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The Existence of Solution for a k-dimensional System of Fractional Differential Inclusions with Anti-Periodic Boundary Value Conditions

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Abstract. We investigate the existence of solutions for a k-dimensional systems of fractional differential inclusions with anti-periodic boundary conditions. We provide two results via different conditions for obtaining solutions of the k-dimensional inclusion problem. We provide some examples to illustrate our results.

1. Introduction

The theory of fractional differential equations has a complete and interesting history. There are many published many works about the existence of solutions for many fractional differential equations by using fixed point theory (see for example, [8]-[12], [25]-[28] and the references there in). Also, some researchers tried to obtain solutions of fractional differential inclusions by using different conditions (see for example, [1]-[7], [14]-[17], [20], [25], [27]-[28], [32] and the references there in). One can find more details elementary notions on fractional differential equations in [22], [26] and [30]. Recently, Chai investigated the existence of solutions for the fractional differential equation

$$ {}^cD^{\alpha}u(t) = f(t, u(t), {}^cD^{\gamma_1}u(t), {}^cD^{\gamma_2}u(t)) \quad (1.1)$$

with the anti-periodic boundary value conditions $t^{\beta_1-1}\,{}^cD^{\beta_1}u(t)|_{t\to 0} = -t^{\beta_1-1}\,{}^cD^{\beta_1}u(t)|_{t=1}$, $u(0) = -u(1)$ and $t^{\beta_2-2}\,{}^cD^{\beta_2}u(t)|_{t\to 0} = -t^{\beta_2-2}\,{}^cD^{\beta_2}u(t)|_{t=1}$, where $t \in J = [0,1]$, $2 < \alpha \le 3$, $0 < \gamma_1 \le 1$, $1 < \gamma_2 \le 2$, $0 < \beta_1 < 1 < \beta_2 < 2$ and ${}^cD$ is the Caputo fractional differentiation ([17]). We use the basic idea of these boundary value conditions and combine it with the basic idea of [5] for investigating a k-dimensional systems of fractional differential inclusions. Also, we use Let $k \ge 2$ be a natural number. In this paper, we investigate the existence solution for a k-dimensional systems of fractional differential inclusions

$$\begin{cases} {}^cD^{\alpha_1}u_1(t) \in F_1(t, u_1(t), \cdots, u_k(t), {}^cD^{\gamma^1_{11}}u_1(t), \cdots, {}^cD^{\gamma^1_{1k}}u_k(t), {}^cD^{\gamma^2_{11}}u_1(t), \cdots, {}^cD^{\gamma^2_{1k}}u_k(t)), \\ {}^cD^{\alpha_2}u_2(t) \in F_2(t, u_1(t), \cdots, u_k(t), {}^cD^{\gamma^1_{21}}u_1(t), \cdots, {}^cD^{\gamma^1_{2k}}u_k(t), {}^cD^{\gamma^2_{21}}u_1(t), \cdots, {}^cD^{\gamma^2_{2k}}u_k(t)), \\ \cdot \\ \cdot \\ \cdot \\ {}^cD^{\alpha_k}u_k(t) \in F_k(t, u_1(t), \cdots, u_k(t), {}^cD^{\gamma^1_{k1}}u_1(t), \cdots, {}^cD^{\gamma^1_{kk}}u_k(t), {}^cD^{\gamma^2_{k1}}u_1(t), \cdots, {}^cD^{\gamma^2_{kk}}u_k(t)), \end{cases} \quad (1.2)$$

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with the anti-periodic boundary value conditions $t^{\beta_{1i}-1c}D^{\beta_{1i}}u_i(t)|_{t\to 0} = -t^{\beta_{1i}-1c}D^{\beta_{1i}}u_i(t)|_{t=1}$, $u_i(0) = -u_i(1)$ and $t^{\beta_{2i}-2c}D^{\beta_{2i}}u_i(t)|_{t\to 0} = -t^{\beta_{2i}-2c}D^{\beta_{2i}}u_i(t)|_{t=1}$ for $i = 1, \cdots, k$, where $t \in J$, $2 < \alpha_i \leq 3$, $0 < \gamma_{ij}^1 \leq 1$, $1 < \gamma_{ij}^2 \leq 2$, $0 < \beta_{1i} < 1 < \beta_{2i} < 2$ and $F_i : J \times \mathbb{R}^{3k} \to 2^{\mathbb{R}}$ is a multifunction for all $1 \leq i, j \leq k$.

2. Preliminaries

Here, we give some needed notions. Let (X, d) be a metric space. It is well known that the Pompeiu-Hausdorff metric $H : 2^X \times 2^X \to [0, \infty)$ is defined by

$$H(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\},$$

where $d(A, b) = \inf_{a\in A} d(a, b)$ and 2^X is the set of all nonempty subsets of X ([13]). Then $(CB(X), H)$ is a metric space and $(C(X), H)$ is a generalized metric space, where $CB(X)$ is the set of closed and bounded subsets of X and $C(X)$ is the set of closed subsets of X ([23]). Denote the set of compact and convex subsets of X by $P_{cp,cv}(X)$. Let $T : X \to 2^X$ be a multifunction. An element $x \in X$ is called an fixed point of T whenever $x \in Tx$ ([21]). A multifunction $T : X \to C(X)$ is called a contraction whenever there exists $\gamma \in (0, 1)$ such that $H(N(x), N(y)) \leq \gamma d(x, y)$ for all $x, y \in X$ ([21]). A multifunction $T : X \to 2^X$ is called lower semi-continuous whenever for each open set A of X, the set $T^{-1}(A) := \{x \in X : Tx \cap A \neq \emptyset\}$ is open in X ([21]). We say that T is upper semi- continuous whenever for each open set A of X, the set $\{x \in X : Tx \subset A\}$ is open in X ([21]). Also, $T : X \to 2^X$ is called compact whenever for each bounded subsets S of X, $\overline{T(S)}$ is a compact set of X ([21]). A multifunction $T : X \to 2^X$ is said to be completely continuous whenever for each bounded subset B of X, $T(B)$ is relatively compact ([21]). A multifunction $T : J \to 2^{\mathbb{R}}$ is said to be measurable whenever the function $t \mapsto d(y, T(t))$ is measurable for all $y \in \mathbb{R}$, where $J = [0, 1]$ ([19]). We say that $F : J \times \mathbb{R}^{3k} \to 2^{\mathbb{R}}$ is a Caratheodory multifunction whenever the map $t \mapsto F(t, x_1, \cdots, x_{3k})$ is measurable for all $x_1, \cdots, x_{3k} \in \mathbb{R}$ and the map $(x_1, \cdots, x_{3k}) \mapsto F(t, x_1, \cdots, x_{3k})$ is upper semi-continuous for almost all $t \in J$ (see [7], [19] and [23]). Also, a Caratheodory multifunction $F : J \times \mathbb{R}^{3k} \to 2^{\mathbb{R}}$ is called L^1-Caratheodory whenever for each $\rho > 0$ there exists $\phi_\rho \in L^1(J, \mathbb{R}^+)$ such that

$$\| F(t, x_1, \cdots, x_{3k}) \| = \sup\{|v| : v \in F(t, x_1, \cdots, x_{3k})\} \leq \phi_\rho(t)$$

for all $x_1, \cdots, x_{3k} \in \mathbb{R}$ with $|x_1|, \cdots, |x_{3k}| \leq \rho$ and almost all $t \in J$ (see [7], [19] and [23]).

Define the space $\mathcal{X}_i = \{x : x, {}^cD^{\gamma_{ii}^1}x, {}^cD^{\gamma_{ii}^2}x \in C(J, \mathbb{R})\}$ endowed with the norm

$$\|x\|_i = \sup_{t\in J} |x(t)| + \sup_{t\in J} |{}^cD^{\gamma_{ii}^1}x(t)| + \sup_{t\in J} |{}^cD^{\gamma_{ii}^2}x(t)|$$

for all $i \in \{1, \cdots, k\}$. Also, define the product space $\mathcal{X} = \mathcal{X}_1 \times ... \times \mathcal{X}_k$ endowed with the norm $\|(u_1, \cdots, u_k)\| = \sum_{i=1}^k \|x_i\|_i$. Then, $(\mathcal{X}, \|.\|)$ is a Banach space ([31]). By using the idea of the papers [4], [6], [27] and [32], we define the set of the selections of F_i at x by

$$S_{F_i,x} = \Big\{v \in L^1[0, 1] : v(t) \in F_i(t, x_1(t), \cdots, x_k(t), {}^c D^{\gamma_{i1}^1}x_1(t), \cdots, {}^c D^{\gamma_{ik}^1}x_k(t),$$

$$ {}^cD^{\gamma_{i1}^2}x_1(t), \cdots, {}^c D^{\gamma_{ik}^2}x_k(t)) \text{ for almost all } t \in J\Big\}$$

for all $x = (x_1, ..., x_k) \in \mathcal{X}$ and $1 \leq i \leq k$. We need the following results in the sequel. One can find the Ascoli-Arzela theorem in [19] and [21].

