good bounds on these parameters in terms of other parameters of G. For instance, it has been proved that for any connected graph G of order n, $\mathrm{sd}_{\gamma_t}(G) \le n - \gamma_t(G) + 1$ [7], $\mathrm{sd}_{\gamma_t}(G) \le 2n/3$ [8], $\mathrm{sd}_{\gamma_c}(G) \le \lfloor \frac{n}{2} \rfloor$ [10], $\mathrm{sd}_{\gamma_R}(G) \le \lceil \frac{n}{2} \rceil - 1$ [2] and $\mathrm{sd}_{\gamma_{r2}}(G) \le n - \Delta(G) + 2$ [13].

The (*weakly convex, connected*) *domination subdivision number* $\mathrm{sd}_\gamma(G)$ ($\mathrm{sd}_{\gamma_{\mathrm{wcon}}}(G)$, $\mathrm{sd}_{\gamma_c}(G)$) of a graph G is the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the (weakly convex, connected) domination number. (We say that an edge $e = uv \in E(G)$ is *subdivided* with a vertex x if the edge uv is deleted, but a new vertex x is added, along with two new edges ux and vx. The vertex x is called a *subdivision vertex* and obtained graph is denoted by G_e). Since the (weakly convex, connected) domination number of the graph K_2 does not change when its only edge is subdivided, we consider weakly convex domination subdivision number for all graphs G satisfying $\Delta(G) \ge 2$.

For any unexplained terms see [16].

Our purpose in this paper is to initialize the study of the weakly convex domination subdivision number $\mathrm{sd}_{\gamma_{\mathrm{wcon}}}(G)$. In particular, we establish some sharp upper bounds on $\mathrm{sd}_{\gamma_{\mathrm{wcon}}}(G)$.

Next result shows that subdividing an edge can decrease or increase the weakly convex domination number.

Theorem 1.1. The differences $\gamma_{\mathrm{wcon}}(G) - \gamma_{\mathrm{wcon}}(G_e)$ and $\gamma_{\mathrm{wcon}}(G_e) - \gamma_{\mathrm{wcon}}(G)$ can be arbitrarily large.

Proof. First we show that for some edge e the difference $\gamma_{\mathrm{wcon}}(G) - \gamma_{\mathrm{wcon}}(G_e)$ can be arbitrarily large. Let $k \ge 3$ and let G' be the graph obtained from a $(2k+1)$–cycle

$$C_{2k+1} = (v_1, v_2, \ldots, v_{2k+1})$$

(where $v_1v_2, v_2v_3, \ldots, v_{2k}v_{2k+1}, v_{2k+1}v_1$ are edges of this cycle) by adding the edge v_kv_{k+2}, the edges v_iv_{2k-i+1} for $i = 1, \ldots, k-2$ and adding the pendant edges v_iw_i for $i = 1, \ldots, k+1, 2k+1$. For any $\gamma_{\mathrm{wcon}}(G')$–set D

of G', we must have $|D \cap \{v_i, w_i\}| \geq 1$ for $i = 1, \ldots, k+1, 2k+1$. Since D is weakly convex, we deduce that $v_i \in D$. In particular, $v_{k+1}, v_{2k+1} \in D$. Since $v_{k+1}, v_{k+2}, \ldots, v_{2k+1}$ is the only $v_{k+1}v_{2k+1}$–geodesic in G', we have $v_i \in D$ for each $i = 1, 2, \ldots, 2k+1$ and hence $\gamma_{\mathrm{wcon}}(G') \geq 2k+1$. On the other hand, $V(C_{2k+1})$ is obviously a weakly convex dominating set of G' and so $\gamma_{\mathrm{wcon}}(G') = 2k+1$. Let G'_e be a graph obtained from G' by subdividing the edge $e = v_{2k+1}v_{2k}$. It is easy to see that $\gamma_{\mathrm{wcon}}(G'_e) = k+2$ (note that the support vertices of G'_e form a $\gamma_{\mathrm{wcon}}(G'_e)$–set). The case $k = 3$ is illustrated in Figure 1.

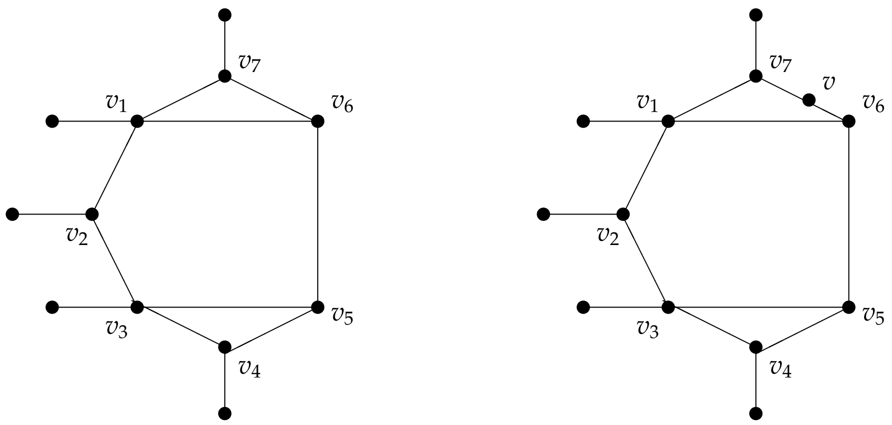

Figure 1: Graph G' and G'_e for $k = 3$.

