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On Equitorsion Concircular Tensors of Generalized Riemannian Spaces

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Abstract. In this paper we consider concircular vector fields of manifolds with non-symmetric metric tensor. The subject of our paper is an equitorsion concircular mapping. A mapping $f : \mathbb{GR}_N \to \mathbb{G}\overline{\mathbb{R}}_N$ is an equitorsion if the torsion tensors of the spaces $\mathbb{GR}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$ are equal.

For an equitorsion concircular mapping of two generalized Riemannian spaces $\mathbb{GR}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$, we obtain some invariant curvature tensors of this mapping $\underset{\theta}{Z}$, $\theta = 1, 2, \ldots, 5$, given by equations (3.14, 3.21, 3.28, 3.31, 3.38). These quantities are generalizations of the concircular tensor Z given by equation (2.5).

1. Introduction

The use of non-symmetric basic tensors and non-symmetric connection became especially actual after appearance of the works of A. Einstein [2]-[4] related to the Unified Field Theory (UFT). Remark that in the UFT the symmetric part $g_{\underline{ij}}$ of the basic tensor g_{ij} is related to gravitation, and antisymmetric one $g_{\underset{\vee}{ij}}$ to electromagnetism.

A generalized Riemannian space $\mathbb{GR}_N$ in the sense of Eisenhart's definition [5] is a differentiable N-dimensional manifold, equipped with non-symmetric basic tensor g_{ij}.

Let us consider two N-dimensional generalized Riemannian spaces $\mathbb{GR}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$ with basic tensors g_{ij} and $\overline{g}_{ij}$, respectively. Generalized Christoffel symbols of the first kind of the spaces $\mathbb{GR}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$ are given by

$$\Gamma_{i.jk} = \frac{1}{2}(g_{ji,k} - g_{jk,i} + g_{ik,j}) \quad \text{and} \quad \overline{\Gamma}_{i.jk} = \frac{1}{2}(\overline{g}_{ji,k} - \overline{g}_{jk,i} + \overline{g}_{ik,j}), \tag{1.1}$$

where, for example, $g_{ij,k} = \partial g_{ij}/\partial x^k$. Connection coefficients of these spaces are generalized Christoffel symbols of the second kind $\Gamma^i_{jk} = g^{\underline{ip}}\Gamma_{p.jk}$ and $\overline{\Gamma}^i_{jk} = \overline{g}^{\underline{ip}}\overline{\Gamma}_{p.jk}$ respectively, where $(g^{\underline{ij}}) = (g_{\underline{ij}})^{-1}$ and $\underline{ij}$ denotes symmetrization with division of the indices i and j. Generally the generalized Christoffel symbols

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are not symmetric, i.e. $\Gamma^i_{jk} \neq \Gamma^i_{kj}$. We suppose that $g = \det(g_{ij}) \neq 0$, $\overline{g} = \det(\overline{g}_{ij}) \neq 0$, $\underline{g} = \det(g_{\underline{ij}}) \neq 0$, $\underline{\overline{g}} = \det(\overline{g}_{\underline{ij}}) \neq 0$.

A diffeomorpism $f : \mathbb{GR}_N \to \mathbb{G}\overline{\mathbb{R}}_N$ is a *conformal mapping* if for the basic tensors g_{ij} and $\overline{g}_{ij}$ of these spaces the condition

$$\overline{g}_{ij} = e^{2\psi} g_{ij} \tag{1.2}$$

is satisfied, where ψ is an arbitrary function of x, and the spaces are considered in the common system of local coordinates x^i.

In this case for the Christoffel symbols of the first kind of the spaces $\mathbb{GR}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$ the relation

$$\overline{\Gamma}_{i.jk} = e^{2\psi}(\Gamma_{i.jk} + g_{ji}\psi_{,k} - g_{jk}\psi_{,i} + g_{ik}\psi_{,j}) \tag{1.3}$$

is satisfied and for the Christoffel symbols of the second kind we have

$$\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + g^{\underline{ip}}(g_{jp}\psi_{,k} - g_{jk}\psi_{,p} + g_{pk}\psi_{,j}), \tag{1.4}$$

where $\psi_{,k} = \partial\psi/\partial x^k$. Let us denote $\psi_k = \psi_{,k}$ and $\psi^i = g^{\underline{ip}}\psi_p$. Now, from (1.4) we have

$$\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + g^{\underline{ip}}(g_{\underline{jp}}\,\psi_k - g_{\underline{jk}}\,\psi_p + g_{\underline{pk}}\,\psi_j) + g^{\underline{ip}}(g_{\underset{\vee}{jp}}\psi_k - g_{\underset{\vee}{jk}}\psi_p + g_{\underset{\vee}{pk}}\psi_j),$$

i.e.