Lemma 2.1. *Let X be compact metric space and A a closed, equi-continuous and bounded subset of $C(X)$. Then A is relatively compact.*

In 1970, Covitz and Nadler proved next result ([18]).

Lemma 2.2. *Let (X, d) be a complete metric space. If $N : X \to C(X)$ is a contractive multifunction, then N has a fixed point.*

Lemma 2.3. *([24]) Let X be a Banach space, $F : J \times X \to P_{cp,cv}(X)$ an L^1-Caratheodory multifunction and Θ a linear continuous mapping from $L^1(J, X)$ to $C(J, X)$. Then the operator $\Theta o S_F : C(J, X) \to P_{cp,cv}(C(J), X)$ defined by $(\Theta o S_F)(x) = \Theta(S_{F,x})$ is a closed graph operator in $C(J, X) \times C(J, X)$.*

Lemma 2.4. *([21]) Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to P_{cp,cv}(C)$ is a upper semi-continuous compact map, where $P_{cp,cv}(C)$ denotes the family of nonempty, compact convex subsets of C. Then either F has a fixed point in $\overline{U}$ or there exist $u \in \partial U$ and $\lambda \in (0, 1)$ such that $u \in \lambda F(u)$.*

Lemma 2.5. *([19]) Let X and Y be Banach spaces and $F : X \to P(Y)$ a completely continuous multifunction via nonempty compact values. Then F is upper semi-continuous if and only if F has a closed graph, that is, $x_n \to x_0$, $y_n \to y_0$ and $y_n \in F(x_n)$ for all n imply that $y_0 \in F(x_0)$.*

Lemma 2.6. *([17]) Let $v \in C(J, \mathbb{R})$, $\alpha \in (2, 3]$ and $0 < \beta_1 < 1 < \beta_2 < 2$. Then a function $u \in C(J, \mathbb{R})$ is a solution of the fractional boundary value problem ${}^cD^{\alpha}u(t) = v(t)$ with the boundary value conditions $u(0) = -u(1)$, $t^{\beta_1-1c}D^{\beta_1}u(t)|_{t\to 0} = -t^{\beta_1-1c}D^{\beta_1}u(t)|_{t=1}$ and $t^{\beta_2-2c}D^{\beta_2}u(t)|_{t\to 0} = -t^{\beta_2-2c}D^{\beta_2}u(t)|_{t=1}$ if and only if u is a solution of the integral equation*

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds + \int_0^1 \Big[\frac{-1}{2\Gamma(\alpha)}(1-s)^{\alpha-1} + \frac{(1-2t)\Gamma(2-\beta_1)}{4\Gamma(\alpha-\beta_1)}(1-s)^{\alpha-\beta_1-1}$$
$$+\frac{1-\beta_1+2t-2(2-\beta_1)t^2}{8(2-\beta_1)}\frac{\Gamma(3-\beta_2)}{\Gamma(\alpha-\beta_2)}(1-s)^{\alpha-\beta_2-1}\Big]v(s)ds.$$

3. Main Results

Now, we are ready to provide our results about the existence of solution for the k-dimensional system of fractional differential inclusions (1.2). We say that a function $(u_1, u_2, ..., u_k) \in \mathcal{X}$ is a solution for the k-dimensional system of inclusions (1.2) whenever there exists functions $v_1, \cdots, v_k$ in $L^1[0, 1]$ such that

$$v_i(t) \in F_i(t, u_1(t), \cdots, u_k(t), {}^cD^{\gamma^1_{i1}}u_1(t), \cdots, {}^cD^{\gamma^1_{ik}}u_k(t), {}^cD^{\gamma^2_{i1}}u_1(t), \cdots, {}^cD^{\gamma^2_{ik}}u_k(t))$$

for almost all $t \in J$,

$$u_i(t) = \frac{1}{\Gamma(\alpha_i)}\int_0^t (t-s)^{\alpha_i-1}v_i(s)ds$$
$$+\int_0^1 \Big[\frac{-1}{2\Gamma(\alpha_i)}(1-s)^{\alpha_i-1} + \frac{(1-2t)\Gamma(2-\beta_{1i})}{4\Gamma(\alpha_i-\beta_{1i})}(1-s)^{\alpha_i-\beta_{1i}-1}$$
$$+\frac{1-\beta_{1i}+2t-2(2-\beta_{1i})t^2}{8(2-\beta_{1i})}\frac{\Gamma(3-\beta_{2i})}{\Gamma(\alpha_i-\beta_{2i})}(1-s)^{\alpha_i-\beta_{2i}-1}\Big]v_i(s)ds$$

and $t^{\beta_{1i}-1c}D^{\beta_{1i}}u_i(t)|_{t\to 0} = -t^{\beta_{1i}-1c}D^{\beta_{1i}}u_i(t)|_{t=1}$, $t^{\beta_{2i}-2c}D^{\beta_{2i}}u_i(t)|_{t\to 0} = -t^{\beta_{2i}-2c}D^{\beta_{2i}}u_i(t)|_{t=1}$ and $u_i(0) = -u_i(1)$ for all $i = 1, \cdots, k$.

3.1. The Caratheodory case

Theorem 3.1. *Suppose that $F_1, \cdots, F_k : J \times \mathbb{R}^{3k} \to P_{cp,cv}(\mathbb{R})$ are Caratheodory multifunctions, there exist a non-decreasing, bounded and continuous map $\psi : [0, \infty) \to (0, \infty)$ and continuous functions $p_1, \cdots, p_k : J \to (0, \infty)$ such that*

$$\|F_i(t, u_1(t), \cdots, u_k(t), {}^cD^{\gamma^1_{i1}}u_1(t), \cdots, {}^cD^{\gamma^1_{ik}}u_k(t), {}^cD^{\gamma^2_{i1}}u_1(t), \cdots, {}^cD^{\gamma^2_{ik}}u_k(t))\| =$$

$$\sup\{|y| : y \in F_i(t, u_1(t), \cdots, u_k(t), {}^cD^{\gamma^1_{i1}}u_1(t), \cdots, {}^cD^{\gamma^1_{ik}}u_k(t), {}^cD^{\gamma^2_{i1}}u_1(t), \cdots, {}^cD^{\gamma^2_{ik}}u_k(t))\} \le p_i(t)\psi(\|(u_1, ..., u_k)\|)$$

for all $1 \le i \le k$, $(u_1, ..., u_k) \in \mathcal{X}$ *and almost all* $t \in J$. *Assume that there exist constants* $L_i > 0$ *such that* $\frac{L_i}{(\Lambda^i_1+\Lambda^i_2+\Lambda^i_3)\|p_i\|_\infty \psi(\|(u_1,u_2,...,u_k)\|)} > 1$ *for all* $(u_1, ..., u_k) \in \mathcal{X}$, *where*

$$\Lambda^i_1 = \frac{3}{\Gamma(\alpha_i + 1)} + \frac{3\Gamma(2 - \beta_{1i})}{4\Gamma(\alpha_i - \beta_{1i} + 1)} + \frac{5\Gamma(3 - \beta_{2i})}{4\Gamma(\alpha_i - \beta_{2i} + 1)},$$

$$\Lambda^i_2 = \frac{1}{\Gamma(\alpha_i - \gamma^1_{ii} + 1)} + \frac{\Gamma(2 - \beta_{1i})}{2\Gamma(\alpha_i - \beta_{1i} + 1)\Gamma(2 - \gamma^1_{ii})} + \frac{\Gamma(3 - \beta_{2i})}{4\Gamma(\alpha_i - \beta_{1i} + 1)\Gamma(2 - \gamma^1_{ii})(2 - \beta_{1i})} + \frac{\Gamma(3 - \beta_{2i})}{2\Gamma(\alpha_i - \beta_{2i} + 1)\Gamma(3 - \gamma^1_{ii})},$$

$\Lambda^i_3 = \frac{1}{\Gamma(\alpha_i-\gamma^2_{ii}+1)} + \frac{\Gamma(3-\beta_{2i})}{2\Gamma(\alpha_i-\beta_{2i}+1)\Gamma(3-\gamma^2_{ii})}$ *and* $\|p_i\|_\infty = \sup_{t\in J} |p_i(t)|$ *for all* $i = 1, ..., k$. *Then the k-dimensional system of inclusions with the boundary value conditions* (1.2) *has at least one solution.*

Proof. Define the operator $N : \mathcal{X} \to 2^{\mathcal{X}}$ by $N(u_1, \cdots, u_k) = \begin{pmatrix} N_1(u_1, \cdots, u_k) \\ N_2(u_1, \cdots, u_k) \\ \vdots \\ N_k(u_1, \cdots, u_k) \end{pmatrix}$, where

$$N_i(u_1, \cdots, u_k) = \Big\{h \in \mathcal{X}_i : \text{ there exists } v \in S_{F_{i,(u_1,\cdots,u_k)}} \text{ such that}$$

$$h(t) = \frac{1}{\Gamma(\alpha_i)} \int_0^t (t - s)^{\alpha_i-1} v(s)ds + \int_0^1 \Big[\frac{-1}{2\Gamma(\alpha_i)}(1 - s)^{\alpha_i-1} + \frac{(1 - 2t)\Gamma(2 - \beta_{1i})}{4\Gamma(\alpha_i - \beta_{1i})}(1 - s)^{\alpha_i-\beta_{1i}-1} + \frac{1 - \beta_{1i} + 2t - 2(2 - \beta_{1i})t^2}{8(2 - \beta_{1i})} \frac{\Gamma(3 - \beta_{2i})}{\Gamma(\alpha_i - \beta_{2i})}(1 - s)^{\alpha_i-\beta_{2i}-1}\Big]v(s)ds\Big\}.$$