Now we show that $\gamma_{\mathrm{wcon}}(G_e) - \gamma_{\mathrm{wcon}}(G)$ can be arbitrarily large for some edge e. Let $k \geq 1$ be an integer and let G'' be obtained from a $(2k+2)$–cycle $C_{2k+2} = (v_1, v_2, \ldots, v_{2k+2})$ by adding the edges $v_i v_{2k+2-i}$ for $i = 1, \ldots, k$ and adding the pendant edges $v_i w_i$ for $i = 1, \ldots, k+1, 2k+2$. As above we can see that $\gamma_{\mathrm{wcon}}(G'') = k+2$ (note that the support vertices form a minimum weakly convex dominating set of G''). After subdividing the edge $e = v_1 v_{2k+2}$ we obtain G''_e, for which $\gamma_{\mathrm{wcon}}(G''_e) = 2k+3$ (all of the vertices except leaves form a minimum weakly convex dominating set for G''_e). Figure 2 demonstrates the case $k = 3$. □

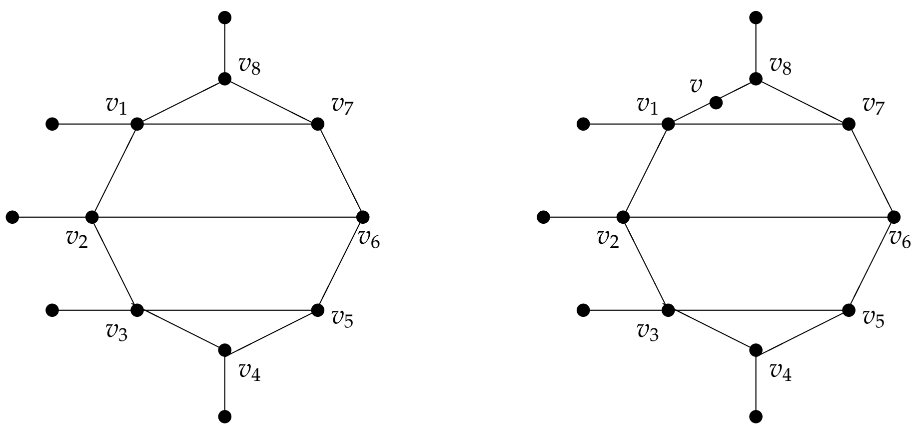

Figure 2: Graph G'' and G''_e for $k = 3$.

In the next theorem we give an upper bound for weakly convex domination number.

Theorem 1.2. Let G be a connected graph of order $n \geq 3$ with $\delta(G) > \lfloor \frac{n}{2} \rfloor$. Then

$$\gamma_{\mathrm{wcon}}(G) \leq \max\{3, 2\left\lfloor \frac{n}{2} \right\rfloor - \delta(G)\}.$$

Proof. Let us denote $c = \delta(G) - \left\lfloor \frac{n}{2} \right\rfloor$. If $\mathrm{diam}(G) \geq 3$, then let x and y be the vertices such that $d_G(x, y) = 3$. Then $1 + d_G(x) + 1 + d_G(y) \leq n$ that implies $2\lfloor \frac{n}{2} \rfloor + 2 < 2\delta(G) + 2 \leq n$, which is impossible. Thus $\mathrm{diam}(G) \leq 2$. If $\mathrm{diam}(G) = 1$, then $\gamma_{\mathrm{wcon}}(G) = 1$. Suppose now that $\mathrm{diam}(G) = 2$. Let $v \in V(G)$ be a vertex with minimum degree $\delta(G)$ and let $N(v) = \{v_1, v_2 \ldots, v_{\delta(G)}\}$. Since $\mathrm{diam}(G) = 2$ and $d_G(v) = \delta(G)$, $V(G) - N[v] \neq \emptyset$. Assume first there is a vertex $u \in V(G) - N[v]$ such that $V(G) - N[v] \subseteq N[u]$. Since $\mathrm{diam}(G) = 2$, u and v have a common neighbor, say w. In the case, $\{u, v, w\}$ is clearly a weakly convex dominating set of G and hence $\gamma_{\mathrm{wcon}}(G) \leq 3$. Now let $V(G) - N[v] \nsubseteq N[u]$ for every $u \in V(G) - N[v]$. It follows that $|V(G) - N[v]| \geq 2$. Let $x \in V(G) - N[v]$. Then

$$d_G(v) + d_G(x) - |N(v) \cap N(x)| + 3 \leq n.$$

Since $d_G(x) \geq \delta(G)$, $d_G(v) \geq \delta(G)$, $\delta(G) = \lfloor \frac{n}{2} \rfloor + c$ and $2\lfloor \frac{n}{2} \rfloor \geq n - 1$ we have

$$|N(v) \cap N(x)| \geq 2c + 2.$$

Hence each vertex in $V(G) - N[v]$ is adjacent to at least $2c + 2$ vertices in $N(v)$. Moreover, the set $N(v) - \{v_1, v_2, \ldots, v_{\lfloor \frac{n}{2} \rfloor - c - 1}\}$ consists of $2c + 1$ vertices. Hence, each vertex in $V(G) - N[v]$ is adjacent to at least one of the vertices in $\{v_1, \ldots, v_{\lfloor \frac{n}{2} \rfloor - c - 1}\}$. This implies that the set $\{v, v_1, \ldots, v_{\lfloor \frac{n}{2} \rfloor - c - 1}\}$ is a weakly convex dominating set of G and so $\gamma_{\mathrm{wcon}}(G) \leq \lfloor \frac{n}{2} \rfloor - c = 2\lfloor \frac{n}{2} \rfloor - \delta(G)$. This completes the proof. □

2. Bounds on Weakly Convex Domination Subdivision Number

In this section, we establish some upper bounds on weakly convex domination subdivision number. Given $S, T \subseteq V(G)$, we write $[S, T]$ for the set of edges having one end–point in S and the other in T. An edge cut is an edge set of the form $[S, \overline{S}]$, where S is a nonempty proper subset of $V(G)$ and $\overline{S}$ denotes $V(G) - S$.