$$\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j\,\psi_k + \delta^i_k\,\psi_j - \psi^i g_{\underline{jk}} + \xi^i_{jk}, \tag{1.5}$$

where

$$\xi^i_{jk} = g^{\underline{ip}}(g_{\underset{\vee}{jp}}\,\psi_k - g_{\underset{\vee}{jk}}\,\psi_p + g_{\underset{\vee}{pk}}\,\psi_j) = -\xi^i_{kj}, \qquad \psi_i = \frac{1}{N}(\overline{\Gamma}^p_{\underline{jp}} - \Gamma^p_{\underline{jp}}). \tag{1.6}$$

and $\underset{\vee}{ij}$ denotes an antisymmetrisation with division. In the corresponding points $M(x)$ and $\overline{M}(x)$ of a conformal mapping we can put

$$\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + P^i_{jk} \quad (i, j, k = 1, ..., N), \tag{1.7}$$

where P^i_{jk} is *the deformation tensor* of the connection Γ of $\mathbb{GR}_N$ according to the conformal mapping $f : \mathbb{GR}_N \to \mathbb{G}\overline{\mathbb{R}}_N$.

Notice that in $\mathbb{GR}_N$ we have

$$\Gamma^p_{\underset{\vee}{ip}} = 0, \tag{1.8}$$

(eq. (2.10) in [14]).

Based on the non-symmetry of the connection in a generalized Riemannian space one can define four kinds of covariant derivatives. For example, for a tensor a^i_j in $\mathbb{GR}_N$ we have

$$a^i_{j\underset{1}{|}m} = a^i_{j,m} + \Gamma^i_{pm}a^p_j - \Gamma^p_{jm}a^i_p, \qquad a^i_{j\underset{2}{|}m} = a^i_{j,m} + \Gamma^i_{mp}a^p_j - \Gamma^p_{mj}a^i_p,$$
$$a^i_{j\underset{3}{|}m} = a^i_{j,m} + \Gamma^i_{pm}a^p_j - \Gamma^p_{mj}a^i_p, \qquad a^i_{j\underset{4}{|}m} = a^i_{j,m} + \Gamma^i_{mp}a^p_j - \Gamma^p_{jm}a^i_p.$$

Here we denoted by $\underset{\theta}{|}$ a covariant derivative of the kind θ ($\theta \in \{1, 2, 3, 4\}$) in $\mathbb{GR}_N$.

In the case of the space $\mathbb{GR}_N$ we have five independent curvature tensors [24]:

$$\begin{aligned}
\underset{1}{K}{}^i_{jmn} &= \Gamma^i_{jm,n} - \Gamma^i_{jn,m} + \Gamma^p_{jm}\Gamma^i_{pn} - \Gamma^p_{jn}\Gamma^i_{pm},\\
\underset{2}{K}{}^i_{jmn} &= \frac{1}{2}(\Gamma^i_{jm,n} - \Gamma^i_{jn,m} + \Gamma^i_{mj,n} - \Gamma^i_{nj,m} + \Gamma^p_{jm}\Gamma^i_{np} + \Gamma^p_{mj}\Gamma^i_{pn} - \Gamma^p_{jn}\Gamma^i_{mp} - \Gamma^p_{nj}\Gamma^i_{pm}),\\
\underset{3}{K}{}^i_{jmn} &= \Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^p_{jm}\Gamma^i_{np} - \Gamma^p_{nj}\Gamma^i_{pm} + \Gamma^p_{nm}(\Gamma^i_{pj} - \Gamma^i_{jp}),\\
\underset{4}{K}{}^i_{jmn} &= \frac{1}{2}(\Gamma^i_{jm,n} - \Gamma^i_{jn,m} + \Gamma^i_{mj,n} - \Gamma^i_{nj,m} + \Gamma^p_{mj}\Gamma^i_{pn} + \Gamma^p_{jm}\Gamma^i_{np} - \Gamma^p_{jn}\Gamma^i_{pm} - \Gamma^p_{nj}\Gamma^i_{mp}),\\
\underset{5}{K}{}^i_{jmn} &= \frac{1}{2}(\Gamma^i_{jm,n} + \Gamma^i_{mj,n} - \Gamma^i_{jn,m} - \Gamma^i_{nj,m} + 2\Gamma^p_{\underline{jm}}\Gamma^i_{\underline{pn}} - 2\Gamma^p_{\underline{jn}}\Gamma^i_{\underline{mp}} + \Gamma^p_{\underset{\vee}{nm}}\Gamma^i_{\underset{\vee}{pm}}).
\end{aligned}$$