Now, we show that the operator N has a fixed point. First, we show that $N(u_1, \cdots, u_k)$ is convex for all $(u_1, \cdots, u_k) \in \mathcal{X}$. Let $(h_1, \cdots, h_k), (h'_1, \cdots, h'_k) \in N(u_1, \cdots, u_k)$. Choose $v_i, v'_i \in S_{F_{i,(u_1,\cdots,u_k)}}$ such that

$$h_i(t) = \frac{1}{\Gamma(\alpha_i)} \int_0^t (t - s)^{\alpha_i-1} v_i(s)ds + \int_0^1 \Big[\frac{-1}{2\Gamma(\alpha_i)}(1 - s)^{\alpha_i-1} + \frac{(1 - 2t)\Gamma(2 - \beta_{1i})}{4\Gamma(\alpha_i - \beta_{1i})}(1 - s)^{\alpha_i-\beta_{1i}-1} + \frac{1 - \beta_{1i} + 2t - 2(2 - \beta_{1i})t^2}{8(2 - \beta_{1i})} \frac{\Gamma(3 - \beta_{2i})}{\Gamma(\alpha_i - \beta_{2i})}(1 - s)^{\alpha_i-\beta_{2i}-1}\Big]v_i(s)ds$$

and

$$h'_i(t) = \frac{1}{\Gamma(\alpha_i)} \int_0^t (t - s)^{\alpha_i-1} v'_i(s)ds + \int_0^1 \Big[\frac{-1}{2\Gamma(\alpha_i)}(1 - s)^{\alpha_i-1} + \frac{(1 - 2t)\Gamma(2 - \beta_{1i})}{4\Gamma(\alpha_i - \beta_{1i})}(1 - s)^{\alpha_i-\beta_{1i}-1} + \frac{1 - \beta_{1i} + 2t - 2(2 - \beta_{1i})t^2}{8(2 - \beta_{1i})} \frac{\Gamma(3 - \beta_{2i})}{\Gamma(\alpha_i - \beta_{2i})}(1 - s)^{\alpha_i-\beta_{2i}-1}\Big]v'_i(s)ds$$

for all $t \in J$ and $1 \le i \le k$. Let $0 \le w \le 1$. Then, we have

$$[wh_i + (1 - w)h'_i](t) = \frac{1}{\Gamma(\alpha_i)} \int_0^t (t - s)^{\alpha_i-1}[wv_i(s) + (1 - w)v'_i(s)ds + \int_0^1 \Big[\frac{-1}{2\Gamma(\alpha_i)}(1 - s)^{\alpha_i-1} + \frac{(1 - 2t)\Gamma(2 - \beta_{1i})}{4\Gamma(\alpha_i - \beta_{1i})}(1 - s)^{\alpha_i-\beta_{1i}-1} + \frac{1 - \beta_{1i} + 2t - 2(2 - \beta_{1i})t^2}{8(2 - \beta_{1i})}$$

$$\frac{\Gamma(3-\beta_{2i})}{\Gamma(\alpha_i-\beta_{2i})}(1-s)^{\alpha_i-\beta_{2i}-1}\Big][wv_i(s)+(1-w)v_i'(s)ds.$$

Since F_i is convex valued for all $1 \le i \le k$, $[wh_i + (1-w)h_i'] \in N_i(u_1, \cdots, u_k)$. Thus,

$$w(h_1, \cdots, h_k) + (1-w)(h_1', \cdots, h_k')$$

$$= (wh_1 + (1-w)h_1', \cdots, wh_k + (1-w)h_k') \in N(u_1, \cdots, u_k).$$

Now, we show that N maps bounded sets of $\mathcal{X}$ into bounded sets. Let $r > 0$, $B_r = \{(u_1, \cdots, u_k) \in \mathcal{X} : \|(u_1, \cdots, u_k)\| \le r\}$, $(u_1, \cdots, u_k) \in B_r$ and $(h_1, \cdots, h_k) \in N(u_1, \cdots, u_k)$. Choose $(v_1, \cdots, v_k) \in S_{F_1,(u_1,\cdots,u_k)} \times \cdots \times S_{F_k,(u_1,\cdots,u_k)}$ such that

$$h_i(t) = \frac{1}{\Gamma(\alpha_i)}\int_0^t (t-s)^{\alpha_i-1}v_i(s)ds + \int_0^1 \Big[\frac{-1}{2\Gamma(\alpha_i)}(1-s)^{\alpha_i-1} + \frac{(1-2t)\Gamma(2-\beta_{1i})}{4\Gamma(\alpha_i-\beta_{1i})}(1-s)^{\alpha_i-\beta_{1i}-1}$$

$$+\frac{1-\beta_{1i}+2t-2(2-\beta_{1i})t^2}{8(2-\beta_{1i})}\frac{\Gamma(3-\beta_{2i})}{\Gamma(\alpha_i-\beta_{2i})}(1-s)^{\alpha_i-\beta_{2i}-1}\Big]v_i(s)ds$$

for all $t \in J$ and $1 \le i \le k$. Hence, we get

$$^cD^{\gamma_{ii}^1}h_i(t) = \frac{1}{\Gamma(\alpha_i-\gamma_{ii}^1)}\int_0^t (t-s)^{\alpha_i-\gamma_{ii}^1-1}v_i(s)ds$$

$$-\frac{t^{1-\gamma_{ii}^1}\Gamma(2-\beta_{1i})}{2\Gamma(2-\gamma_{ii}^1)\Gamma(\alpha_i-\beta_{1i})}\int_0^1 (1-s)^{\alpha_i-\beta_{1i}-1}v_i(s)ds$$

$$+\frac{t^{1-\gamma_{ii}^1}\Gamma(3-\beta_{2i})}{4\Gamma(2-\gamma_{ii}^1)\Gamma(\alpha_i-\beta_{1i})(2-\beta_{1i})}\int_0^1 (1-s)^{\alpha_i-\beta_{2i}-1}v_i(s)ds$$

$$-\frac{t^{2-\gamma_{ii}^1}\Gamma(3-\beta_{2i})}{2\Gamma(3-\gamma_{ii}^1)\Gamma(\alpha_i-\beta_{2i})}\int_0^1 (1-s)^{\alpha_i-\beta_{2i}-1}v_i(s)ds$$

and

$$^cD^{\gamma_{ii}^2}h_i(t) = \frac{1}{\Gamma(\alpha_i-\gamma_{ii}^2)}\int_0^t (t-s)^{\alpha_i-\gamma_{ii}^2-1}v_i(s)ds$$

$$-\frac{t^{2-\gamma_{ii}^2}\Gamma(3-\beta_{2i})}{2\Gamma(3-\gamma_{ii}^2)\Gamma(\alpha_i-\beta_{2i})}\int_0^1 (1-s)^{\alpha_i-\beta_{2i}-1}v_i(s)ds.$$

Hence,

$$|h_i(t)| \le \frac{1}{\Gamma(\alpha_i)}\int_0^t (t-s)^{\alpha_i-1}|v_i(s)|ds + \int_0^1 \Big|\Big[\frac{-1}{2\Gamma(\alpha_i)}(1-s)^{\alpha_i-1}$$

$$+\frac{(1-2t)\Gamma(2-\beta_{1i})}{4\Gamma(\alpha_i-\beta_{1i})}(1-s)^{\alpha_i-\beta_{1i}-1} + \frac{1-\beta_{1i}+2t-2(2-\beta_{1i})t^2}{8(2-\beta_{1i})}$$

$$\frac{\Gamma(3-\beta_{2i})}{\Gamma(\alpha_i-\beta_{2i})}(1-s)^{\alpha_i-\beta_{2i}-1}\Big]v_i(s)\Big|ds \le \|p_i\|_\infty \psi(\|(u_1, \cdots, u_k)\|)\Lambda_1^i,$$

$$|^cD^{\gamma_{ii}^1}h_i(t)| \le \frac{1}{\Gamma(\alpha_i-\gamma_{ii}^1)}\int_0^t (t-s)^{\alpha_i-\gamma_{ii}^1-1}|v_i(s)|ds$$

$$+\frac{t^{1-\gamma_{ii}^1}\Gamma(2-\beta_{1i})}{2\Gamma(2-\gamma_{ii}^1)\Gamma(\alpha_i-\beta_{1i})}\int_0^1 (1-s)^{\alpha_i-\beta_{1i}-1}|v_i(s)|ds$$