Theorem 2.1. For any connected graph G of order $n \geq 3$, $\mathrm{sd}_{\gamma_{\mathrm{wcon}}}(G) \leq \kappa'(G)$.

Proof. Let $[S, \overline{S}]$ be an edge cut of G of size $\kappa'(G)$, and let G_1 and G_2 be the components of $G - [S, \overline{S}]$. Assume G' is the graph obtained from G by subdividing the edges of $[S, \overline{S}]$ and T be the set of all subdivision vertices. Let D be a minimum weakly convex dominating set of G' and $D_i = D \cap V(G_i)$ for $i = 1, 2$. If $D \cap T = \emptyset$, then $D_i \neq \emptyset$ for $i = 1, 2$, and $D = D_1 \cup D_2$. Now for vertices $x_1 \in D_1$ and $x_2 \in D_2$, any x_1x_2–geodesic intersects T implying that $D \cap T \neq \emptyset$ which leads to a contradiction. Therefore $D \cap T \neq \emptyset$. Since D is weakly convex set of G' and $d_G(x, y) \leq d_{G'}(x, y)$ for the vertices $x, y \in D - T$, the set $D - T$ is a weakly convex dominating set of G. Moreover $|D - T| < \gamma_{\mathrm{wcon}}(G')$. This yields $\mathrm{sd}_{\gamma_{\mathrm{wcon}}}(G) \leq \kappa'(G)$ and the proof is completed. □

According to Theorem 1.1, the subdividing an edge may decrease the weakly convex domination number. Hence, it is not immediately obvious that the weakly convex domination subdivision number is defined for all connected graphs G with $\Delta(G) \geq 2$. However, since every connected graph of order at least 3 has an edge cut, we conclude from Theorem 2.1 that the weakly convex domination number is well-defined for all connected graphs G with $\Delta(G) \geq 2$.

Moreover, from Theorem 2.1 we also obtain two corollaries.

Corollary 2.2. If there exists a cut edge in G, then $\mathrm{sd}_{\gamma_{\mathrm{wcon}}}(G) = 1$.

Corollary 2.3. For any connected simple graph G of order $n \geq 3$, $\mathrm{sd}_{\gamma_{\mathrm{wcon}}}(G) \leq \delta(G)$.

Next result presents the necessary condition for a graph to have $\mathrm{sd}_{\gamma_{\mathrm{wcon}}}(G) > 1$.

Proposition 2.4. If $\mathrm{sd}_{\gamma_{\mathrm{wcon}}}(G) > 1$, then every edge of G belongs to a cycle C_3, C_4 or C_5.

Proof. Assume that G has an edge e such that e does not belong neither to a 3-cycle nor to 4-cycle nor to 5-cycle. We will show that $\mathrm{sd}_{\gamma_{\mathrm{wcon}}}(G) = 1$. If e is a cut–edge, then from Corollary 2.2, $\mathrm{sd}_{\gamma_{\mathrm{wcon}}}(G) = 1$. Suppose now e belongs to a cycle. Let C be a smallest cycle containing e. From our assumption $C = C_p$, where $p \geq 6$. Let us subdivide the edge $e = uv$ with a vertex w and let D' be a $\gamma_{\mathrm{wcon}}(G_e)$–set. If $\{u, v\} \subseteq D'$, then $w \in D'$

and clearly $D' - \{w\}$ is a weakly convex dominating set in G that implies $\mathrm{sd}_{\gamma_{\mathrm{wcon}}}(G) = 1$. Let $|D' \cap \{u, v\}| = 1$. Assume, without loss of generality, that $\{u, v\} \cap D' = \{u\}$. First let $w \in D'$. Suppose v' is the neighbor of v on C other than u. Since $p \geq 6$, $v' \notin D'$. So v' is dominated by $v'' \in D'$. Then $d_{G_e}(w, v'') \leq 3$. Since v and v' does not belong to D' and D' is weakly convex, there is another wv''–path, say P_1. Then the induced subgraph $G[V(P_1) - \{w\} \cup \{v, v', v''\}]$ gives a cycle of length at most 5, a contradiction. Now let $w \notin D'$. Then v is dominated by $z \in D'$ and $d_{G_e}(u, z) \leq 3$. Since $w \notin D'$ and $v \notin D'$ and D' is weakly convex, there is another uz–path, say P_2, of length at most 3. Then the induced subgraph $G[V(P_2) \cup \{v\}]$ is a cycle of length at most 5, a contradiction. □

In [10] the following Proposition was shown.

Proposition 2.5. [10] If G is a connected graph of order $n \geq 3$, then $\mathrm{sd}_{\gamma_c}(G) \leq \gamma_c(G) + 1$.

We prove similar relation for weakly convex domination. Let $\alpha'(G)$ be the maximum number of edges in a matching in G.

Proposition 2.6. If G contains a matching M such that $\gamma_{\mathrm{wcon}}(G) < |M|$, then $\mathrm{sd}_{\gamma_{\mathrm{wcon}}}(G) \leq |M|$. In particular, if $\alpha'(G) > \gamma_{\mathrm{wcon}}(G)$, then $\mathrm{sd}_{\gamma_{\mathrm{wcon}}}(G) \leq \gamma_{\mathrm{wcon}}(G) + 1$.