We use the conformal mapping $f : \mathbb{GR}_N \to \mathbb{G}\overline{\mathbb{R}}_N$ to obtain the tensors $\underset{\theta}{\overline{K}}{}^i_{jmn}$ $(\theta = 1, ..., 5)$, where for example

$$\underset{1}{\overline{K}}{}^i_{jmn} = \overline{\Gamma}^i_{jm,n} - \overline{\Gamma}^i_{jn,m} + \overline{\Gamma}^p_{jm}\overline{\Gamma}^i_{pn} - \overline{\Gamma}^p_{jn}\overline{\Gamma}^i_{pm}. \tag{1.9}$$

2. Concircular vector field

In 1940. K. Yano [23] considered the conformal mapping $\overline{g}_{ij} = \psi^2 g_{ij}$ of two Riemannian spaces. In this case, he proved that geodesics are invariant under this mapping if and only if

$$\psi_{;ij} - \psi_i\psi_j = \omega g_{ij}, \tag{2.1}$$

where $(;)$ is a covariant derivative, g_{ij} a symmetric metric tensor, ω an invariant and ψ_i is a gradient vector.

When N. S. Sinyukov studied geodesic mappings of symmetric spaces [18], he wrote this condition in terms of $\xi = e^{-\psi}$. It is easy to see that the formula (2.1) transformes to

$$\xi_{i;j} = \rho g_{ij}, \tag{2.2}$$

where $\rho = -\omega e^{-\psi}$, $\xi_{;i} = \xi_i$. The vector field ξ_i, was called *concircular* vector field by K. Yano [23] . In the case when $\rho = const.$, ξ is called *convergent*, and in the case $\rho = B\xi + C$, $(B, C = const.)$, ξ is called *special concircular*. A space with concircular vector field was called *equidistant space* by N.S. Sinyukov.

Definition 2.1. [1] *A generalized Riemannian space $\mathbb{GR}_N$ with a non-symmetric metric tensor g_{ij} is called an* **equidistant space**, *if its adjoint Riemannian space $\mathbb{R}_N$ is an equidistant space, i.e. if there exists a non-vanishing one-form φ in $\mathbb{GR}_N$, $\varphi_i \neq 0$ satisfying*

$$\varphi_{i;j} = \rho g_{\underline{ij}}, \tag{2.3}$$

where $(;)$ denotes the covariant derivative with respect to the symmetric part of the connection of the space $\mathbb{GR}_N$. For $\rho \neq 0$ equidistant spaces belong to the **primary type**, *and for $\rho \equiv 0$ to the* **particular**.

The following definition is a consequence of the previous definition

Definition 2.2. A Concircular mapping $f : \mathbb{GR}_N \to \mathbb{G}\overline{\mathbb{R}}_N$ *is a conformal mapping if the following equation is valid*

$$\psi_{ij} = \psi_{;ij} - \psi_i\psi_j = \omega g_{\underline{ij}}, \tag{2.4}$$

where $\psi_i = \frac{1}{N}(\overline{\Gamma}^p_{\underline{jp}} - \Gamma^p_{\underline{jp}})$, ω is an invariant, and $(;)$ is the covariant derivative with respect to the connection $\Gamma^i_{\underline{jk}}$.

In the case of a concircular mapping $f : \mathbb{R}_N \to \overline{\mathbb{R}}_N$ of two Riemannian spaces $\mathbb{R}_N$ and $\overline{\mathbb{R}}_N$, we have an invariant geometric object

$$\boxed{Z^i{}_{jmn} = R^i{}_{jmn} - \frac{R}{N(N-1)}(\delta^i_n g_{jm} - \delta^i_m g_{jn}),} \tag{2.5}$$

where $R^i{}_{jmn}$ is the Riemann-Christoffel curvature tensor of the space $\mathbb{R}_N$, R_{jm} the Ricci tensor and R the scalar curvature. The object $Z^i{}_{jmn}$ is called the *concircular curvature tensor*.

3. Equitorsion concircular curvature tensors

For a concircular mapping $f : \mathbb{GR}_N \to \mathbb{G}\overline{\mathbb{R}}_N$, it is not possible to find a generalization of the concircular curvature tensor. For that reason, we define a special concircular mapping.