$$\frac{t^{1-\gamma_{ii}^1}\Gamma(3-\beta_{2i})}{4\Gamma(2-\gamma_{ii}^1)\Gamma(\alpha_i-\beta_{1i})(2-\beta_{1i})}\int_0^1(1-s)^{\alpha_i-\beta_{2i}-1}|v_i(s)|ds$$
$$+\frac{t^{2-\gamma_{ii}^1}\Gamma(3-\beta_{2i})}{2\Gamma(3-\gamma_{ii}^1)\Gamma(\alpha_i-\beta_{2i})}\int_0^1(1-s)^{\alpha_i-\beta_{2i}-1}|v_i(s)|ds\le\|p_i\|_\infty\psi(\|(u_1,\cdots,u_k)\|)\Lambda_2^i$$

and

$$|{}^cD^{\gamma_{ii}^2}h_i(t)|\le\frac{1}{\Gamma(\alpha_i-\gamma_{ii}^2)}\int_0^t(t-s)^{\alpha_i-\gamma_{ii}^2-1}|v_i(s)|ds$$
$$\frac{t^{2-\gamma_{ii}^2}\Gamma(3-\beta_{2i})}{2\Gamma(3-\gamma_{ii}^2)\Gamma(\alpha_i-\beta_{2i})}\int_0^1(1-s)^{\alpha_i-\beta_{2i}-1}|v_i(s)|ds\le\|p_i\|_\infty\psi(\|(u_1,\cdots,u_k)\|)\Lambda_3^i$$

for all $t\in J$ and $1\le i\le k$. Thus,

$$\|h_i\|_i\le(\Lambda_1^i+\Lambda_2^i+\Lambda_2^i)\|p_i\|_\infty\psi(r)$$

and so $\|(h_1,\cdots,h_k)\|=\sum_{i=1}^k\|h_i\|_i\le\psi(r)\sum_{i=1}^k(\Lambda_1^i+\Lambda_2^i+\Lambda_2^i)\|p_i\|_\infty$. Now, we show that N maps bounded sets to equi-continuous subsets of $\mathcal{X}$. Let $(u_1,\cdots,u_k)\in B_r$, $t_1,t_2\in J$ with $t_1<t_2$ and $(h_1,\cdots,h_k)\in N(u_1,\cdots,u_k)$. Then, we have

$$|h_i(t_2)-h_i(t_1)|=|\frac{1}{\Gamma(\alpha_i)}\int_0^{t_2}(t_2-s)^{\alpha_i-1}v_i(s)ds+\int_0^1\Big[\frac{-1}{2\Gamma(\alpha_i)}(1-s)^{\alpha_i-1}$$
$$+\frac{(1-2t_2)\Gamma(2-\beta_{1i})}{4\Gamma(\alpha_i-\beta_{1i})}(1-s)^{\alpha_i-\beta_{1i}-1}+\frac{1-\beta_{1i}+2t_2-2(2-\beta_{1i})t_2^2}{8(2-\beta_{1i})}$$
$$\frac{\Gamma(3-\beta_{2i})}{\Gamma(\alpha_i-\beta_{2i})}(1-s)^{\alpha_i-\beta_{2i}-1}\Big]v_i(s)ds-\frac{1}{\Gamma(\alpha_i)}\int_0^{t_1}(t_1-s)^{\alpha_i-1}v_i(s)ds$$
$$+\int_0^1\Big[\frac{-1}{2\Gamma(\alpha_i)}(1-s)^{\alpha_i-1}+\frac{(1-2t_1)\Gamma(2-\beta_{1i})}{4\Gamma(\alpha_i-\beta_{1i})}(1-s)^{\alpha_i-\beta_{1i}-1}$$
$$+\frac{1-\beta_{1i}+2t_1-2(2-\beta_{1i})t_1^2}{8(2-\beta_{1i})}\frac{\Gamma(3-\beta_{2i})}{\Gamma(\alpha_i-\beta_{2i})}(1-s)^{\alpha_i-\beta_{2i}-1}\Big]v_i(s)ds|$$
$$\le\|p_i\|_\infty\psi(\|(u_1,\cdots,u_k)\|)\Big[\frac{t_2^{\alpha_i}-t_1^{\alpha_i}}{\Gamma(\alpha_i+1)}-\frac{2(t_2-t_1)\Gamma(2-\beta_{1i})}{4\Gamma(\alpha_i-\beta_{1i}+1)}$$
$$+\frac{2(t_2-t_1)\Gamma(3-\beta^2-i)}{8(2-\beta_{1i})\Gamma(\alpha_i-\beta_{2i}+1)}\Big],$$

$$|{}^cD^{\gamma_{ii}^1}h_i(t_2)-{}^cD^{\gamma_{ii}^1}h_i(t_1)|\le\|p_i\|_\infty\psi(\|(u_1,\cdots,u_k)\|)\Big[\frac{t_2^{\alpha_i-\gamma_{ii}^1}-t_1^{\alpha_i-\gamma_{ii}^1}}{\Gamma(\alpha_i-\gamma_{ii}^1+1)}$$
$$+\frac{(t_2^{1-\gamma_{ii}^1}-t_1^{1-\gamma_{ii}^1})\Gamma(2-\beta_{1i})}{2\Gamma(\alpha_i-\beta_{1i}+1)\Gamma(2-\gamma_{ii}^1)}+\frac{(t_2^{1-\gamma_{ii}^1}-t_1^{1-\gamma_{ii}^1})\Gamma(3-\beta_{2i})}{4\Gamma(\alpha_i-\beta_{1i}+1)\Gamma(2-\gamma_{ii}^1)(2-\beta_{1i})}$$
$$+\frac{(t_2^{2-\gamma_{ii}^1}-t_1^{2-\gamma_{ii}^1})\Gamma(3-\beta_{2i})}{2\Gamma(\alpha_i-\beta_{2i}+1)\Gamma(3-\gamma_{ii}^1)}\Big]$$

and

$$|{}^cD^{\gamma_{ii}^2}h_i(t_2)-{}^cD^{\gamma_{ii}^2}h_i(t_1)|$$
$$\le\|p_i\|_\infty\psi(\|(u_1,\cdots,u_k)\|)\Big[\frac{t_2^{\alpha_i-\gamma_{ii}^2}-t_1^{\alpha_i-\gamma_{ii}^2}}{\Gamma(\alpha_i-\gamma_{ii}^2+1)}+\frac{(t_2^{2-\gamma_{ii}^2}-t_1^{2-\gamma_{ii}^2})\Gamma(3-\beta_{2i})}{2\Gamma(\alpha_i-\beta_{2i}+1)\Gamma(3-\gamma_{ii}^2)}\Big]$$

for all $1 \leq i \leq k$. This implies that $\lim_{t_2\to t_1} |(h_1(t_2) - h_1(t_1), \cdots, h_k(t_2) - h_k(t_1))| = 0$, $\lim_{t_2\to t_1} |({}^cD^{\gamma^1_{ii}} h_1(t_2) - {}^cD^{\gamma^1_{ii}} h_1(t_1), \cdots, {}^cD^{\gamma^1_{ii}} h_k(t_2) - {}^cD^{\gamma^1_{ii}} h_k(t_1))| = 0$ and

$$\lim_{t_2\to t_1} |({}^cD^{\gamma^2_{ii}} h_1(t_2) - {}^cD^{\gamma^2_{ii}} h_1(t_1), \cdots, {}^cD^{\gamma^2_{ii}} h_k(t_2) - {}^cD^{\gamma^2_{ii}} h_k(t_1))| = 0.$$

Hence by using the Arzela-Ascoli theorem for each bounded subset B of $\mathcal{X}$, $N(B)$ is relatively compact. Thus, N is completely continuous. Now, we show that N has a closed graph. Let $(u_1^n, \cdots, u_k^n) \in \mathcal{X}$ and $(h_1^n, \cdots, h_k^n) \in N(u_1^n, \cdots, u_k^n)$ $(n \geq 1)$ with $(u_1^n, \cdots, u_k^n) \to (u_1^0, \cdots, u_k^0)$ and $(h_1^n, \cdots, h_k^n) \to (h_1^0, \cdots, h_k^0)$. We show that $(h_1^0, \cdots, h_k^0) \in N(u_1^0, \cdots, u_k^0)$. For each natural number n, choose $(v_1^n, \cdots, v_k^n) \in S_{F_1,(u_1^n,\cdots u_k^n)} \times \cdots \times S_{F_k,(u_1^n,\cdots,u_k^n)}$ such that

$$h_i^n(t) = \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i - 1} v_i^n(s) ds + \int_0^1 \Big[\frac{-1}{2\Gamma(\alpha_i)} (1-s)^{\alpha_i - 1}$$
$$+ \frac{(1-2t)\Gamma(2-\beta_{1i})}{4\Gamma(\alpha_i - \beta_{1i})} (1-s)^{\alpha_i - \beta_{1i} - 1} + \frac{1 - \beta_{1i} + 2t - 2(2-\beta_{1i})t^2}{8(2-\beta_{1i})} \frac{\Gamma(3-\beta_{2i})}{\Gamma(\alpha_i - \beta_{2i})} (1-s)^{\alpha_i - \beta_{2i} - 1} \Big] v_i^n(s) ds$$

for all $t \in J$ and $1 \leq i \leq k$. Define the continuous linear operator $\theta_i : L^1(J, \mathbb{R}) \to \mathcal{X}_i$ by

$$\theta_i(v)(t) = \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i - 1} v(s) ds + \int_0^1 \Big[\frac{-1}{2\Gamma(\alpha_i)} (1-s)^{\alpha_i - 1}$$
$$+ \frac{(1-2t)\Gamma(2-\beta_{1i})}{4\Gamma(\alpha_i - \beta_{1i})} (1-s)^{\alpha_i - \beta_{1i} - 1} + \frac{1 - \beta_{1i} + 2t - 2(2-\beta_{1i})t^2}{8(2-\beta_{1i})} \frac{\Gamma(3-\beta_{2i})}{\Gamma(\alpha_i - \beta_{2i})} (1-s)^{\alpha_i - \beta_{2i} - 1} \Big] v(s) ds.$$