Proof. Let G' be obtained by subdividing every edge of M. Each weakly convex dominating set of G' has order at least $|M|$. Hence $\gamma_{\mathrm{wcon}}(G') > \gamma_{\mathrm{wcon}}(G)$ and thus $\mathrm{sd}_{\gamma_{\mathrm{wcon}}}(G) \leq |M|$. If $\alpha'(G) > \gamma_{\mathrm{wcon}}(G)$, then G contains a matching M of size $\gamma_{\mathrm{wcon}}(G) + 1$, which leads to the result. □

Theorem 2.7. If G is a connected graph of order $n \geq 3$, then

$$\mathrm{sd}_{\gamma_{\mathrm{wcon}}}(G) \leq \gamma_{\mathrm{wcon}}(G) + 1.$$

Proof. The result is immediate for $n = 3, 4, 5$. Let $n \geq 6$. If $\delta(G) \leq \gamma_{\mathrm{wcon}}(G) + 1$, then by Corollary 2.3 we have $\mathrm{sd}_{\gamma_{\mathrm{wcon}}}(G) \leq \delta(G) \leq \gamma_{\mathrm{wcon}}(G) + 1$. If $\gamma_{\mathrm{wcon}}(G) > \lfloor \frac{n}{2} \rfloor$, then by Theorem 1.2, $\delta(G) \leq \lfloor \frac{n}{2} \rfloor$ and again by Corollary 2.3 we have $\mathrm{sd}_{\gamma_{\mathrm{wcon}}}(G) \leq \gamma_{\mathrm{wcon}}(G)$. Moreover, if $\gamma_c(G) = \gamma_{\mathrm{wcon}}(G)$, then from Proposition 3.2 and Proposition 2.5 we obtain $\mathrm{sd}_{\gamma_{\mathrm{wcon}}}(G) \leq \mathrm{sd}_{\gamma_c}(G) \leq \gamma_c(G) + 1 = \gamma_{\mathrm{wcon}}(G) + 1$. For $\alpha'(G) > \gamma_{\mathrm{wcon}}(G)$ the result follows from Proposition 2.6.

In the remaining cases we assume that $\gamma_{\mathrm{wcon}}(G) \leq \lfloor \frac{n}{2} \rfloor$, $\delta(G) \geq \gamma_{\mathrm{wcon}}(G) + 1$, $\alpha'(G) \leq \gamma_{\mathrm{wcon}}(G)$ and $\gamma_c(G) < \gamma_{\mathrm{wcon}}(G)$. It is known from [6] that the matching number of every graph is at least $\min\{\delta(G), \lfloor \frac{n}{2} \rfloor\}$. Hence, since $\delta(G) > \alpha'(G)$, we have $\alpha'(G) = \lfloor \frac{n}{2} \rfloor$. This implies $\gamma_{\mathrm{wcon}}(G) = \lfloor \frac{n}{2} \rfloor$. So $\delta(G) \geq \lfloor \frac{n}{2} \rfloor + 1$, what gives $\mathrm{diam}(G) \leq 2$. Hence every $\gamma_c(G)$–set is also $\gamma_{\mathrm{wcon}}(G)$–set, a contradiction with $\gamma_c(G) < \gamma_{\mathrm{wcon}}(G)$. □

In [4] and [18] the following results were shown.

Proposition 2.8. ([4]) For any connected simple graph G of order $n \geq 3$ with $g(G) \geq 5$, $\gamma(G) \geq \delta(G)$.

Proposition 2.9. [18] If G is a connected graph of order n, then $\gamma(G) \leq \gamma_c(G) \leq \gamma_{\mathrm{wcon}}(G)$.

From above Propositions and Corollary 2.3 we obtain the next result.

Proposition 2.10. For any connected graph G of order $n \geq 3$ with $g(G) = 5$,

$$\mathrm{sd}_{\gamma_{\mathrm{wcon}}}(G) \leq \gamma_{\mathrm{wcon}}(G).$$

Proof. By Propositions 2.8 and 2.9, we obtain $\gamma_{\mathrm{wcon}}(G) \geq \delta(G)$ and the result follows from Corollary 2.3. □

3. Graphs with Small Weakly Convex Domination Subdivision Number

In this section, we consider graphs with small weakly convex domination subdivision number. We make use of the following results in this section.

Proposition 3.1. [10] If a connected graph G of order $n \geq 3$ satisfies one of the following properties

(i) $\gamma_c(G) = 1$;

(ii) $\gamma_c(G) = 2$ and G contains a $\gamma_c(G)$–set $\{a, b\}$ such that $N(a) \cap N(b) = \emptyset$,

then $\mathrm{sd}_{\gamma_c}(G) = 1$.

Proposition 3.2. If $\gamma_c(G) = \gamma_{\mathrm{wcon}}(G)$, then $\mathrm{sd}_{\gamma_{\mathrm{wcon}}}(G) \leq \mathrm{sd}_{\gamma_c}(G)$.

Proof. After subdividing $\mathrm{sd}_{\gamma_c}(G)$ edges of G, the resulting graph G' satisfies $\gamma_c(G') > \gamma_c(G) = \gamma_{\mathrm{wcon}}(G)$. Hence $\gamma_{\mathrm{wcon}}(G') \geq \gamma_c(G') > \gamma_{\mathrm{wcon}}(G)$ and $\mathrm{sd}_{\gamma_{\mathrm{wcon}}}(G) \leq \mathrm{sd}_{\gamma_c}(G)$. □

Now we present sufficient conditions for a graph to have weakly convex domination subdivision number equal to 1.