Definition 3.1. *A concircular mapping $f : \mathbb{GR}_N \to \mathbb{G}\overline{\mathbb{R}}_N$ is* **equitorsion** *if the torsion tensors of the spaces $\mathbb{GR}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$ are equal at corresponding points.*

According to (1.7), this means that

$$\overline{\Gamma}^i_{\underset{\vee}{jk}} - \Gamma^i_{\underset{\vee}{jk}} = \xi^i_{jk} = 0. \tag{3.1}$$

3.1. *Equitorsion concircular curvature tensor of the first kind*

Using (1.7), we get a relation between the first kind curvature tensors of the spaces $\mathbb{GR}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$:

$$\underset{1}{\overline{K}}{}^i_{jmn} = \underset{1}{K}{}^i_{jmn} + P^i_{\underline{jm};n} - P^i_{\underline{jn};m} + P^p_{\underline{jm}}P^i_{\underline{pn}} - P^p_{\underline{jn}}P^i_{\underline{pm}} + P^i_{\underline{pn}}\Gamma^p_{\underset{\vee}{jm}} - P^p_{\underline{jn}}\Gamma^i_{\underset{\vee}{pm}} - P^i_{\underline{pm}}\Gamma^p_{\underset{\vee}{jn}} + P^p_{\underline{jm}}\Gamma^i_{\underset{\vee}{pn}}\ . \tag{3.2}$$

Substituting the deformation tensor P with respect to (1.5, 1.7), and using (2.4), we obtain

$$\begin{aligned}\underset{1}{\overline{K}}{}^i_{jmn} = {} & \underset{1}{K}{}^i_{jmn} + 2\delta^i_m \omega g_{\underline{jn}} - 2\delta^i_n \omega g_{\underline{jm}} + (\delta^i_m g_{\underline{jn}} - \delta^i_n g_{\underline{jm}})\triangle\psi \\ & + \psi_p\delta^i_n\Gamma^p_{\underset{\vee}{jm}} - 2\psi_j\Gamma^i_{\underset{\vee}{nm}} - \psi_p\delta^i_m\Gamma^p_{\underset{\vee}{jn}} - 2\psi^i g_{\underline{pn}}\Gamma^p_{\underset{\vee}{jm}} + \psi^p g_{\underline{jn}}\Gamma^i_{\underset{\vee}{pm}} - \psi^p g_{\underline{jm}}\Gamma^i_{\underset{\vee}{pn}},\end{aligned} \tag{3.3}$$

where we denoted

$$\psi^i_j = g^{\underline{ip}}\psi_{pj}, \quad \triangle\psi = g^{\underline{pq}}\psi_p\psi_q = \psi_p\psi^p. \tag{3.4}$$

Contracting with respect to the indices i and n in (3.3) we get

$$\underset{1}{\overline{K}}_{jm} = \underset{1}{K}_{jm} - 2(N-1)\omega g_{\underline{jm}} - (N-1)\triangle\psi g_{\underline{jm}} + (N-2)\psi_p\Gamma^p_{\underset{\vee}{jm}} + 2\psi^p\Gamma_{\underset{\vee}{m.jp}}, \tag{3.5}$$

In case of concircular mappings, it is easy to prove the following formula

$$\overline{g}^{\underline{ij}} = e^{-2\psi}\, g^{\underline{ij}}\ . \tag{3.6}$$

In (3.5) multiplying by $g^{\underline{jm}}$ and contracting with respect to the indices j and then m we get

$$e^{2\psi}\underset{1}{\overline{K}} = \underset{1}{K} + 2N(1-N)\omega + N(1-N)\triangle\psi, \tag{3.7}$$

where $\underset{1}{\overline{K}} = \overline{g}^{\underline{pq}}\underset{1}{\overline{K}}_{pq}$, and $\underset{1}{K} = g^{\underline{pq}}\underset{1}{K}_{pq}$ are scalar curvatures of the first kind of the spaces $\mathbb{G}\overline{\mathbb{R}}_N$ and $\mathbb{GR}_N$ respectively. From (3.7), we have

$$\omega = \frac{1}{2N(1-N)}(e^{2\psi}\underset{1}{\overline{K}} - \underset{1}{K}) - \frac{1}{2}\triangle\psi. \tag{3.8}$$

It is easy to see that for concircular mappings the following formula is valid

$$g^{\underline{pi}} g_{\underline{jn}} = \overline{g}^{\underline{pi}} \overline{g}_{\underline{jn}}. \tag{3.9}$$

