Filomat 30:10 (2016), 2623–2629
DOI 10.2298/FIL1610623C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Singular Value Inequalities for Real and Imaginary Parts of Matrices

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Abstract. Let $A = \mathrm{Re}\,A + i\,\mathrm{Im}\,A$ be the Cartesian decomposition of square matrix A of order n with $\mathrm{Re}\,A = \frac{A+A^*}{2}$ and $\mathrm{Im}\,A = \frac{A-A^*}{2i}$. Fan-Hoffman's result asserts that

$$\lambda_j(\mathrm{Re}A) \leq s_j(A),\ j = 1, \ldots, n,$$

where $\lambda_j(M)$ and $s_j(M)$ stand for the jth largest eigenvalue of M and the jth largest singular value of M, respectively. We investigate singular value inequalities for real and imaginary parts of matrices and prove the following inequalities:

$$s_j(\mathrm{Re}\,A) \leq \frac{1}{4} s_j\left([(|A| + |A^*|) - (A + A^*)] \oplus [(|A| + |A^*|) + (A + A^*)]\right),$$

and

$$s_j(\mathrm{Im}\,A) \leq \frac{1}{4} s_j\left([(|A| + |A^*|) - i(A^* - A)] \oplus [(|A| + |A^*|) + i(A^* - A)]\right),\ j = 1, \ldots, n.$$

In particular, we have

$$s_j(\mathrm{Re}\,A) \leq \frac{1}{2} s_j\left((|A| + |A^*|) \oplus (|A| + |A^*|)\right),$$

and

$$s_j(\mathrm{Im}\,A) \leq \frac{1}{2} s_j\left((|A| + |A^*|) \oplus (|A| + |A^*|)\right),\ j = 1, \ldots, n.$$

Moreover, we also show that these inequalities are sharp.

1. Introduction

Let M_n denote the vector space of all complex $n \times n$ matrices and let H_n be the set of all Hermitian matrices of order n. We always denote the eigenvalues of $A \in H_n$ in decreasing order by $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$. For $A, B \in H_n$, we use the notation $A \leq B$ or $B \geq A$ to mean that $B - A$ is positive semidefinite. Clearly,

2010 *Mathematics Subject Classification*. 47A63; 47B05; 47B15; 15A60; 47A30; 15A42; 15A18
Keywords. Hermitian matrices, Positive semidefinite matrices, Singular values, Cartesian decomposition
Received: 14 April 2014; Accepted: 11 June 2014
Communicated by Mohammad Sal Moslehian
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Research supported by the Anhui Provincial Natural Science Foundation(1408085MA08,1508085MA14) and the Natural Science Foundation of Anhui Higher Education Institutions of China (KJ2016A633,KJ2016A633,KJ2016B001,KJ2015A035).
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" $\le$ " and " $\ge$ " define two partial orders on H_n, each of which is called *Löwner partial order*. In particular, $B \ge 0$ (res. $B > 0$) means that B is positive semidefinite (res. B is positive definite). For $T \in M_n$, the *singular values* of T, denoted by $s_1(T), s_2(T), \ldots, s_n(T)$ are the eigenvalues of the positive semidefinite matrix $|T| = (T^*T)^{\frac{1}{2}}$, enumerated as $s_1(T) \ge s_2(T) \ge \cdots \ge s_n(T)$ and repeated according to multiplicity. It follows that the singular values of a normal matrix are just the moduli of its eigenvalues. In particular, if $T \in M_n$ is positive semidefinite, then singular values and eigenvalues of T are the same. For more information on this related topic, we refer to [1, 6, 7]. Let $A \in M_n$. Then $A = \mathrm{Re}\, A + i\, \mathrm{Im}\, A$, where $\mathrm{Re}\, A = \frac{A+A^*}{2}$ and $\mathrm{Im}\, A = \frac{A-A^*}{2i}$. This is called the *Cartesian decomposition* of A. It is clear that both $\mathrm{Re}\, A$ and $\mathrm{Im}\, A$ are Hermitian. Here we denote the block matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ by $A \oplus B$.