Proposition 3.3. Let G be a connected graph of order $n \geq 3$. If G satisfies one of the following properties:

(i) $\gamma_{\mathrm{wcon}}(G) = 1$;

(ii) $\gamma_{\mathrm{wcon}}(G) = 2$ and G contains a $\gamma_{\mathrm{wcon}}(G)$–set $\{a, b\}$ such that $N(a) \cap N(b) = \emptyset$;

(iii) G contains two adjacent vertices of degree 2;

(iv) $g(G) \geq 6$;

(v) G has an edge e such that if e is subdivided with a vertex w, then G_e has a $\gamma_{\mathrm{wcon}}(G_e)$-set containing w,

then $\mathrm{sd}_{\gamma_{\mathrm{wcon}}}(G) = 1$.

Proof. (i, ii) Clearly $\gamma_c(G) = \gamma_{\mathrm{wcon}}(G)$ and the result follows from Proposition 3.2 and Proposition 3.1.

(iii) Let x_1 and y_1 be two adjacent vertices of degree 2 in G and let G' be obtained from G by subdividing the edge x_1y_1 with a vertex z. Then G' contains an induced path x, x_1, z, y_1, y (possibly a cycle if x and y are adjacent or if $x = y$). Let D be a $\gamma_{\mathrm{wcon}}(G')$–set. If $z \in D$, then $D - \{z\}$ is a weakly convex dominating set of G. Let $z \notin D$. To dominate z, without loss of generality we can suppose that $x_1 \in D$. Since D is weakly convex, $\{x, x_1\}$ is a subset of D. Since x_1 was in D only to dominate the vertex z, the set $D - \{x_1\}$ is a weakly convex dominating set of G. Therefore $\gamma_{\mathrm{wcon}}(G) < \gamma_{\mathrm{wcon}}(G')$. In particular for paths and cycles, $\mathrm{sd}_{\gamma_{\mathrm{wcon}}}(P_n) = \mathrm{sd}_{\gamma_{\mathrm{wcon}}}(C_n) = 1$.

(iv) Let $e = u_1u_2$ be an arbitrary edge of G. If e is a cut edge, then clearly $\gamma_{\mathrm{wcon}}(G_e) > \gamma_{\mathrm{wcon}}(G)$. Let $C = (u_1, u_2, \ldots, u_k)$ be a cycle containing e. Assume G_e is obtained from G by subdividing the edge e with subdivision vertex w and D is a $\gamma_{\mathrm{wcon}}(G_e)$–set. We show that $w \in D$ which implies $D - \{w\}$ is a weakly convex dominating set of G, as desired. Assume to the contrary that $w \notin D$. It follows that $\{u_1, u_2\} \not\subseteq D$. Assume without loss of generality that $u_2 \notin D$. Then to dominate w and u_2, we must have $u_1 \in D$ and $D \cap N_G(u_2) \neq \emptyset$. Let $v \in D \cap N_G(u_2)$. Since $g(G) \geq 6$, then u_1, w, u_2, v is the unique u_1v–path in G' that implies $u_1, w, u_2, v \in D$, a contradiction. Therefore, $w \in D$ and $\gamma_{\mathrm{wcon}}(G_e) > \gamma_{\mathrm{wcon}}(G)$. Thus $\mathrm{sd}_{\gamma_{\mathrm{wcon}}}(G) = 1$.

(v) Let $e = ab$ be an edge of G such that if e is subdivided with a vertex w then G_e has a $\gamma_{\mathrm{wcon}}(G_e)$-set D' containing w. Since D' is a weakly convex dominating set, $D' \cap \{a, b\} \neq \emptyset$. If $a, b \in D'$, then $w \in D'$ and $D' - \{w\}$ is obviously a weakly convex dominating set of G implying that $\gamma_{\mathrm{wcon}}(G) \leq |D'| - 1 < \gamma_{\mathrm{wcon}}(G_e)$. Let $\{a, b\} \not\subseteq D'$. Assume without loss of generality that $a \in D'$ and $b \notin D'$. If $D' = \{a, w\}$, then obviously $D' - \{w\}$ is a weakly convex dominating set of G and hence $\gamma_{\mathrm{wcon}}(G) < \gamma_{\mathrm{wcon}}(G_e)$ again. Let $\{a, w\} \subsetneq D'$. Since D' is a weakly convex dominating set, we deduce that for any $x \in D' - \{a, w\}$ there exists a xw–geodesic P_{xw} in G' such that $V(P_{xw}) \subseteq D'$. Obviously $P_{xw} - \{w\}$ is a xa–geodesic in G for each $x \in D'$. It follows that $D' - \{w\}$ is a weakly convex dominating set of G implying that $\gamma_{\mathrm{wcon}}(G) < \gamma_{\mathrm{wcon}}(G_e)$. Thus $\mathrm{sd}_{\gamma_{\mathrm{wcon}}}(G) = 1$. □

Note that the case (i) includes the complete graphs and the case (ii) includes the complete bipartite graph $K_{p,q}$ with $p, q \geq 2$, and the graph obtained from K_4 by subdividing one edge once.

Now we give upper bounds for weakly convex domination subdivision number of graphs with weakly convex domination number 2 or 3.