From (1.2) follows

$$\psi_i = \frac{1}{2N}\left(\frac{\partial}{\partial x^i}\ln \overline{g} - \frac{\partial}{\partial x^i}\ln g\right), \tag{3.10}$$

where $g = \det(g_{ij})$, $\overline{g} = \det(\overline{g}_{ij})$. From (3.1) and (3.10) we obtain

$$\Gamma_{j.\underset{\vee}{nm}}\psi^i = \frac{1}{2N}\overline{\Gamma}_{j.\underset{\vee}{nm}}\overline{g}^{\underline{ip}}\frac{\partial}{\partial x^p}\ln \overline{g} - \frac{1}{2N}\Gamma_{j.\underset{\vee}{nm}} g^{\underline{ip}}\frac{\partial}{\partial x^p}\ln g \tag{3.11}$$

and

$$\Gamma^i_{\underset{\vee}{qn}} g_{\underline{mj}}\,\psi^q = \frac{1}{2N}\overline{\Gamma}^i_{\underset{\vee}{qn}}\overline{g}_{\underline{mj}}\overline{g}^{\underline{pq}}\frac{\partial}{\partial x^p}\ln \overline{g} - \frac{1}{2N}\Gamma^i_{\underset{\vee}{qn}} g_{\underline{mj}} g^{\underline{pq}}\frac{\partial}{\partial x^p}\ln g. \tag{3.12}$$

Taking into account (3.10, 3.11, 3.12), we can write the relation (3.3) in the form

$$\underset{1}{\overline{Z}}{}^i_{jmn} = \underset{1}{Z}{}^i_{jmn}, \tag{3.13}$$

where

$$\begin{aligned}\underset{1}{Z}{}^i_{imn} = \underset{1}{K}{}^i_{jmn} &- \frac{1}{N(N-1)}K(\delta^i_n\, g_{jm} - \delta^i_m\, g_{jn})\\ &+ \frac{1}{2N}\Big(-\delta^i_n\,\Gamma^p_{\underset{\vee}{jm}} + 2\delta^p_j\,\Gamma^i_{\underset{\vee}{nm}} + \delta^i_m\,\Gamma^p_{\underset{\vee}{jn}} + 2g^{\underline{ip}}g_{\underline{qn}}\,\Gamma^q_{\underset{\vee}{jm}} - g^{\underline{pq}}g_{\underline{jn}}\,\Gamma^i_{\underset{\vee}{qm}} + g^{\underline{pq}}g_{\underline{jm}}\,\Gamma^i_{\underset{\vee}{qn}}\Big)\frac{\partial}{\partial x^p}\ln g.\end{aligned} \tag{3.14}$$

and analogously for the geometrical object $\underset{1}{\overline{Z}}{}^i_{jmn} \in \mathbb{G}\overline{\mathbb{R}}_N$. The tensor $\underset{1}{Z}{}^i{}_{jmn}$ is an invariant of equitorsion concircular mappings, and one can call it **the equitorsion concircular curvature tensor of the first kind.** So, the following theorem is proved:

Theorem 3.1. *Let the generalized Riemannian spaces $\mathbb{GR}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$ be defined by virtue of their non-symmetric basic tensors g_{ij} and $\overline{g}_{ij}$ respectively. The equitorsion concircular curvature tensor of the first kind $\underset{1}{Z}{}^i{}_{jmn}$* (3.14) *is an invariant of the equitorsion concircular mapping $f : \mathbb{GR}_N \to \mathbb{G}\overline{\mathbb{R}}_N$.*

3.2. Equitorsion concircular curvature tensor of the second kind

For the second kind curvature tensors of the spaces $\mathbb{GR}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$ we get the relation

$$\underset{2}{\overline{K}}{}^i_{jmn} = \underset{2}{K}{}^i_{jmn} + P^i_{\underline{jm};n} - P^i_{\underline{jn};m} + P^p_{\underline{jm}}P^i_{\underline{pn}} - P^p_{\underline{jn}}P^i_{\underline{pm}} \tag{3.15}$$

i.e., using (1.5, 1.7, 2.4) one obtains

$$\underset{2}{\overline{K}}{}^i_{jmn} = \underset{2}{K}{}^i_{jmn} + 2\delta^i_m\,\omega g_{\underline{jn}} - 2\delta^i_n\,\omega g_{\underline{jm}} + (\delta^i_m\, g_{\underline{jn}} - \delta^i_n\, g_{\underline{jm}})\triangle\psi. \tag{3.16}$$

Contracting with respect to the indices i and n in (3.16) we get

$$\underset{2}{\overline{K}}_{jm} = \underset{2}{K}_{jm} - 2(N-1)\omega g_{\underline{jm}} - (N-1)\triangle\psi g_{\underline{jm}}. \tag{3.17}$$

