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A New Factor Theorem for Generalized Cesàro Summability

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Abstract. In [6], we proved a theorem dealing with an application of quasi-f-power increasing sequences. In this paper, we prove that theorem under less and weaker conditions. This theorem also includes several new results.

1. Introduction

A positive $X = (X_n)$ is said to be a quasi-f-power increasing sequence if there exists a constant $K = K(X, f) \geq 1$ such that $Kf_nX_n \geq f_mX_m$ for all $n \geq m \geq 1$, where $f = (f_n) = \{n^\sigma(\log n)^\eta, \eta \geq 0, 0 < \sigma < 1\}$ (see [13]). If we take η=0, then we get a quasi-σ-power increasing sequence (see [12]). We write $\mathcal{BV}_O = \mathcal{BV} \cap \mathcal{C}_O$, where $\mathcal{C}_O = \{ x = (x_k) \in \Omega \ : \ \lim_k |x_k| = 0 \}$, $\mathcal{BV}$=$\{ x = (x_k) \in \Omega \ : \ \sum_k |x_k - x_{k+1}| < \infty \}$ and Ω being the space of all real or complex-valued sequences. Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by $u_n^{\alpha,\delta}$ and $t_n^{\alpha,\delta}$ the nth Cesàro means of order (α, δ), with $\alpha + \delta > -1$, of the sequence (s_n) and (na_n), respectively, that is (see [8])

$$u_n^{\alpha,\delta} = \frac{1}{A_n^{\alpha+\delta}} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} A_v^\delta s_v \tag{1}$$

$$t_n^{\alpha,\delta} = \frac{1}{A_n^{\alpha+\delta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^\delta v a_v, \tag{2}$$

where

$$A_n^{\alpha+\delta} = O(n^{\alpha+\delta}), \quad \alpha+\delta > -1, \quad A_0^{\alpha+\delta} = 1 \quad and \quad A_{-n}^{\alpha+\delta} = 0 \quad for \quad n > 0. \tag{3}$$

Let (φ_n) be a sequence of complex numbers. The series $\sum a_n$ is said to be summable $\varphi - | C, \alpha, \delta |_k$, $k \geq 1$ and $\alpha + \delta > -1$, if (see [4],[9])

$$\sum_{n=1}^{\infty} | \varphi_n(u_n^{\alpha,\delta} - u_{n-1}^{\alpha,\delta}) |^k = \sum_{n=1}^{\infty} n^{-k} | \varphi_n t_n^{\alpha,\delta} |^k < \infty. \tag{4}$$

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In the special case when $\varphi_n = n^{1-\frac{1}{k}}$, $\varphi - \mid C,\alpha,\delta \mid_k$ summability is the same as $\mid C,\alpha,\delta \mid_k$ summability (see [9]). Also, if we set $\varphi_n = n^{\gamma+1-\frac{1}{k}}$, then $\varphi - \mid C,\alpha,\delta \mid_k$ summability reduces to $\mid C,\alpha,\delta;\gamma \mid_k$ summability (see [4]). If we take $\delta = 0$, then we have $\varphi - \mid C,\alpha \mid_k$ summability (see [1]). Furthermore, if we take $\varphi_n = n^{1-\frac{1}{k}}$ and $\beta = 0$, then we get $\mid C,\alpha \mid_k$ summability (see [10]). Finally, if we take $\varphi_n = n^{\gamma+1-\frac{1}{k}}$ and $\delta = 0$, then we obtain $\mid C,\alpha;\gamma \mid_k$ summability (see [11]).

2. The known result. The following theorem is known.

Theorem A ([6]). Let $(\lambda_n) \in \mathcal{BV}_O$ and let (X_n) be a quasi-f-power increasing sequence for some σ $(0 < \sigma < 1)$. Suppose also that there exist sequences (β_n) and (λ_n) such that

$$\mid \Delta\lambda_n \mid\leq \beta_n, \tag{5}$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{6}$$

$$\sum_{n=1}^{\infty} n \mid \Delta\beta_n \mid X_n < \infty, \tag{7}$$

$$\mid \lambda_n \mid X_n = O(1) \quad as \quad n \to \infty. \tag{8}$$

If there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k} \mid \varphi_n \mid^k)$ is non-increasing and if the sequence $(\theta_n^{\alpha,\delta})$ is defined by

$$\theta_n^{\alpha,\delta} = \begin{cases} \left|t_n^{\alpha,\delta}\right|, & \alpha = 1, \delta > -1 \\ \max_{1\leq v\leq n} \left|t_v^{\alpha,\delta}\right|, & 0 < \alpha < 1, \delta > -1 \end{cases} \tag{9}$$