Proposition 3.4. Let G be a connected graph of order $n \geq 3$ with $\gamma_{\rm wcon}(G) = 2$. Then

$$\mathrm{sd}_{\gamma_{\rm wcon}}(G) \leq 2.$$

Proof. Let G be a connected graph of order $n \geq 3$ with $\gamma_{\rm wcon}(G) = 2$. Then $\Delta(G) \leq n - 2$. Let $S = \{u, v\}$ be a $\gamma_{\rm wcon}(G)$–set, u' a private neighbor of u with respect to S and v' a private neighbor of v with respect to S. Let G' be the graph obtained from G by subdividing the edges uu', vv' with subdivision vertices x and y, respectively, and let D be a $\gamma_{\rm wcon}(G')$–set. We show that $|D| \geq 3$ that implies $\mathrm{sd}_{\gamma_{\rm wcon}}(G) \leq 2$. Suppose to the contrary that $|D| \leq 2$. To dominate x, y, we must have $|D \cap \{u, u'\}| \geq 1$ and $|D \cap \{v, v'\}| \geq 1$. Since $|D| \leq 2$, we have $|D \cap \{u, u'\}| = 1$ and $|D \cap \{v, v'\}| = 1$. Since $G[D]$ is connected, $uv' \notin E(G)$ and $vu' \notin E(G)$, we deduce that either $D = \{u, v\}$ or $D = \{u', v'\}$. In each case, D is not a dominating set of G' which is a contradiction. □

Proposition 3.5. Let $k \geq 2$ be an integer. For the complete k-partite graph $G = K_{p_1,p_2,\ldots p_k}$ with $2 \leq p_1 \leq p_2 \leq \ldots \leq p_k$,

$$\mathrm{sd}_{\gamma_{\rm wcon}}(G) = \begin{cases} 1 & \text{if } k = 2 \\ 2 & \text{otherwise.} \end{cases}$$

Proof. It is clear that any two adjacent vertices form a minimum weakly convex dominating set of G which implies $\gamma_{\rm wcon}(G) = 2$. If $k = 2$, the result follows from Proposition 3.3 (ii). Let $k \geq 3$ and let $V_1, V_2, \ldots, V_k$ be the partite sets of $V(G)$. By Proposition 3.4, $\mathrm{sd}_{\gamma_{\rm wcon}}(G) \leq 2$. Now we show that $\mathrm{sd}_{\gamma_{\rm wcon}}(G) \geq 2$. Let $e = ab$ be an edge of G. Hence $a \in V_i, b \in V_j$, where $i \neq j$. In this case the set $\{a, v\}$, where v is a vertex belonging to V_k and $k \notin \{i, j\}$, forms a minimum weakly convex dominating set of G. Thus $\mathrm{sd}_{\gamma_{\rm wcon}}(G) = 2$ and the proof is completed. □

Proposition 3.5 shows that the bound in Proposition 3.4 is sharp.

In [10] the following Proposition was presented.

Proposition 3.6. If G is a connected graph of order $n \geq 3$ and $\gamma_c(G) = 3$, then $1 \leq \mathrm{sd}_{\gamma_c}(G) \leq 3$.

We use above result when we consider graphs with weakly convex domination subdivision number equal to 3.

Proposition 3.7. For every connected graph G of order $n \geq 3$, if $\gamma_{\rm wcon}(G) = 3$, then $\mathrm{sd}_{\gamma_{\rm wcon}}(G) \leq 3$.

Proof. If $\gamma_{\rm wcon}(G) = 3$, then the $\gamma_c(G)$–sets and the $\gamma_{\rm wcon}(G)$–sets are the same and so $\gamma_c(G) = \gamma_{\rm wcon}(G)$. By Proposition 3.2, $\mathrm{sd}_{\gamma_{\rm wcon}}(G) \leq \mathrm{sd}_{\gamma_c}(G)$ and the result follows from Proposition 3.6. □

Using Propositions 3.3, 3.4 and 3.7, we obtain other two general upper bounds for weakly convex domination subdivision number.

Theorem 3.8. For any connected graph G of order $n \geq 3$,

$$\mathrm{sd}_{\gamma_{\rm wcon}}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

Proof. The result is immediate for $n = 3$. Let $n \geq 4$. If $\delta(G) \leq \lfloor \frac{n}{2} \rfloor$, the result is true by Corollary 2.3. Let $\delta(G) > \lfloor \frac{n}{2} \rfloor$. If $n = 4$, then $\gamma_{\rm wcon}(G) = 1$ and it follows from Proposition 3.3 that $\mathrm{sd}_{\gamma_{\rm wcon}}(G) = 1 < \left\lfloor \frac{n}{2} \right\rfloor$. If $n = 5$, then clearly $\gamma_{\rm wcon}(G) \leq 2$ and by Propositions 3.3 and 3.4 we have $\mathrm{sd}_{\gamma_{\rm wcon}}(G) \leq 2 = \left\lfloor \frac{n}{2} \right\rfloor$. Let $n \geq 6$. We deduce from Theorem 1.2 that $\gamma_{\rm wcon}(G) \leq \max\{3, 2\lfloor \frac{n}{2} \rfloor - \delta(G)\}$. Now the result follows from Proposition 3.7 and Theorem 2.7. □

Corollary 3.9. For any connected graph G of order $n \geq 3$,

$$\mathrm{sd}_{\gamma_{\mathrm{wcon}}}(G) \leq \alpha'(G).$$

Proof. By Corollary 2.3 and Theorem 3.8, we have $\mathrm{sd}_{\gamma_{\mathrm{wcon}}}(G) \leq \min\{\delta(G), \lfloor \frac{n}{2} \rfloor\}$. On the other hand, it is known from [6] that the matching number of every graph is at least $\min\{\delta(G), \lfloor \frac{n}{2} \rfloor\}$. Thus $\mathrm{sd}_{\gamma_{\mathrm{wcon}}}(G) \leq \alpha'(G)$. □

Next result gives a bound for the weakly convex domination subdivision number of a triangle–free graph graph G with $\gamma_{\mathrm{wcon}}(G) = 4$.