Fan and Hoffman [2] asserts that for $A \in M_n$,

$$\lambda_j(\mathrm{Re}\, A) \le s_j(A) \tag{1}$$

for $j = 1, \ldots, n$. In the book [4] of page 327, it is said that (1) implies that

$$|\lambda_j(\mathrm{Re}\, A)| \le s_j(A) \tag{2}$$

for $j = 1, \ldots, n$. Since the singular values of a Hermitian matrix are just the moduli of its eigenvalues., it seems that (2) is presented singular value inequalities between the real part $\mathrm{Re}\, A$ and A.

But, there exists a gap in (2). We shall point out that through an example. Consider the square matrix $A = \begin{pmatrix} i & 0 \\ 0 & -2 \end{pmatrix}$. Then $\mathrm{Re}\, A = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}$. It is obvious that

$$\lambda_1(\mathrm{Re}\, A) = 0,\ \lambda_2(\mathrm{Re}\, A) = -2 \text{ and } s_1(A) = 2,\ s_2(A) = 1.$$

However, $|\lambda_2(\mathrm{Re}\, A)| = 2 > 1 = s_2(A)$. This contradicts (2). More details on the monograph [4] review, we refer to the helpful paper by Zhang [8].

In this paper, our main consideration is singular value inequalities involving real and imaginary parts of matrices and themselves. We prove the following inequalities, i.e.,

$$s_j(\mathrm{Re}\, A) \le \frac{1}{4} s_j\left([(|A| + |A^*|) - (A + A^*)] \oplus [(|A| + |A^*|) + (A + A^*)]\right)$$

and

$$s_j(\mathrm{Im}\, A) \le \frac{1}{4} s_j\left([(|A| + |A^*|) - i(A^* - A)] \oplus [(|A| + |A^*|) + i(A^* - A)]\right),$$

for all $j = 1, \ldots, n$. In particular, the following inequalities hold:

$$s_j(\mathrm{Re}\, A) \le \frac{1}{2} s_j\left((|A| + |A^*|) \oplus (|A| + |A^*|)\right)$$

and

$$s_j(\mathrm{Im}\, A) \le \frac{1}{2} s_j\left((|A| + |A^*|) \oplus (|A| + |A^*|)\right),$$

for all $j = 1, \ldots, n$. Furthermore, we show that these inequalities are sharp. Some applications of these results and other related inequalities will be also obtained. Finally, we give new revision form of (2) between the absolute values of the eigenvalues of $\mathrm{Re}\, A$ and the singular values of A.

2. Main Results

We start with some lemmas.

Lemma 1 *If $A_1, A_2, B_1, B_2 \in M_n$ such that*

$$s_j(A_1) \le s_j(B_1), \ \ s_j(A_2) \le s_j(B_2), \ \text{for all} \ \ j = 1, \ldots, n,$$

then

$$s_j(A_1 \oplus A_2) \le s_j(B_1 \oplus B_2), \ \text{for all} \ \ j = 1, \ldots, 2n.$$

Moreover,

$$s_j(S) \le s_j(T), \ \text{for all} \ \ j = 1, \ldots, n,$$

if and only if

$$s_j(S \oplus S) \le s_j(T \oplus T), \ \text{for all} \ \ j = 1, \ldots, 2n.$$

Proof. Note that singular values are unitarily invariant: For any $A \in M_n$ and unitary $U, V \in M_n$, $s(UAV) = s(A)$. In particular, for positive semidefinite matrices, singular values and eigenvalues are the same. Let

$$A = \text{diag}(s_1(A_1), s_2(A_1), \cdots, s_n(A_1), s_1(A_2), s_2(A_2), \cdots, s_n(A_2))$$

and

$$B = \text{diag}(s_1(B_1), s_2(B_1), \cdots, s_n(B_1), s_1(B_2), s_2(B_2), \cdots, s_n(B_2)).$$

Hence

$$s(A_1 \oplus A_2) \ = \ s(A) \ = \lambda(A), s(B_1 \oplus B_2) \ = \ s(B) \ = \lambda(B).$$

Since $s_j(A_1) \le s_j(B_1)$ *and* $s_j(A_2) \le s_j(B_2)$, $1 \le j \le n,$, it follows that $0 \le A \le B$. By Weyl's Monotonicity Theorem,

$$\lambda_j(A) \le \lambda_j(B), \ 1 \le j \le 2n,$$

i.e.,

$$s_j(A_1 \oplus A_2) = \lambda_j(A) \ \le \lambda_j(B) = \ s_j(B_1 \oplus B_2), \ \ 1 \le j \le 2n.$$

The left part of the lemma is trivial. This completes the proof. □

The following useful result can be founded in [1, 6, 7].