By using Lemma 2.3, $\theta_i o S_{F_i}$ is a closed graph operator. Since $h_i^n \in \theta_i(S_{F_i,(u_1^n,\cdots,u_k^n)})$ for all n, $1 \leq i \leq k$ and $(u_1^n, \cdots, u_k^n) \to (u_1^0, \cdots, u_k^0)$, there exists $v_i^0 \in S_{F_i,(u_1^0,\cdots,u_k^0)}$ such that

$$h_i^0(t) = \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i - 1} v_i^0(s) ds + \int_0^1 \Big[\frac{-1}{2\Gamma(\alpha_i)} (1-s)^{\alpha_i - 1}$$
$$+ \frac{(1-2t)\Gamma(2-\beta_{1i})}{4\Gamma(\alpha_i - \beta_{1i})} (1-s)^{\alpha_i - \beta_{1i} - 1} + \frac{1 - \beta_{1i} + 2t - 2(2-\beta_{1i})t^2}{8(2-\beta_{1i})}$$
$$\frac{\Gamma(3-\beta_{2i})}{\Gamma(\alpha_i - \beta_{2i})} (1-s)^{\alpha_i - \beta_{2i} - 1} \Big] v_i^0(s) ds.$$

Hence, $h_i^0 \in N_i(u_1^0, \cdots, u_k^0)$ for all $1 \leq i \leq k$. This implies that N_i has a closed graph for all $1 \leq i \leq k$ and so N has a closed graph. Now, suppose that there exists $\lambda \in (0,1)$ such that $(u_1, \cdots, u_k) \in \lambda N(u_1, \cdots, u_k)$. Then there exists

$$(v_1, \cdots, v_k) \in S_{F_1,(u_1,\cdots,u_k)} \times \cdots \times S_{F_k,(u_1,\cdots,u_k)}$$

such that

$$u_i(t) = \frac{\lambda}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i - 1} v_i(s) ds + \int_0^1 \lambda \Big[\frac{-1}{2\Gamma(\alpha_i)} (1-s)^{\alpha_i - 1}$$
$$+ \frac{(1-2t)\Gamma(2-\beta_{1i})}{4\Gamma(\alpha_i - \beta_{1i})} (1-s)^{\alpha_i - \beta_{1i} - 1} + \frac{1 - \beta_{1i} + 2t - 2(2-\beta_{1i})t^2}{8(2-\beta_{1i})}$$
$$\frac{\Gamma(3-\beta_{2i})}{\Gamma(\alpha_i - \beta_{2i})} (1-s)^{\alpha_i - \beta_{2i} - 1} \Big] v_i(s) ds$$

for all $t \in J$ and $1 \leq i \leq k$. Since $\frac{\|u\|_i}{(\Lambda_1^i + \Lambda_2^i + \Lambda_3^i)\|p_i\|_\infty \psi(\|(u_1, u_2, \ldots, u_k)\|)} \leq 1$, $\|u_i\|_i < L_i$ for all $i = 1, \cdots, k$. Now, put $U = \{(u_1, \cdots, u_k) \in \mathcal{X} : \|(u_1, \cdots, u_k)\| < \sum_{i=1}^k L_i + 1\}$. Thus, there are no $(u_1, \cdots, u_k) \in \partial U$ and $\lambda \in (0,1)$ such that $(u_1, \cdots, u_k) \in \lambda N(u_1, \cdots, u_k)$. Note that, the operator $N : \overline{U} \to P_{cp,cv}(\overline{U})$ is upper semi-continuous because it is completely continuous and has closed graph. By considering definition of U, there is no $(u_1,...,u_k) \in \partial U$ such that $(u_1,...,u_k) \in \lambda N(u_1,...,u_k)$ for some $\lambda \in (0,1)$. Now by using Lemma 2.4, N has a fixed point in $\overline{U}$. It is easy to check that each fixed point of N is a solution of the k-dimensional system of inclusions (1.2). This completes the proof. □

Example 3.1. *Consider the 2-dimensional system of fractional differential inclusions*

$$\begin{cases} {}^cD^{\frac{5}{2}}u_1(t) \in F_1(t,u_1(t),u_2(t),{}^cD^{\frac{1}{2}}u_1(t),{}^cD^{\frac{1}{3}}u_2(t),{}^cD^{\frac{5}{4}}u_1(t),{}^cD^{\frac{3}{2}}u_2(t)), \\ {}^cD^{\frac{8}{3}}u_2(t) \in F_2(t,u_1(t),u_2(t),{}^cD^{\frac{1}{4}}u_1(t),{}^cD^{\frac{3}{4}}u_2(t),{}^cD^{\frac{5}{4}}u_1(t),{}^cD^{\frac{7}{4}}u_2(t)), \end{cases} \tag{3.1}$$

with the boundary value conditions $t^{\frac{-1}{2}}\,{}^cD^{\frac{1}{2}}u_1(t)|_{t\to 0} = -t^{\frac{-1}{2}}\,{}^cD^{\frac{1}{2}}u_1(t)|_{t=1}$, $u_1(0) = -u_1(1)$, $u_2(0) = -u_2(1)$, $t^{\frac{-2}{3}}\,{}^cD^{\frac{1}{3}}u_2(t)|_{t\to 0} = -t^{\frac{-2}{3}}\,{}^cD^{\frac{1}{3}}u_2(t)|_{t=1}$, $t^{\frac{-3}{4}}\,{}^cD^{\frac{5}{4}}u_1(t)|_{t\to 0} = -t^{\frac{-3}{4}}\,{}^cD^{\frac{5}{4}}u_1(t)|_{t=1}$ *and* $t^{\frac{-1}{2}}\,{}^cD^{\frac{3}{2}}u_2(t)|_{t\to 0} = -t^{\frac{-1}{2}}\,{}^cD^{\frac{3}{2}}u_2(t)|_{t=1}$. *In fact,* $k = 2$, $\alpha_1 = \frac{5}{2}$, $\alpha_2 = \frac{8}{3}$, $\gamma_{11}^1 = \frac{1}{2}$, $\gamma_{12}^1 = \frac{1}{3}$, $\gamma_{11}^2 = \frac{5}{4}$, $\gamma_{12}^2 = \frac{3}{2}$, $\gamma_{21}^1 = \frac{1}{4}$, $\gamma_{22}^1 = \frac{3}{4}$, $\gamma_{21}^2 = \frac{5}{4}$, $\gamma_{22}^2 = \frac{7}{4}$, $\beta_{11} = \frac{1}{2}$, $\beta_{12} = \frac{1}{3}$, $\beta_{21} = \frac{5}{4}$, $\beta_{22} = \frac{3}{2}$ *and the multifunction* $F_1, F_2 : J \times \mathbb{R}^6 \to 2^{\mathbb{R}}$ *are given by*

$$F_1(t,x_1,...,x_6) = \left[\sin t + \frac{|x_1|}{1+|x_1|} + \cos x_2, 4 + 3t^2 + \frac{1}{1+e^{|x_3|}} + \sin x_4 + \frac{|x_6|^3}{|x_6|^3+2}\right]$$

and $F_2(t,x_1,...,x_6) = \left[0, \frac{e^t}{\pi}\left(\frac{|x_1|+|x_2|+|x_3|+|x_4|+|x_6|}{1+|x_1|+|x_2|+|x_3|+|x_4|+|x_6|}\right) + \sin x_5\right]$. *Thus,*

$$\|F_1(t,x_1,...,x_6)\| = \sup\{|y| : y \in F_1(t,x_1,...,x_6)\} \le 10$$

and $\|F_2(t,x_1,...,x_6)\| = \sup\{|y| : y \in F_2(t,x_1,...,x_6)\} \le 2$. *We show that* F_1 *and* F_2 *are Caratheodory multifunctions. In this way, consider the continuous maps*

$$f_1(t) = \sin t + \frac{|x_1|}{1+|x_1|} + \cos x_2,$$

$g_1(t) = 4 + 3t^2 + \frac{1}{1+e^{|x_3|}} + \sin x_4 + \frac{|x_6|^3}{|x_6|^3+2}$, $f_2(t) = 0$ *and*

$$g_2(t) = \frac{e^t}{\pi}\left(\frac{|x_1|+|x_2|+|x_3|+|x_4|+|x_6|}{1+|x_1|+|x_2|+|x_3|+|x_4|+|x_6|}\right) + \sin x_5$$