satisfies the condition

$$\sum_{n=1}^{m} \frac{(\mid \varphi_n \mid \theta_n^{\alpha,\delta})^k}{n^k} = O(X_m) \quad \text{as} \quad m \to \infty, \tag{10}$$

then the series $\sum a_n\lambda_n$ is summable $\varphi- \mid C,\alpha,\delta \mid_k$, $k \geq 1$, $0 < \alpha \leq 1$, $\delta > -1$,$\eta \geq 0$ and $(\alpha + \delta)k + \epsilon > 1$.

Remark 2.2 It should ne noted that in the statement of Theorem 2.1, a different notation has been used for the quasi-f-power increasing sequences. If we take $\eta = 0$, then we obtain a known theorem (see [5]).

3. The main result

The aim of this paper is to prove Theorem 2.1 under weaker conditions. Now, we shall prove the following theorem :

Theorem 3.1 Let (X_n) be a quasi-f-power increasing sequence and the sequences (λ_n) and (β_n) such that conditions (5)-(8) are satisfied. If there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k} \mid \varphi_n \mid^k)$ is non-increasing and if the condition

$$\sum_{n=1}^{m} \frac{(\mid \varphi_n \mid \theta_n^{\alpha,\delta})^k}{n^k X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{11}$$

satisfies, then the series $\sum a_n\lambda_n$ is summable $\varphi- \mid C,\alpha,\delta \mid_k$, $k \geq 1$, $0 < \alpha \leq 1$, $\delta > -1$, $\eta \geq 0$ and $(\alpha + \delta - 1)k + \epsilon > 0$.

Remark 3.2 It should be noted that condition (11) is the same as condition (10) when k=1. When $k > 1$, condition (11) is weaker than condition (10), but the converse is not true. As in [14] we can show that if (10) is satisfied, then we get that

$$\sum_{n=1}^{m} \frac{(\mid \varphi_n \mid \theta_n^{\alpha,\delta})^k}{n^k X_n^{k-1}} = O(\frac{1}{X_1^{k-1}}) \sum_{n=1}^{m} \frac{(\mid \varphi_n \mid \theta_n^{\alpha,\delta})^k}{n^k} = O(X_m).$$

If (11) is satisfied, then for $k > 1$ we obtain that

$$\sum_{n=1}^{m} \frac{(\mid \varphi_n \mid \theta_n^{\alpha,\delta})^k}{n^k} = \sum_{n=1}^{m} X_n^{k-1} \frac{(\mid \varphi_n \mid \theta_n^{\alpha,\delta})^k}{n^k X_n^{k-1}} = O(X_m^{k-1}) \sum_{n=1}^{m} \frac{(\mid \varphi_n \mid \theta_n^{\alpha,\delta})^k}{n^k X_n^{k-1}} = O(X_m^k) \neq O(X_m).$$

Also it should be noted that the condition "$(\lambda_n) \in \mathcal{BV}_O$" has been removed.
We need the following lemmas for the proof of our theorem.
Lemma 3.3 ([3]) If $0 < \alpha \leq 1$, $\delta > -1$ and $1 \leq v \leq n$, then

$$\left| \sum_{p=0}^{v} A_{n-p}^{\alpha-1} A_p^{\delta} a_p \right| \leq \max_{1\leq m\leq v} \left| \sum_{p=0}^{m} A_{m-p}^{\alpha-1} A_p^{\delta} a_p \right|. \tag{12}$$

Lemma 3.4 ([7]) Under the conditions on (X_n), (β_n) and (λ_n) as expressed in the statement of the theorem, we have the following ;

$$nX_n\beta_n = O(1), \tag{13}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{14}$$

4. Proof of Theorem 3.1 Let $(T_n^{\alpha,\delta})$ be the nth (C, α, δ) mean of the sequence $(na_n\lambda_n)$. Then, by (2), we have that

$$T_n^{\alpha,\delta} = \frac{1}{A_n^{\alpha+\delta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\delta} v a_v \lambda_v.$$