Lemma 2 *The partitioned block matrix* $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ *is positive semidefinite if and only if both A and C are positive semidefinite and there exists a contraction W such that* $B = A^{\frac{1}{2}} W C^{\frac{1}{2}}$.

The following useful singular inequality was given by Zhan [5].

Lemma 3 *Let $A, B \in H_n$. If $A \ge 0$ and $B \ge 0$, then*

$$s_j \left(A - B\right) \le s_j(A \oplus B), \tag{3}$$

for $j \ = 1, \ 2, \ldots, \ n.$

The following inequalities are due to Hirzallah and Kittaneh[3].

Lemma 4 *Let $X, Y \in M_n$. Then*

$$s_j \left(\frac{X+Y}{2}\right) \le s_j(X \oplus Y), \ j \ = 1, \ 2, \ldots, \ n. \tag{4}$$

As a consequence,

$$s_j(\text{Re}\ X) \le s_j \left(X \oplus X\right), \ j \ = 1, \ 2, \ldots, \ n. \tag{5}$$

Next, we shall prove our main results about singular value inequalities involving real and imaginary parts of matrices and themselves.

Theorem 5 *Let $A \in M_n$. Then*

$$s_j(\mathrm{Re}\,A) \le \frac{1}{4} s_j\left([(|A| + |A^*|) - (A + A^*)] \oplus [(|A| + |A^*|) + (A + A^*)]\right) \tag{6}$$

and

$$s_j(\mathrm{Im}\,A) \le \frac{1}{4} s_j\left([(|A| + |A^*|) - i(A^* - A)] \oplus [(|A| + |A^*|) + i(A^* - A)]\right), \tag{7}$$

for all $j = 1, \ldots, n$.

Proof. Note that $A = |A^*|^{\frac{1}{2}} U |A|^{\frac{1}{2}}$ with unitary U. By Lemma 2, we have

$$\begin{pmatrix} |A| & A^* \\ A & |A^*| \end{pmatrix} \ge 0, \quad \begin{pmatrix} |A^*| & A \\ A^* & |A| \end{pmatrix} \ge 0.$$

Then

$$\begin{pmatrix} |A| + |A^*| & A + A^* \\ A + A^* & |A| + |A^*| \end{pmatrix} \ge 0.$$

Using Lemma 2 again, there exists a contraction $W \in M_n$ such that

$$A + A^* = (|A| + |A^*|)^{\frac{1}{2}}\, W\, (|A| + |A^*|)^{\frac{1}{2}}.$$

Next, we shall show that we can choose W such that W is Hermitian. We divide into two cases.

First, consider the case that A is invertible. Then both $|A|$ and $|A^*|$ are invertible, i.e., $|A| > 0$, $|A^*| > 0$. Thus $W = (|A| + |A^*|)^{-\frac{1}{2}} (A + A^*) (|A| + |A^*|)^{-\frac{1}{2}}$ is Hermitian.

In general, we have

$$\begin{pmatrix} |A| + |A^*| + m^{-1} I & A + A^* \\ A + A^* & |A| + |A^*| + m^{-1} I \end{pmatrix} > 0,$$

for any positive integer m. By proved case above, for each m there is a Hermitian contraction $W_m \in M_n$ such that

$$A + A^* = \left(|A| + |A^*| + m^{-1} I\right)^{\frac{1}{2}} W_m \left(|A| + |A^*| + m^{-1} I\right)^{\frac{1}{2}}. \tag{8}$$

Since M_n is a finite-dimensional space, the unit ball $\{X \in M_n : \|X\|_\infty \le 1\}$ of the spectral norm is compact. By Bolzano-Weierstrass theorem, $\{W_m\}_{m=1}^{\infty}$ has a convergent subsequence $\{W_{m_k}\}_{k=1}^{\infty}$, i.e., $\lim\limits_{k\to\infty} W_{m_k} = W$. Since W_{m_k} are Hermitian, it follows that

$$W = \lim_{k\to\infty} W_{m_k} = \lim_{k\to\infty} \left(W_{m_k}^*\right)^* = \left(\lim_{k\to\infty} W_{m_k}^*\right)^* = W^*.$$

In (8), letting $k \to \infty$ yields

$$A + A^* = (|A| + |A^*|)^{\frac{1}{2}}\, W\, (|A| + |A^*|)^{\frac{1}{2}},$$

where W is a Hermitian contraction.