In the previous equation multiplying by $g^{\underline{jm}}$ and contracting with respect to j and then to m, we get

$$e^{2\psi}\underset{2}{\overline{K}} = \underset{2}{K} + 2N(1-N)\omega + N(1-N)\Delta\psi, \tag{3.18}$$

where $\underset{2}{\overline{K}} = \overline{g}^{\underline{pq}}\underset{2}{\overline{K}}_{pq}$, and $\underset{2}{K} = g^{\underline{pq}}\underset{2}{K}_{pq}$ are scalar curvatures of the second kind of the spaces $\mathbb{G}\overline{\mathbb{R}}_N$ and $\mathbb{G}\mathbb{R}_N$ respectively. From (3.18), we have

$$\omega = \frac{1}{2N(1-N)}(e^{2\psi}\underset{2}{\overline{K}} - \underset{2}{K}) - \frac{1}{2}\Delta\psi. \tag{3.19}$$

And finally, taking into account (3.10, 3.11, 3.12), we can write the relation (3.16) in the form

$$\underset{2}{\overline{Z}}{}^i_{jmn} = \underset{2}{Z}{}^i{}_{jmn}, \tag{3.20}$$

where

$$\underset{2}{Z}{}^i{}_{jmn} = \underset{2}{K}{}^i_{jmn} - \frac{1}{N(N-1)}\underset{2}{K}(\delta^i_n\, g_{jm} - \delta^i_m\, g_{jn}) \tag{3.21}$$

and analogously for $\underset{2}{\overline{Z}}{}^i_{jmn} \in \mathbb{G}\overline{\mathbb{R}}_N$. The tensor $\underset{2}{Z}{}^i{}_{jmn}$ is an invariant of equitorsion concircular mappings, and one can call it **the equitorsion concircular curvature tensor of the second kind.** So, we have:

Theorem 3.2. *Starting from the curvature tensor* $\underset{2}{K}{}^i_{jmn}$, *one obtains an invariant tensor* $\underset{2}{Z}{}^i{}_{jmn}$ *with respect to the equitorsion concircular mapping* $f : \mathbb{G}\mathbb{R}_N \to \mathbb{G}\overline{\mathbb{R}}_N$ *in the form* (3.21).

3.3. Equitorsion concircular curvature tensor of the third kind

In the case of the third kind curvature tensors of the spaces $\mathbb{G}\mathbb{R}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$ we get the relation

$$\begin{aligned}\underset{3}{\overline{K}}{}^i_{jmn} = {} & \underset{3}{K}{}^i_{jmn} + P^i_{\underline{jm};n} - P^i_{\underline{jn};m} + P^p_{\underline{jm}}P^i_{\underline{pn}} - P^p_{\underline{jn}}P^i_{\underline{pm}}\\ & + P^i_{\underline{pn}}\Gamma^p_{\underset{\vee}{jm}} - P^p_{\underline{jn}}\Gamma^i_{\underset{\vee}{pm}} + P^i_{\underline{pm}}\Gamma^p_{\underset{\vee}{jn}} - P^p_{\underline{jm}}\Gamma^i_{\underset{\vee}{pn}} - 2P^p_{\underline{nm}}\Gamma^i_{\underset{\vee}{jp}},\end{aligned} \tag{3.22}$$

i.e., using (1.5, 1.7, 2.4) one obtains

$$\begin{aligned}\underset{3}{\overline{K}}{}^i_{jmn} = {} & \underset{3}{K}{}^i_{jmn} + 2\delta^i_m\,\omega g_{\underline{jn}} - 2\delta^i_n\,\omega g_{\underline{jm}} + (\delta^i_m\, g_{\underline{jn}} - \delta^i_n\, g_{\underline{jm}})\triangle\psi\\ & - 2\psi_n\Gamma^i_{\underset{\vee}{jm}} + \psi_p\delta^i_n\,\Gamma^p_{\underset{\vee}{jm}} - 2\psi_m\Gamma^i_{\underset{\vee}{jn}} + \psi_p\delta^i_m\,\Gamma^p_{\underset{\vee}{jn}} + \psi^p g_{\underline{jn}}\,\Gamma^i_{\underset{\vee}{pm}} + 2\psi^p g_{\underline{mn}}\,\Gamma^i_{\underset{\vee}{jp}} + \psi^p g_{\underline{jm}}\,\Gamma^i_{\underset{\vee}{pn}}.\end{aligned} \tag{3.23}$$