for all t. *Since the maps* f_1, g_1, f_2 *and* g_2 *are continuous, it is easy to check that the maps* $t \mapsto d(y, F_1(t,x_1,...,x_6))$ *and* $t \mapsto d(y, F_2(t,x_1,...,x_6))$ *are continuous and so measurable ones for all* $y \in \mathbb{R}$. *Also, one can check that* $t \mapsto F_1(t,x_1,...,x_6)$ *and* $t \mapsto F_2(t,x_1,...,x_6)$ *are completely continuous multifunctions via compact values and closed graph for all* $t \in J$. *Now by using Lemma 2.5, we get* $t \mapsto F_1(t,x_1,...,x_6)$ *and* $t \mapsto F_2(t,x_1,...,x_6)$ *are upper semi-continuous multifunctions for all* $x_1,...,x_6 \in \mathbb{R}$. *Thus,* F_1 *and* F_2 *are Caratheodory multifunctions. Now, put* $p_1(t) = 5$, $p_2(t) = 1$ *and* $\psi(t) = 2$. *If* $L_1 > 55.37$ *and* $L_2 > 10.046$, *then we get* $\frac{L_1}{(\Lambda_1^1+\Lambda_2^1+\Lambda_3^1)\|p_1\|_\infty \psi(\|(u_1,u_2)\|)} > 1$ *and* $\frac{L_2}{(\Lambda_1^2+\Lambda_2^2+\Lambda_3^2)\|p_2\|_\infty \psi(\|(u_1,u_2)\|)} > 1$. *Thus, the assumptions of Theorem 3.1 hold and so the 2-dimensional system of fractional differential inclusions* (3.1) *has at least one solution.*

3.2. The Lipschitz case

Now, we provide our next result about the existence of solution for the k-dimensional system of inclusions (1.2) by deleting the assumption of convex values for the multifunctions.

Theorem 3.2. *Let* $m_1, \cdots, m_k \in C(J, \mathbb{R}^+)$ *be such that* $L = \sum_{i=1}^k \|m_i\|_\infty(\Lambda_1^i + \Lambda_2^i + \Lambda_3^i) < 1$, *where* $\Lambda_1^i = \frac{3}{\Gamma(\alpha_i+1)} + \frac{3\Gamma(2-\beta_{1i})}{4\Gamma(\alpha_i-\beta_{1i}+1)} + \frac{5\Gamma(3-\beta_{2i})}{4\Gamma(\alpha_i-\beta_{2i}+1)}$,

$$\Lambda_2^i = \frac{1}{\Gamma(\alpha_i - \gamma_{ii}^1 + 1)} + \frac{\Gamma(2-\beta_{1i})}{2\Gamma(\alpha_i - \beta_{1i} + 1)\Gamma(2-\gamma_{ii}^1)} + \frac{\Gamma(3-\beta_{2i})}{4\Gamma(\alpha_i - \beta_{2i} + 1)\Gamma(2-\gamma_{ii}^1)(2-\beta_{1i})}$$

$$+\frac{\Gamma(3-\beta_{2i})}{2\Gamma(\alpha_i-\beta_{2i}+1)\Gamma(3-\gamma_{ii}^1)}$$

and $\Lambda_3^i = \frac{1}{\Gamma(\alpha_i-\gamma_{ii}^2+1)} + \frac{\Gamma(3-\beta_{2i})}{2\Gamma(\alpha_i-\beta_{2i}+1)\Gamma(3-\gamma_{ii}^2)}$ *for* $i = 1, \cdots, k$. *Suppose that* $F_i : J \times \mathbb{R}^{3k} \to P_{cp}(\mathbb{R})$ *is a multifunction such that the map* $t \vdash F_i(t, x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k)$ *is integrable bounded, measurable and*

$$H_d(F_i(t, u_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k), F_i(t, x'_1, \cdots, x'_k, y'_1, \cdots, y'_k, z'_1, \cdots, z'_k))$$

$$\leq m_i(t)\Big(\sum_{i=1}^{k} |x_i - x'_i| + |y_i - y'_i| + |z_i - z'_i|\Big)$$

for almost all $t \in J$, $x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k, x'_1, \cdots, x'_k, y'_1, \cdots, y'_k, z'_1, \cdots, z'_k, \in \mathbb{R}$ *and* $i = 1, \cdots, k$. *Then the k-dimensional system of inclusions with the boundary value conditions* (1.2) *has a solution.*

Proof. Note that, the multifunction

$$t \to F_i(t, u_1(t), \cdots, u_k(t), {}^cD^{\gamma_{i1}^1}u_1(t), \cdots, {}^cD^{\gamma_{ik}^1}u_k(t), {}^cD^{\gamma_{i1}^2}u_1(t), \cdots, {}^cD^{\gamma_{i1}^2}u_k(t))$$

is measurable and closed valued for all $(u_1, \cdots, u_k) \in \mathcal{X}$ and $i = 1, \cdots, k$. Hence, it has measurable selection and so the set $S_{F_i,(u_1,\cdots,u_k)}$ is nonempty for all $i = 1, \cdots, k$. Again, consider the operator $N : \mathcal{X} \to 2^{\mathcal{X}}$ defined by $N(u_1, \cdots, u_k) = \begin{pmatrix} N_1(u_1, \cdots, u_k) \\ N_2(u_1, \cdots, u_k) \\ \vdots \\ N_k(u_1, \cdots, u_k) \end{pmatrix}$, where

$$N_i(u_1, \cdots, u_k) = \Big\{h \in \mathcal{X}_i : \text{ there exists } v \in S_{F_{i,(u_1,\cdots,u_k)}} \text{ such that}$$

$$h(t) = \frac{1}{\Gamma(\alpha_i)}\int_0^t (t-s)^{\alpha_i-1}v(s)ds + \int_0^1 \Big[\frac{-1}{2\Gamma(\alpha_i)}(1-s)^{\alpha_i-1} + \frac{(1-2t)\Gamma(2-\beta_{1i})}{4\Gamma(\alpha_i-\beta_{1i})}(1-s)^{\alpha_i-\beta_{1i}-1}$$
$$+\frac{1-\beta_{1i}+2t-2(2-\beta_{1i})t^2}{8(2-\beta_{1i})}\frac{\Gamma(3-\beta_{2i})}{\Gamma(\alpha_i-\beta_{2i})}(1-s)^{\alpha_i-\beta_{2i}-1}\Big]v(s)ds, \Big\}.$$

First, we show that $N(u_1, \cdots, u_k)$ is a closed subset of $\mathcal{X}$ for all $(u_1, \cdots, u_k) \in \mathcal{X}$. Let $\{(u_1^n, \cdots, u_k^n)\}_{n\geq 1}$ be a sequence in $N(u_1, \cdots, u_k)$ such that $(u_1^n, \cdots, u_k^n) \to (u_1^0, \cdots, u_k^0)$. Choose $(v_1^n, \cdots, v_k^n) \in S_{F_1,(u_1,\cdots,u_k)} \times S_{F_2,(u_1,\cdots,u_k)} \times \cdots \times S_{F_k,(u_1,\cdots,u_k)}$ such that

$$u_i^n(t) = \frac{1}{\Gamma(\alpha_i)}\int_0^t (t-s)^{\alpha_i-1}v_i^n(s)ds + \int_0^1 \Big[\frac{-1}{2\Gamma(\alpha_i)}(1-s)^{\alpha_i-1}$$
$$+\frac{(1-2t)\Gamma(2-\beta_{1i})}{4\Gamma(\alpha_i-\beta_{1i})}(1-s)^{\alpha_i-\beta_{1i}-1} + \frac{1-\beta_{1i}+2t-2(2-\beta_{1i})t^2}{8(2-\beta_{1i})}$$
$$\frac{\Gamma(3-\beta_{2i})}{\Gamma(\alpha_i-\beta_{2i})}(1-s)^{\alpha_i-\beta_{2i}-1}\Big]v_i^n(s)ds$$

for all $t \in J$ and $i = 1, \cdots, k$. Since F_i is compact valued for all i, $\{v_i^n\}_{n\geq 1}$ has a subsequence which converges to some $v_i^0 \in L^1(J, \mathbb{R})$. Denote the subsequence again by $\{v_i^n\}_{n\geq 1}$. It is easy to check that $v_i^0 \in S_{F_i,(u_1,\cdots,u_k)}$ and

$$u_i^n(t) \to u_i^0(t) = \frac{1}{\Gamma(\alpha_i)}\int_0^t (t-s)^{\alpha_i-1}v_i^0(s)ds + \int_0^1 \Big[\frac{-1}{2\Gamma(\alpha_i)}(1-s)^{\alpha_i-1}$$
$$+\frac{(1-2t)\Gamma(2-\beta_{1i})}{4\Gamma(\alpha_i-\beta_{1i})}(1-s)^{\alpha_i-\beta_{1i}-1} + \frac{1-\beta_{1i}+2t-2(2-\beta_{1i})t^2}{8(2-\beta_{1i})}\frac{\Gamma(3-\beta_{2i})}{\Gamma(\alpha_i-\beta_{2i})}(1-s)^{\alpha_i-\beta_{2i}-1}\Big]v_i^0(s)ds$$