Since W is Hermitian contraction, it follows that $\pm W \le I$. Then we have

$$\pm(A + A^*) = (|A| + |A^*|)^{\frac{1}{2}} (\pm W) (|A| + |A^*|)^{\frac{1}{2}} \le |A| + |A^*|.$$

Note that

$$(|A| + |A^*|) + (A + A^*) \ge 0, \quad (|A| + |A^*|) - (A + A^*) \ge 0$$

and

$$\operatorname{Re} A = \frac{1}{4}\{[(|A| + |A^*|) + (A + A^*)] - [(|A| + |A^*|) - (A + A^*)]\}.$$

By Lemma 3, for each $j = 1, \ldots, n$, we have

$$s_j(\operatorname{Re} A) \leq \frac{1}{4} s_j \left([(|A| + |A^*|) - (A + A^*)] \oplus [(|A| + |A^*|) + (A + A^*)]\right).$$

Note that $\operatorname{Im} A = \operatorname{Re}(-iA)$. Replacing A by $-iA$, we obtain (7). This completes the proof. □

Remark 1 In the proof of Theorem 5, we know that

$$\pm(A + A^*) \leq |A| + |A^*|, \ \pm i(A^* - A) \leq |A| + |A^*|.$$

Using Theorem 5 and the above remark, we can obtain the following inequality.

Theorem 6 *Let $A \in M_n$. Then*

$$s_j(\operatorname{Re} A) \leq \frac{1}{2} s_j \left((|A| + |A^*|) \oplus (|A| + |A^*|)\right) \tag{9}$$

and

$$s_j(\operatorname{Im} A) \leq \frac{1}{2} s_j \left((|A| + |A^*|) \oplus (|A| + |A^*|)\right), \tag{10}$$

for all $j = 1, \ldots, n$..

Proof. Note that

$$|A| + |A^*| + (A + A^*) \geq 0, \ |A| + |A^*| - (A + A^*) \geq 0$$

and

$$(|A| + |A^*|) - i(A^* - A) \geq 0, \ (|A| + |A^*|) + i(A^* - A) \geq 0.$$

We have

$$0 \leq |A| + |A^*| \pm (A + A^*) \leq 2(|A| + |A^*|)$$

and

$$0 \leq |A| + |A^*| \pm i(A^* - A) \leq 2(|A| + |A^*|).$$

Using the fact that for positive semidefinite matrices, singular values and eigenvalues are the same and Weyl's Monotonicity Principle, we have

$$s_j\left(|A| + |A^*| \pm (A + A^*)\right) \leq 2 s_j(|A| + |A^*|)$$

and

$$s_j\left(|A| + |A^*| \pm i(A^* - A)\right) \leq 2 s_j(|A| + |A^*|)$$

for all $j = 1, \ldots, n$. By Lemma 1 and Theorem 5, (9) and (10) hold. This completes the proof. □

Remark 2 It should be mentioned here that the inequality

$$s_j(\operatorname{Re} A) \leq \frac{1}{2} s_j \left(|A| + |A^*|\right),$$

is false for $j = 1, \ldots, n$. To see this, consider the matrix $A = \begin{pmatrix} i & i \\ 0 & 0 \end{pmatrix}$. Then

$$s_2(\operatorname{Re} A) = \frac{1}{2} > \frac{\sqrt{2} - 1}{2} = \frac{1}{2} s_2(|A| + |A^*|).$$

Next, we shall show that (6), (7), (9) and (10) are sharp.