Contracting (3.23) with respect to the indices i and n, the previous equation becomes

$$\underset{3}{\overline{K}}_{jm} = \underset{3}{K}_{jm} - 2(N-1)\omega g_{\underline{jm}} - (N-1)\triangle\psi g_{\underline{jm}} + (N-2)\psi_p\Gamma^p_{\underset{\vee}{jm}} + 2\psi^p\Gamma_{\underset{\vee}{m.jp}}, \tag{3.24}$$

Multiplying (3.24) by $\overline{g}^{\underline{jm}} = e^{-2\psi}g_{\underline{jm}}$ and contracting we get

$$e^{2\psi}\underset{3}{\overline{K}} = \underset{3}{K} + 2N(1-N)\omega + N(1-N)\Delta\psi, \tag{3.25}$$

where $\underset{3}{\overline{K}} = \overline{g}^{\underline{pq}}\underset{3}{\overline{K}}_{pq}$, and $\underset{3}{K} = g^{\underline{pq}}\underset{3}{K}_{pq}$ are scalar curvatures of the third kind of the spaces $\mathbb{G}\overline{\mathbb{R}}_N$ and $\mathbb{G}\mathbb{R}_N$ respectively. From (3.25), we have

$$\omega = \frac{1}{2N(1-N)}(e^{2\psi}\underset{3}{\overline{R}} - \underset{3}{R}) - \frac{1}{2}\Delta\psi, \tag{3.26}$$

Finally,

$$\underset{3}{\overline{Z}}{}^i_{jmn} = \underset{3}{Z}{}^i{}_{jmn} \tag{3.27}$$

where

$$\begin{aligned}\underset{3}{Z}{}^{i}_{imn} &= \underset{3}{R}{}^{i}{}_{jmn} - \frac{1}{N(N-1)}\underset{3}{K}(\delta^i_n\, g_{jm} - \delta^i_m\, g_{jn})\\ &+ \frac{1}{2N}\Big(2\delta^p_n\,\Gamma^i_{\underset{\vee}{jm}} - \delta^i_n\,\Gamma^p_{\underset{\vee}{jm}} + 2\delta^p_m\,\Gamma^i_{\underset{\vee}{jn}} - \delta^i_m\,\Gamma^p_{\underset{\vee}{jn}} - g^{\underline{pq}} g_{\underline{jn}}\,\Gamma^i_{\underset{\vee}{qm}} - 2g^{\underline{pq}} g_{\underline{mn}}\,\Gamma^i_{\underset{\vee}{jq}} - g^{\underline{pq}} g_{\underline{jm}}\,\Gamma^i_{\underset{\vee}{qn}}\Big)\frac{\partial}{\partial x^p}\ln g.\end{aligned} \tag{3.28}$$

And analogously for $\underset{3}{\overline{Z}}{}^{i}_{jmn}$ of the space $\mathbb{G}\overline{\mathbb{R}}_N$. The tensor $\underset{3}{Z}{}^{i}{}_{jmn}$ is an invariant of equitorsion concircular mappings, and one can call it **the equitorsion concircular curvature tensor of the third kind.** Now we have proved

Theorem 3.3. *From the curvature tensor* $\underset{3}{K}{}^{i}_{jmn}$*, we obtain an invariant tensor* $\underset{3}{Z}{}^{i}{}_{jmn}$ *according to the equitorsion concircular mapping* $f: \mathbb{G}\mathbb{R}_N \to \mathbb{G}\overline{\mathbb{R}}_N$ *in the form* (3.28).

3.4. Equitorsion concircular curvature tensor of the fourth kind

For curvature tensors of the fourth kind we get

$$\underset{4}{\overline{K}}{}^{i}_{jmn} = \underset{4}{K}{}^{i}_{jmn} + P^i_{\underline{jm};n} - P^i_{\underline{jn};m} + P^p_{\underline{jm}} P^i_{\underline{pn}} - P^p_{\underline{jn}} P^i_{\underline{pm}} \tag{3.29}$$

i.e.

$$\underset{4}{\overline{K}}{}^{i}_{jmn} = \underset{4}{K}{}^{i}_{jmn} + 2\delta^i_m\,\omega g_{\underline{jn}} - 2\delta^i_n\,\omega g_{\underline{jm}} + (\delta^i_m\, g_{\underline{jn}} - \delta^i_n\, g_{\underline{jm}})\triangle\psi. \tag{3.30}$$