First applying Abel's transformation and then use of Lemma 3.3, we have that

$$T_n^{\alpha,\delta} = \frac{1}{A_n^{\alpha+\delta}} \sum_{v=1}^{n-1} \Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\delta} p a_p + \frac{\lambda_n}{A_n^{\alpha+\delta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\delta} v a_v,$$

$$\begin{aligned} | T_n^{\alpha,\delta} | &\leq \frac{1}{A_n^{\alpha+\delta}} \sum_{v=1}^{n-1} | \Delta\lambda_v | \left| \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\delta} p a_p \right| + \frac{| \lambda_n |}{A_n^{\alpha+\delta}} \left| \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\delta} v a_v \right| \\ &\leq \frac{1}{A_n^{\alpha+\delta}} \sum_{v=1}^{n-1} A_v^{\alpha} A_v^{\delta} \theta_v^{\alpha,\delta} | \Delta\lambda_v | + | \lambda_n | \theta_n^{\alpha,\delta} \\ &= T_{n,1}^{\alpha,\delta} + T_{n,2}^{\alpha,\delta}. \end{aligned}$$

To complete the proof of the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{-k} | \varphi_n T_{n,r}^{\alpha,\delta} |^k < \infty \quad for \quad r = 1, 2.$$

Now, when $k > 1$, applying Hölder's inequality with indices k and k', where $\frac{1}{k} + \frac{1}{k'} = 1$, we get that

$$\begin{aligned}
\sum_{n=2}^{m+2} n^{-k} \mid \varphi_n T_{n,1}^{\alpha,\delta} \mid^k &\leq \sum_{n=2}^{m+1} n^{-k} (A_n^{\alpha+\delta})^{-k} \mid \varphi_n \mid^k \{\sum_{v=1}^{n-1} A_v^{\alpha+\delta} \theta_v^{\alpha,\delta} \mid \Delta\lambda_v \mid\}^k \\
&\leq \sum_{n=2}^{m+1} n^{-k} n^{-(\alpha+\delta)k} \mid \varphi_n \mid^k \sum_{v=1}^{n-1} v^{\alpha k} (\theta_v^{\alpha,\delta})^k \mid \Delta\lambda_v \mid^k \times \left\{ \sum_{v=1}^{n-1} 1 \right\}^{k-1} \\
&= O(1) \sum_{v=1}^{m} v^{(\alpha+\delta)^k} (\theta_v^{\alpha,\delta})^k (\beta_v)^k \sum_{n=v+1}^{m+1} \frac{n^{-k} \mid \varphi_n \mid^k}{n^{(\alpha+\delta)k+1-k}} \\
&= O(1) \sum_{v=1}^{m} v^{(\alpha+\delta)k} (\theta_v^{\alpha,\delta})^k {\beta_v}^k \sum_{n=v+1}^{m+1} \frac{n^{\epsilon-k} \mid \varphi_n \mid^k}{n^{1+(\alpha+\delta-1)k+\epsilon}} \\
&= O(1) \sum_{v=1}^{m} v^{(\alpha+\delta)k} (\theta_v^{\alpha,\delta})^k (\beta_v)^k v^{\epsilon-k} \mid \varphi_v \mid^k \sum_{n=v+1}^{m+1} \frac{1}{n^{1+(\alpha+\delta-1)k+\epsilon}} \\
&= O(1) \sum_{v=1}^{m} v^{(\alpha+\delta)k} (\theta_v^{\alpha,\delta})^k (\beta_v)^k v^{\epsilon-k} \mid \varphi_v \mid^k \int_v^\infty \frac{dx}{x^{1+(\alpha+\delta-1)k+\epsilon}} \\
&= O(1) \sum_{v=1}^{m} \beta_v (\beta_v)^{k-1} (\theta_v^{\alpha,\delta} \mid \varphi_v \mid)^k \\
&= O(1) \sum_{v=1}^{m} \beta_v \left(\frac{1}{vX_v} \right)^{k-1} (\theta_v^{\alpha,\delta} \mid \varphi_v \mid)^k \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v\beta_v) \sum_{r=1}^{v} \frac{(\mid \varphi_r \mid \theta_r^{\alpha,\delta})^k}{r^k X_r^{k-1}} + O(1) m\beta_m \sum_{v=1}^{m} \frac{(\mid \varphi_v \mid \theta_v^{\alpha,\delta})^k}{v^k X_v^{k-1}} \\
&= O(1) \sum_{v=1}^{m-1} \mid \Delta(v\beta_v) \mid X_v + O(1) m\beta_m X_m \\
&= O(1) \sum_{v=1}^{m-1} \mid (v+1)\Delta\beta_v - \beta_v \mid X_v + O(1) m\beta_m X_m \\
&= O(1) \sum_{v=1}^{m-1} v \mid \Delta\beta_v \mid X_v + O(1) \sum_{v=1}^{m-1} \beta_v X_v + O(1) m\beta_m X_m \\
&= O(1) \quad as \quad m \to \infty,
\end{aligned}$$