Example Consider the square matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Let

$$H_1 = |A| + |A^*| - (A + A^*),\ H_2 = |A| + |A^*| + (A + A^*) \text{ and } H_3 = |A| + |A^*|.$$

Then

$$\operatorname{Re} A = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix},\ \operatorname{Im} A = \begin{pmatrix} 0 & \frac{1}{2i} \\ -\frac{1}{2i} & 0 \end{pmatrix}$$

and

$$H_1 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},\ H_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\ H_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that

$$s_1(\operatorname{Re} A) = \frac{1}{4} s_1(H_1 \oplus H_2) = \frac{1}{2} s_1(H_3 \oplus H_3) = \frac{1}{2}$$

and

$$s_2(\operatorname{Re} A) = \frac{1}{4} s_2(H_1 \oplus H_2) = \frac{1}{2} s_2(H_3 \oplus H_3) = \frac{1}{2}.$$

Similarly, let $H_4 = |A| + |A^*| - i(A^* - A), H_5 = |A| + |A^*| + i(A^* - A)$, we also have

$$s_1(\operatorname{Im} A) = \frac{1}{4} s_1(H_4 \oplus H_5) = \frac{1}{2} s_1(H_3 \oplus H_3) = \frac{1}{2}$$

and

$$s_2(\operatorname{Im} A) = \frac{1}{4} s_2(H_4 \oplus H_5) = \frac{1}{2} s_2(H_3 \oplus H_3) = \frac{1}{2}.$$

This example shows that the inequalities (6), (7), (9) and (10) are sharp.

On the other hand, using this example we could see that (6) and (9) seem sharper than (5) in Lemma 4, since

$$s_1 (A \oplus A) = s_2 (A \oplus A) = 1.$$

In [3, Corollary 2.4], Hirzallah and Kittaneh show that let $X, Y \in M_n$. Then

$$s_j(XY^* + YX^*) \leq s_j \left((|X|^2 + |Y|^2) \oplus (|X|^2 + |Y|^2) \right),\ j = 1, \ldots, n.$$

Replacing A in (9) of Corollary 6 by XY^*, we have following related inequality.

Corollary 7 *Let $X, Y \in M_n$. Then*

$$s_j(XY^* + YX^*) \leq s_j \left((|XY^*| + |YX^*|) \oplus (|XY^*| + |YX^*|) \right),\ j = 1, \ldots, n. \tag{11}$$

An immediate consequence of Theorem 6 gives the following inequality related to normal matrices.

Corollary 8 *Let $T \in M_n$ be normal matrix. Then*

$$s_j(\operatorname{Re} T) \leq s_j (T \oplus T),\ s_j(\operatorname{Im} T) \leq s_j (T \oplus T),\ j = 1, \ldots, n.$$

In the end, we shall give a new revision of (2).

Theorem 9 *Let $A \in M_n$ and let j be positive integer with $1 \leq j \leq n$. If $j \geq \frac{n+1}{2}$, then*

$$|\lambda_j(\operatorname{Re} A)| \leq s_j(A), \tag{12}$$

$$|\lambda_j(\operatorname{Im} A)| \leq s_j(A). \tag{13}$$

Otherwise, we have

$$|\lambda_j(\operatorname{Re} A)| \leq s_{n-j+1}(A), \tag{14}$$

$$|\lambda_j(\operatorname{Im} A)| \leq s_{n-j+1}(A). \tag{15}$$

Proof. Replacing A by $-A$ in (1) and using the fact $s_j(-A) = s_j(A)$, $j = 1, \ldots, n$, we have

$$\lambda_{n-j+1}(\text{Re }(-A)) \leq s_{n-j+1}(A), \ j = 1, \ldots, n.$$

Note that $\lambda_j(\text{Re } A) = -\lambda_{n-j+1}(\text{Re }(-A))$, $j = 1, \ldots, n$. Then

$$\lambda_j(\text{Re } A) = -\lambda_{n-j+1}(\text{Re }(-A)) \geq -s_{n-j+1}(A), \ j = 1, \ldots, n.$$

By (1) due to Fan and Hoffman, we have

$$s_j(A) \geq \lambda_j(\text{Re } A) \geq -s_{n-j+1}(A), \ j = 1, \ldots, n.$$

Therefore

$$|\lambda_j(\text{Re } A)| \leq \max \{s_j(A), \ s_{n-j+1}(A)\}.$$

Note that $\text{Im } A = \text{Re }(-iA)$. Replacing A by $-iA$ and using the fact $s_j(-iA) = s_j(A)$, the inequality (12) holds. Comparing the value between j and $n - j + 1$, this completes the proof. □

Acknowledgement

The authors are grateful to Professor Xingzhi Zhan for his helpful suggestions and the authors also thank Professor Fuzhen Zhang to provide them with the interesting paper [8].

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