for all $t \in J$. This implies that $u_i^0 \in N_i(u_1, \cdots, u_k)$ for all $i = 1, \cdots, k$. This implies that $(u_1^0, \cdots, u_k^0) \in N(u_1, \cdots, u_k)$. Now, we show that N is a contractive multifunction with the constant $L < 1$, where $L = \sum_{i=1}^k \|m_i\|_\infty(\Lambda_1^i + \Lambda_2^i + \Lambda_3^i) < 1$. Let $(x_1, \cdots, x_k), (y_1, \cdots, y_k) \in \mathcal{X}$ and $(h_1, \cdots, h_k) \in N(y_1, \cdots, y_k)$ be given. Then, we can choose

$$(v_1, \cdots, v_k) \in S_{F_1,(y_1,\cdots,y_k)} \times S_{F_2,(y_1,\cdots,y_k)} \times \cdots \times S_{F_k,(y_1,\cdots,y_k)}$$

such that

$$h_i(t) = \frac{1}{\Gamma(\alpha_i)}\int_0^t (t-s)^{\alpha_i-1}v_i(s)ds + \int_0^1 \Big[\frac{-1}{2\Gamma(\alpha_i)}(1-s)^{\alpha_i-1} + \frac{(1-2t)\Gamma(2-\beta_{1i})}{4\Gamma(\alpha_i-\beta_{1i})}(1-s)^{\alpha_i-\beta_{1i}-1}$$
$$+\frac{1-\beta_{1i}+2t-2(2-\beta_{1i})t^2}{8(2-\beta_{1i})}\frac{\Gamma(3-\beta_{2i})}{\Gamma(\alpha_i-\beta_{2i})}(1-s)^{\alpha_i-\beta_{2i}-1}\Big]v_i(s)ds$$

for all $t \in J$ and $i = 1, \cdots, k$. Since

$$H_d\Big(F_i(t, x_1(t), \cdots, x_k(t), {}^cD^{\gamma_{i1}^1}x_1(t), \cdots, {}^cD^{\gamma_{ik}^1}x_k(t), {}^cD^{\gamma_{i1}^2}x_1(t), \cdots, {}^cD^{\gamma_{ik}^2}x_k(t)),$$

$$F_i(t, y_1(t), \cdots, y_k(t), {}^cD^{\gamma_{i1}^1}y_1(t), \cdots, {}^cD^{\gamma_{ik}^1}y_k(t), {}^cD^{\gamma_{i1}^2}y_1(t), \cdots, {}^cD^{\gamma_{ik}^2}y_k(t))\Big)$$

$$\le m_i(t)\sum_{i=1}^k(|x_i(t)-y_i(t)| + |{}^cD^{\gamma_{ii}^1}x_i(t) - {}^cD^{\gamma_{ii}^1}y_i(t)| + |{}^cD^{\gamma_{ii}^2}x_i(t) - {}^cD^{\gamma_{ii}^2}y_i(t)|)$$

for almost all $t \in J$ and $i = 1, \cdots, k$, there exists

$$w_i \in F_i(t, x_1(t), \cdots, x_k(t), {}^cD^{\gamma_{i1}^1}x_1(t), \cdots, {}^cD^{\gamma_{ik}^1}x_k(t), {}^cD^{\gamma_{i1}^2}x_1(t), \cdots, {}^cD^{\gamma_{ik}^2}x_k(t))$$

such that

$$|v_i(t) - w_i| \le m_i(t)\sum_{i=1}^k\Big(|x_i(t)-y_i(t)| + |{}^cD^{\gamma_{ii}^1}x_i(t) - {}^cD^{\gamma_{ii}^1}y_i(t)| + |{}^cD^{\gamma_{ii}^2}x_i(t) - {}^cD^{\gamma_{ii}^2}y_i(t)|\Big)$$

for almost all $t \in J$ and $i = 1, \cdots, k$. Consider the multifunction $U_i : J \to 2^{\mathbb{R}}$ by

$$U_i(t) = \{w \in \mathbb{R} : |v_i(t) - w| \le m_i(t)g(t) \text{ for almost all } t \in J\},$$

where $g(t) = \sum_{i=1}^k(|x_i(t)-y_i(t)| + |{}^cD^{\gamma_{ii}^1}x_i(t) - {}^cD^{\gamma_{ii}^1}y_i(t)| + |{}^cD^{\gamma_{ii}^2}x_i(t) - {}^cD^{\gamma_{ii}^2}y_i(t)|$. Since v_i and $\varphi_i = m_i\sum_{i=1}^k\Big(|x_i - y_i| + |{}^cD^{\gamma_{ii}^1}x_i - {}^cD^{\gamma_{ii}^1}y_i| + |{}^cD^{\gamma_{ii}^2}x_i - {}^cD^{\gamma_{ii}^2}y_i|\Big)$ are measurable for all i, $U_i(.) \cap F_i(t, x_1(.), \cdots, x_k(.), {}^cD^{\gamma_{i1}^1}x_1(.), \cdots, {}^cD^{\gamma_{ik}^1}x_k(.), {}^cD^{\gamma_{i1}^2}x_1(.), ..., {}^cD^{\gamma_{ik}^2}x_k$ is a measurable multifunction. Thus, we can choose

$$v_i'(t) \in F_i(t, x_1(t), \cdots, x_k(t), {}^cD^{\gamma_{i1}^1}x_1(t), \cdots, {}^cD^{\gamma_{ik}^1}x_k(t), {}^cD^{\gamma_{i1}^2}x_1(t), \cdots, {}^cD^{\gamma_{ik}^2}x_k(t))$$

such that

$$h_i'(t) = \frac{1}{\Gamma(\alpha_i)}\int_0^t (t-s)^{\alpha_i-1}v_i'(s)ds + \int_0^1 \Big[\frac{-1}{2\Gamma(\alpha_i)}(1-s)^{\alpha_i-1} + \frac{(1-2t)\Gamma(2-\beta_{1i})}{4\Gamma(\alpha_i-\beta_{1i})}(1-s)^{\alpha_i-\beta_{1i}-1}$$
$$+\frac{1-\beta_{1i}+2t-2(2-\beta_{1i})t^2}{8(2-\beta_{1i})}\frac{\Gamma(3-\beta_{2i})}{\Gamma(\alpha_i-\beta_{2i})}(1-s)^{\alpha_i-\beta_{2i}-1}\Big]v_i'(s)ds$$

for all $t \in J$ and $i = 1, \cdots, k$. Since

$$|h_i(t) - h_i'(t)| \le \frac{1}{\Gamma(\alpha_i)}\int_0^t (t-s)^{\alpha_i-1}|v_i(s) - v_i'(s)|ds$$

$$+\int_0^1 \left\|\left[\frac{-1}{2\Gamma(\alpha_i)}(1-s)^{\alpha_i-1}\frac{(1-2t)\Gamma(2-\beta_{1i})}{4\Gamma(\alpha_i-\beta_{1i})}(1-s)^{\alpha_i-\beta_{1i}-1}+\frac{1-\beta_{1i}+2t-2(2-\beta_{1i})t^2}{8(2-\beta_{1i})}\right.\right.$$
$$\left.\left.\frac{\Gamma(3-\beta_{2i})}{\Gamma(\alpha_i-\beta_{2i})}(1-s)^{\alpha_i-\beta_{2i}-1}\right]\right\||v_i(s)-v_i'(s)|ds \le \Lambda_1^i\|m_i\|_\infty\|(x_1-y_1,\cdots,x_k-y_k)\|,$$

$$|{}^cD^{\gamma_{ii}^1}h_i(t)-{}^cD^{\gamma_{ii}^1}h_i'(t)| \le \frac{1}{\Gamma(\alpha_i-\gamma_{ii}^1)}\int_0^t (t-s)^{\alpha_i-\gamma_{ii}^1-1}|v_i(s)-v_i'(s)|ds$$
$$+\frac{t^{1-\gamma_{ii}^1}\Gamma(2-\beta_{1i})}{2\Gamma(2-\gamma_{ii}^1)\Gamma(\alpha_i-\beta_{1i})}\int_0^1(1-s)^{\alpha_i-\beta_{1i}-1}|v_i(s)-v_i'(s)|ds$$
$$+\frac{t^{1-\gamma_{ii}^1}\Gamma(3-\beta_{2i})}{4\Gamma(2-\gamma_{ii}^1)\Gamma(\alpha_i-\beta_{1i})(2-\beta_{1i})}\int_0^1(1-s)^{\alpha_i-\beta_{2i}-1}|v_i(s)-v_i'(s)|ds$$
$$+\frac{t^{2-\gamma_{ii}^1}\Gamma(3-\beta_{2i})}{2\Gamma(3-\gamma_{ii}^1)\Gamma(\alpha_i-\beta_{2i})}\int_0^1(1-s)^{\alpha_i-\beta_{2i}-1}|v_i(s)-v_i'(s)|ds$$
$$\le \|m_i\|_\infty\|(x_1-y_1,\cdots,x_k-y_k)\|\Lambda_2^i$$