Theorem 3.10. Let G be a connected triangle-free graph G with $\gamma_{\mathrm{wcon}}(G) = 4$. Then $\mathrm{sd}_{\gamma_{\mathrm{wcon}}}(G) \leq 4$.

Proof. Let $D = \{u_1, u_2, u_3, u_4\}$ be a $\gamma_{\mathrm{wcon}}(G)$–set such that the size of $G[D]$ is as large as possible. Since the induced subgraph $G[D]$ is connected, we consider three cases.

Case 1. $G[D] = C_4$ such that $u_1u_2 \in E(C_4)$, $u_2u_3 \in E(C_4)$, $u_3u_4 \in E(C_4)$, $u_4u_1 \in E(C_4)$.
If u_i has no private neighbor with respect to D for some i, then clearly $D - \{u_i\}$ is a weakly convex dominating set of G which is a contradiction. Let v_i be a private neighbor of u_i with respect to D for each i. Assume G' is obtained from G by subdividing the edges u_iv_i with vertices x_i for each i. Suppose S_1 is a $\gamma_{\mathrm{wcon}}(G')$–set. We show that $|S_1| \geq 5$. Assume to the contrary that $|S_1| \leq 4$. To dominate x_i, we must have $S_1 \cap \{u_i, v_i\} \neq \emptyset$. Since $|S_1| \leq 4$, $|S_1 \cap \{u_i, v_i\}| = 1$. Since v_i is a private neighbor of u_i with respect to D for each i, $S_1 \cap \{u_i \mid 1 \leq i \leq 4\} \neq \emptyset$ and $S_1 \cap \{v_i \mid 1 \leq i \leq 4\} \neq \emptyset$. Let $u_i \in S_1, v_j \in S_1$ for some $i \neq j$. Then clearly every u_iv_j–path contains a vertex not in $\{u_i, v_i \mid 1 \leq i \leq 4\}$ which leads to a contradiction.

Case 2. $G[D] = P_4$ such that $u_1u_2 \in E(P_4)$, $u_2u_3 \in E(P_4)$, $u_3u_4 \in E(P_4)$.
Obviously u_1 and u_4 have no common neighbor. If u_1 has no private neighbor with respect to D, then clearly $D - \{u_1\}$ is a weakly convex dominating set of G, a contradiction. Let v_1 be a private neighbor of u_1 with respect to D. Similarly, u_4 has a private neighbor with respect to D, say v_4. Assume G' is obtained from G by subdividing the edges $u_1v_1, u_1u_2, u_2u_3, u_4v_4$ with vertices x_1, x_2, x_3, x_4, respectively. Assume S_2 is a $\gamma_{\mathrm{wcon}}(G')$–set. Now we show that $|S_2| \geq 5$. Let $|S_2| \leq 4$. Clearly, $S_2 \cap \{u_1, v_1\} \neq \emptyset, S_2 \cap \{u_4, v_4\} \neq \emptyset$ and $S_2 \cap \{u_2, u_3\} \neq \emptyset$. If $\{u_1, v_1\} \subseteq S_2$ then $x_1 \in S_2$ implying that $|S_2| \geq 5$ which is a contradiction. Therefore $|S_2 \cap \{u_1, v_1\}| = 1$. Similarly, $|S_2 \cap \{u_4, v_4\}| = 1$. First let $u_1 \notin S_2$. Then we must have $u_2, v_1 \in S_2$. Since G is triangle–free and $|S_2| \leq 4$, we have $2 \leq d_{G'}(v_1, u_2) \leq 3$. If $d_{G'}(v_1, u_2) = 3$, then let v_1, w_1, w_2, u_2 is a geodesic path such that $S_2 = \{v_1, u_2, w_1, w_2\}$. Since G is triangle–free, $u_1w_1 \notin E(G)$ and $u_1w_2 \notin E(G)$ and so S_2 does not dominate u_1, a contradiction. Therefore $d_{G'}(v_1, u_2) = 2$ and so u_2 and v_1 have a common neighbor w not in $\{u_3, u_4, v_4\}$. Hence $S_2 = \{v_1, w, u_2, w'\}$ where $w' \in \{u_4, v_4\}$. But then, to dominate u_1, we must have $wu_1 \in E(G)$ which is a contradiction because G is triangle–free. Assume now $u_1 \in S_2$. If $u_2 \in S_2$, then $x_2 \in S_2$ and clearly $G'[S_2]$ will be not connected which is a contradiction. Let $u_2 \notin S_2$ that yields $u_3 \in S_2$. Since G is triangle–free and $|S_2| \leq 4$, u_3, u_1 have a common neighbor w which belongs to S_2. If $w' \in S_2 \cap \{u_4, v_4\}$, then $S_2 = \{w, w', u_1, u_3\}$. It is easy to see that S_2 does not dominate u_2, a contradiction.