Using the same procedure like in the previous cases, in this case an invariant object of the equitorsion concircular mapping is in the form

$$\underset{4}{Z}{}^{i}{}_{jmn} = \underset{4}{K}{}^{i}_{jmn} - \frac{1}{N(N-1)}\underset{4}{K}(\delta^i_n\, g_{jm} - \delta^i_m\, g_{jn}) \tag{3.31}$$

where $\underset{4}{K}_{jm}$ is the Ricci curvature tensor of the fourth kind and $\underset{4}{K}$ a scalar curvature of the fourth kind. The object $\underset{4}{Z}{}^{i}{}_{jmn}$ is a tensor and we call it **equitorsion concircular curvature tensor of the fourth kind** of the equitorsion mapping. So, the next theorem is valid:

Theorem 3.4. *From the curvature tensor* $\underset{4}{K}{}^{i}_{jmn}$*, one obtains an invariant tensor* $\underset{4}{Z}{}^{i}{}_{jmn}$ (3.31) *of the equitorsion mapping of generalized Riemannian spaces.*

3.5. Equitorsion concircular curvature tensor of the fifth kind

For the curvature tensors of the fifth kind of the spaces $\mathbb{G}\mathbb{R}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$ we have

$$\underset{5}{\overline{K}}{}^{i}_{jmn} = \underset{5}{K}{}^{i}_{jmn} + P^i_{\underline{jm};n} - P^i_{\underline{jn};m} + P^p_{\underline{jm}} P^i_{\underline{pn}} - P^p_{\underline{jn}} P^i_{\underline{pm}} \tag{3.32}$$

i.e.

$$\underset{5}{\overline{K}}{}^{i}_{jmn} = \underset{5}{K}{}^{i}_{jmn} + 2\delta^i_m\,\omega g_{\underline{jn}} - 2\delta^i_n\,\omega g_{\underline{jm}} + (\delta^i_m\, g_{\underline{jn}} - \delta^i_n\, g_{\underline{jm}})\triangle\psi. \tag{3.33}$$

Contracting with respect to the indices i, n and denoting

$$\underset{5}{K}{}^{p}_{jmp} = \underset{5}{K}_{jm}, \quad \underset{5}{\overline{K}}{}^{p}_{jmp} = \underset{5}{\overline{K}}_{jm}, \tag{3.34}$$

we obtain

$$\underset{5}{\overline{K}}_{jm} = \underset{5}{K}_{jm} - 2(N-1)\omega g_{\underline{jm}} - (N-1)\triangle\psi g_{\underline{jm}}\,. \tag{3.35}$$

wherefrom, multiplying by $\overline{g}^{\underline{jm}} = e^{-2\psi} g_{\underline{jm}}$ and contracting with respect to the indices j and m one obtains

$$\omega = \frac{1}{2N(1-N)}(e^{2\psi}\underset{5}{\overline{K}} - \underset{5}{K}) - \frac{1}{2}\Delta\psi. \tag{3.36}$$

After eliminating ω from (3.33) we can write

$$\underset{5}{\overline{Z}}{}^i_{jmn} = \underset{5}{Z}{}^i{}_{jmn}, \tag{3.37}$$

where

$$\underset{5}{Z}{}^i{}_{jmn} = \underset{5}{K}{}^i_{jmn} - \frac{1}{N(N-1)}\underset{5}{K}(\delta^i_n g_{jm} - \delta^i_m g_{jn}). \tag{3.38}$$

The object $\underset{5}{Z}{}^i{}_{jmn}$ is an invariant of the concircular equitorsion mapping. We call it **equitorsion concircular curvature tensor of the fifth kind**. So, the following theorem is proved:

Theorem 3.5. *Starting from the curvature tensor* $\underset{5}{K}{}^i_{jmn}$*, we obtain an invariant tensor* $\underset{5}{Z}{}^i{}_{jmn}$ (3.38) *of the equitorsion concircular mapping* $f : \mathbb{GR}_N \to \mathbb{G}\overline{\mathbb{R}}_N$.

4. Concluding remarks

For $g_{ij}(x) = g_{ji}(x)$ the space $\mathbb{GR}_N$ reduces to the Riemannian space $\mathbb{R}_N$. The curvature tensors $\underset{\theta}{K}$, $\theta = 1, \ldots, 5$ in a generalized Riemannian space reduce to the single curvature tensor R in Riemannian space (in the symmetric case).

In the case of equitorsion concircular mapping of the Riemannian spaces (in the symmetric case) $\underset{\theta}{Z}$, $(\theta = 1, \cdots, 5)$, given by the formulas (3.14, 3.21, 3.28, 3.31, 3.38) reduce to the concircular curvature tensor [18, 23]

$$Z^i{}_{jmn} = R^i{}_{jmn} - \frac{R}{N(N-1)}(\delta^i_n g_{jm} - \delta^i_m g_{jn}). \tag{4.1}$$

All these new quantities can be quite interesting for further investigation.

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