and

$$|{}^cD^{\gamma_{ii}^2}h_i(t)-{}^cD^{\gamma_{ii}^2}h_i'(t)| \le \frac{1}{\Gamma(\alpha_i-\gamma_{ii}^2)}\int_0^t (t-s)^{\alpha_i-\gamma_{ii}^2-1}|v_i(s)-v_i'(s)|ds$$
$$+\frac{t^{2-\gamma_{ii}^2}\Gamma(3-\beta_{2i})}{2\Gamma(3-\gamma_{ii}^2)\Gamma(\alpha_i-\beta_{2i})}\int_0^1(1-s)^{\alpha_i-\beta_{2i}-1}|v_i(s)-v_i'(s)|ds$$
$$\le \|m_i\|_\infty\|(x_1-y_1,\cdots,x_k-y_k)\|\Lambda_3^i,$$

we get $\|h_i-h_i'\|_i \le (\Lambda_1^i+\Lambda_2^i+\Lambda_3^i)\|m_i\|_\infty\|(x_1-y_1,\cdots,x_k-y_k)\|$ for all $i=1,\cdots,k$. Hence,

$$\|(h_1,\cdots,h_k)-(h_1',\cdots,h_k')\| = \sum_{i=1}^k \|h_i-h_i'\|_i$$
$$\le \sum_{i=1}^k(\Lambda_1^i+\Lambda_2^i+\Lambda_3^i)\|m_i\|_\infty\|(x_1-y_1,\cdots,x_k-y_k)\|$$
$$\le L\|(x_1,\cdots,x_k)-(y_1,\cdots,y_k)\|.$$

This implies that $H(N(x_1,\cdots,x_k),N(y_1,\cdots,y_k)) \le L\|(x_1,\cdots,x_k)-(y_1,\cdots,y_k)\|$ and so N is a closed valued contractive multifunction. By using Lemma 2.6, N has a fixed point. One can check that each fixed point of N is a solution for the k-dimensional system of inclusions with the boundary value conditions (1.2). □

Example 3.2. *Consider the 2-dimensional system of fractional differential inclusions*

$$\begin{cases} {}^cD^{\frac{7}{3}}u_1(t) \in F_1(t,u_1(t),u_2(t),{}^cD^{\frac{1}{3}}u_1(t),{}^cD^{\frac{1}{2}}u_2(t),{}^cD^{\frac{5}{3}}u_1(t),{}^cD^{\frac{5}{4}}u_2(t)), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.2) \\ {}^cD^{\frac{13}{6}}u_2(t) \in F_2(t,u_1(t),u_2(t),{}^cD^{\frac{1}{3}}u_1(t),{}^cD^{\frac{6}{7}}u_2(t),{}^cD^{\frac{7}{6}}u_1(t),{}^cD^{\frac{7}{4}}u_2(t)), \end{cases}$$

with the boundary conditions $t^{\frac{-1}{2}}\,{}^cD^{\frac{1}{2}}u_1(t)|_{t\to 0} = -t^{\frac{-1}{2}}\,{}^cD^{\frac{1}{2}}u_1(t)|_{t=1}$, $u_1(0)=-u_1(1)$, $u_2(0)=-u_2(1)$, $t^{\frac{-3}{4}}\,{}^cD^{\frac{1}{4}}u_2(t)|_{t\to 0} = -t^{\frac{-3}{4}}\,{}^cD^{\frac{1}{4}}u_2(t)|_{t=1}$, $t^{\frac{-1}{2}}\,{}^cD^{\frac{3}{2}}u_1(t)|_{t\to 0} = -t^{\frac{-1}{2}}\,{}^cD^{\frac{3}{2}}u_1(t)|_{t=1}$ *and*

$$t^{\frac{-3}{5}}\,{}^cD^{\frac{7}{5}}u_2(t)|_{t\to 0} = -t^{\frac{-3}{5}}\,{}^cD^{\frac{7}{5}}u_2(t)|_{t=1}.$$

In fact, $k = 2$, $\alpha_1 = \frac{7}{3}$, $\alpha_2 = \frac{13}{6}$, $\gamma_{11}^1 = \frac{1}{3}$,$\gamma_{12}^1 = \frac{1}{2}$, $\gamma_{11}^2 = \frac{5}{3}$, $\gamma_{12}^2 = \frac{5}{4}$, $\gamma_{21}^1 = \frac{1}{3}$, $\gamma_{22}^1 = \frac{6}{7}$, $\gamma_{21}^2 = \frac{7}{6}$, $\gamma_{22}^2 = \frac{7}{4}$, $\beta_{11} = \frac{1}{2}$, $\beta_{12} = \frac{1}{4}$, $\beta_{21} = \frac{3}{2}$, $\beta_{22} = \frac{7}{5}$ *and the multifunctions* $F_1, F_2 : J \times \mathbb{R}^6 \to 2^{\mathbb{R}}$ *are given by* $F_1(t, x_1, ..., x_6) = \left[0, \frac{t \sin x_1}{9\pi(2+t^2)} + \frac{t|x_2|}{75(1+|x_2|)} + \frac{|x_3|}{80(1+|x_3|)} + \frac{t \cos x_4}{10e^2} + \frac{t|x_5|}{70(1+|x_5|)} + \frac{t^2|x_6|}{75(1+|x_6|)}\right]$ *and*

$$F_2(t, x_1, ..., x_6) = \left[0, \frac{\cos t(|x_1| + |x_2| + |x_3| + |x_4|)}{25\sqrt{\pi}(1 + |x_1| + |x_2| + |x_3| + |x_4|)} + \frac{|x_5|}{45(1 + |x_5|)} + \frac{t^2 \sin x_6}{30(t^2 + 1)}\right].$$

Since $\|F_1(t, x_1, ..., x_6)\| = \sup\{|y| : y \in F_1(t, x_1, ..., x_6)\} \le \frac{t}{9\pi(2+t^2)} + \frac{t}{75} + \frac{1}{80} + \frac{t}{10e^2} + \frac{t}{70} + \frac{t^2}{75}$ *and* $\|F_2(t, x_1, ..., x_6)\| = \sup\{|y| : y \in F_2(t, x_1, ..., x_6)\} \le \frac{\cos t}{25\sqrt{\pi}} + \frac{1}{45} + \frac{t^2}{30(t^2+1)}$ *for all* $x_1, \cdots, x_6 \in \mathbb{R}$ *and* $t \in J$*, we get* $t \mapsto F_1(t, x_1, ..., x_6)$ *and* $t \mapsto F_2(t, x_1, ..., x_6)$ *are are integrable bounded multifunctions for all* $x_1, \cdots, x_5 \in \mathbb{R}$*. By using a similar argument in Example 3.1, we can show that* $t \mapsto F_1(t, x_1, ..., x_6)$ *and* $t \mapsto F_2(t, x_1, ..., x_6)$ *are measurable. On the other hand, it is easy to see that*

$$H\Big(F_1(t, x_1, ..., x_6), F_1(t, y_1, ..., y_6)\Big) \le \left(\frac{t}{9\pi(2 + t^2)} + \frac{t}{75} + \frac{1}{80} + \frac{t}{10e^2} + \frac{t}{70} + \frac{t^2}{75}\right)\sum_{i=1}^{6} |x_i - y_i|$$

and $H\Big(F_2(t, x_1, ..., x_6), F_2(t, y_1, ..., y_6)\Big) \le \left(\frac{\cos t}{25\sqrt{\pi}} + \frac{1}{45} + \frac{t^2}{30(t^2+1)}\right)\sum_{i=1}^{6} |x_i - y_i|$ *for all* $t \in J$ *and* $x_1, .., x_6, y_1, .., y_6 \in \mathbb{R}$*. Now, put*

$$m_1(t) = \frac{t}{9\pi(2 + t_2)} + \frac{t}{75} + \frac{1}{80} + \frac{t}{10e^2} + \frac{t}{70} + \frac{t^2}{75}$$

and $m_2(t) = \frac{\cos t}{25\sqrt{\pi}} + \frac{1}{45} + \frac{t^2}{30(t^2+1)}$ *for all* $t \in J$*. Then, we have*

$$H\Big(F_1(t, x_1, ..., x_6), F_1(t, y_1, ..., y_6)\Big) \le m_1(t)\sum_{i=1}^{6} |x_i - y_i|$$

and $H\Big(F_2(t, x_1, ..., x_6), F_2(t, y_1, ..., y_6)\Big) \le m_2(t)\sum_{i=1}^{6} |x_i - y_i|$ *for all* $x_1, .., x_6, y_1, .., y_6 \in \mathbb{R}$ *and* $t \in J$*. On the other hand, we have*

$$L = \sum_{i=1}^{2} \|m_i\|_\infty(\Lambda_1^i + \Lambda_2^i + \Lambda_3^i)$$
$$\le (0.09 \times (2.654 + 1.74 + 1.63)) + (0.06 \times (1.962 + 1.15 + 1.46)) \simeq 0.8178 < 1.$$

Thus, the assumptions of Theorem 3.2 hold and so the 2-dimensional system (3.2) *has at least one solution.*

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