virtue of the hypotheses of the theorem and Lemma 3.4. Finally, we have that

$$\begin{aligned}
\sum_{n=1}^{m} n^{-k} \mid \varphi_n T_{n,2}^{\alpha,\delta} \mid^k &= \sum_{n=1}^{m} \mid \lambda_n \mid \mid \lambda_n \mid^{k-1} n^{-k} (\theta_n^{\alpha,\delta} \mid \varphi_n \mid)^k \\
&= O(1) \sum_{n=1}^{m} \mid \lambda_n \mid \left(\frac{1}{X_n} \right)^{k-1} n^{-k} (\theta_n^{\alpha,\delta} \mid \varphi_n \mid)^k \\
&= O(1) \sum_{n=1}^{m-1} \Delta \mid \lambda_n \mid \sum_{r=1}^{n} \frac{(\mid \varphi_r \mid \theta_r^{\alpha,\delta})^k}{r^k X_r^{k-1}} + O(1) \mid \lambda_m \mid \sum_{n=1}^{m} \frac{(\mid \varphi_n \mid \theta_n^{\alpha,\delta})^k}{n^k X_n^{k-1}} \\
&= O(1) \sum_{n=1}^{m-1} \mid \Delta \lambda_n \mid X_n + O(1) \mid \lambda_m \mid X_m \\
&= O(1) \sum_{n=1}^{m-1} \beta_n X_n + O(1) \mid \lambda_m \mid X_m = O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of the theorem and Lemma 3.4. This completes the proof of Theorem 3.1. If we set $\epsilon = 1$ and $\varphi_n = n^{1-\frac{1}{k}}$, then we obtain a new result concerning the $\mid C, \alpha, \delta \mid_k$ summability. Also, if we take $\varphi_n = n^{\gamma+1-\frac{1}{k}}$, then we obtain another new result dealing with the $\mid C, \alpha, \delta; \gamma \mid_k$ summability factors. If we take $\eta = 0$, then we obtain Theorem A under weaker conditions. If we set $\eta = 0$ and δ=0, then we obtain the result of Bor and Özarslan under weaker conditions (see [2]). Furthermore, if we take $\epsilon = 1$, $\delta = 0$ and $\varphi_n = n^{\gamma+1-\frac{1}{k}}$, then we get the result of Bor under weaker conditions (see [7]).

References

[1] M. Balcı, Absolute φ-summability factors, Comm. Fac. Sci. Univ. Ankara, Ser. A_1 29 (1980) 63-80.
[2] H. Bor and H. S. Özarslan, A study on quasi power increasing sequences, Rocky Mountain J. Math. 38 (2008) 801-807.
[3] H. Bor, On a new application of quasi power increasing sequences, Proc. Est. Acad. Sci. 57 (2008) 205-209.
[4] H. Bor, A newer application of almost increasing sequences, Pac. J. Appl. Math. 2(3) (2010) 211-216.
[5] H. Bor, A new application of quasi power increasing sequences, Rev. Un. Mat. Argentina 52 (2011) 27-32.
[6] H. Bor, A study on generalized quasi-power increasing sequences, Russian Math. (Iz. VUZ) 55 (6) (2011) 15-19.
[7] H. Bor, A new application of quasi power increasing sequences, Filomat 26 (2012) 631-635.
[8] D. Borwein, Theorems on some methods of summability, Quart. J. Math. Oxford Ser. (2) 9 (1958) 310-316.
[9] G. Das, A Tauberian theorem for absolute summability, Proc. Camb. Phil. Soc. 67 (1970) 321-326.
[10] T. M. Flett, On an extension of absolute summability and some theorems of Littlewood and Paley, Proc. London Math. Soc. 7 (1957) 113-141.
[11] T. M. Flett, Some more theorems concerning the absolute summability of Fourier series, Proc. London Math. Soc. 8 (1958) 357-387.
[12] L. Leindler, A new application of quasi power increasing sequences, Publ. Math. Debrecen 58 (2001) 791-796.
[13] W. T. Sulaiman, Extension on absolute summability factors of infinite series, J. Math. Anal. Appl. 322 (2006) 1224-1230.
[14] W. T. Sulaiman, A note on $\mid A \mid_k$ summability factors of infinite series, Appl. Math. Comput. 216 (2010) 2645-2648.