Subcase 2.3. $G[D] = K_{1,3}$.
Assume $u = u_4$ is the center of $G[D] = K_{1,3}$ and u_1, u_2, u_3 are leaves adjacent to u. As above, we can see that u_i has a private neighbor with respect to D, say v_i, for each i. Let G' be the graph obtained from G by subdividing the edges u_1v_1, u_2v_2, u_3v_3 with vertices x_1, x_2, x_3, respectively, and let S_3 be a $\gamma_{\mathrm{wcon}}(G')$–set. We show that $|S_3| \geq 5$. Assume to the contrary that $|S_3| \leq 4$. To dominate x_i, we must have $S_3 \cap \{u_i, v_i\} \neq \emptyset$ for each i. If $\{u_i, v_i\} \subseteq S_3$ for some i, then $x_i \in S_3$ implying that $|S_3| \geq 5$, a contradiction. If $u_i, u_j \in S_3$, then u_i, u_j must have a common neighbor $w \in S_3$ that dominates v_i or is adjacent to u_k, v_k ($k \notin \{i, j\}$) which is a contradiction because G is triangle–free. If $u_i, v_j \in S_3$ for some $i \neq j$, then u_i, v_j must have a common neighbor w that dominates u_j, a contradiction again. So we assume $\{v_1, v_2, v_3\} \subseteq S_3$. If $S_3 = \{v_1, v_2, v_3, w\}$, then w must be adjacent to u_i for each i which leads to a contradiction because G is triangle–free. This completes the proof. □

Next result is an immediate consequence of Propositions 2.10, 3.3 (part (iv)), 3.4, 3.7 and Theorem 3.10.

Corollary 3.11. For any connected graph G of order $n \geq 3$ with $g(G) \geq \gamma_{\mathrm{wcon}}(G)$,

$$\mathrm{sd}_{\gamma_{\mathrm{wcon}}}(G) \leq \gamma_{\mathrm{wcon}}(G).$$

4. Graphs with Large Weakly Convex Domination Subdivision Number

In the previous sections, we essentially presented bounds on the weakly convex domination subdivision number in graphs. Our goal in this section is to show that the weakly convex domination subdivision number of a graph can be arbitrarily large. The following graph was introduced by Haynes et al. in [14] to prove a similar result for $\mathrm{sd}_{\gamma_t}(G)$.

Let $X = \{1, 2, \ldots, 3(k-1)\}$ and let $\mathcal{Y} = \{Y \subset X \mid |Y| = k\}$. Thus, $\mathcal{Y}$ consists of all k–subsets of X, and so $|\mathcal{Y}| = \binom{3(k-1)}{k}$. Let G be the graph with vertex set $X \cup \mathcal{Y}$ and with edge set constructed as follows: add an edge joining every two distinct vertices of X and for each $x \in X$ and $Y \in \mathcal{Y}$, add an edge joining x and Y if and only if $x \in Y$. Then, G_k is a connected graph of order $n = \binom{3(k-1)}{k} + 3(k-1)$. The set X induces a clique in G_k, while the set $\mathcal{Y}$ is an independent set and each vertex of $\mathcal{Y}$ has degree k in G_k. Therefore $\delta(G) = k$. Favaron et al. [10] proved that $\gamma_c(G_k) = 2(k-1)$ and $\mathrm{sd}_{\gamma_c}(G_k) = k$.

Proposition 4.1. For any integer $k \geq 2$, $\gamma_{\mathrm{wcon}}(G_k) = 2(k-1)$.

Proof. By Proposition 3.6, $\gamma_{\mathrm{wcon}}(G_k) \geq \gamma_c(G_k) = 2(k-1)$. On the other hand, any subset of X of cardinality $2(k-1)$ is a weakly convex dominating set of G, and so $\gamma_{\mathrm{wcon}}(G_k) \leq 2(k-1)$. Consequently, $\gamma_{\mathrm{wcon}}(G_k) = \gamma_c(G_k) = 2(k-1)$. □

Theorem 4.2. For any integer $k \geq 2$, $\mathrm{sd}_{\gamma_{\mathrm{wcon}}}(G_{2k}) \geq k+1$.

Proof. Assume $F = \{e_1, \ldots, e_k\}$ is an arbitrary subset of k edges of G_{2k} and let G'_{2k} be the graph obtained from G_{2k} by subdividing all edges in F. We show that $\gamma_{\mathrm{wcon}}(G'_{2k}) \leq \gamma_{\mathrm{wcon}}(G_{2k}) = 2(2k-1)$. Assume $e_i = u_i v_i$ for each i and let $S = X \cap \{u_i, v_i \mid 1 \leq i \leq k\}$. Clearly $|X - S| \geq 4k - 3$ and each vertex in $X - S$ is adjacent to all vertices in S. Since every edge of G_{2k} is incident with at least one vertex of X, we may assume that $u_i \in X$ for each i. If $v_i \in \mathcal{Y}$, then since $d_{G_{2k}}(v_i) = 2k$ and $|F| = k$, v_i is adjacent to a vertex of $X - S$, say w_i, such that $v_i w_i \notin F$. If $v_i \in X$, then let w_i be any vertex of $X - S$. Assume $D_F = \{u_i, w_i \mid 1 \leq i \leq k\}$. Then $|D_F| \leq 2k$. Now extend D_F to a set D of size $2(2k-1)$ by adding $4k - 2 - |D_F|$ vertices of $X - S$. Obviously D is a weakly convex dominating set of G'_{2k}, and so $\gamma_{\mathrm{wcon}}(G'_{2k}) \leq 2(2k-1) = \gamma_{\mathrm{wcon}}(G_{2k})$. This implies that $\mathrm{sd}_{\gamma_{\mathrm{wcon}}}(G_{2k}) \geq k+1$. □

We conclude this paper with an open problem.

Problem 4.3. *Prove or disprove: For any connected graph G of order $n \geq 3$,*

$$\mathrm{sd}_{\gamma_{\mathrm{wcon}}}(G) \leq \gamma_{\mathrm{wcon}}(G).$$

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