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On G-Sequential Continuity

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Abstract. Let X be a first countable Hausdorff topological group. The limit of a sequence in X defines a function denoted by lim from the set of all convergent sequences to X. This notion has been modified by Connor and Grosse-Erdmann for real functions by replacing lim with an arbitrary linear functional G defined on a linear subspace of the vector space of all real sequences. Recently Çakallı has extended the concept to the topological group setting and introduced the concepts of G-sequential compactness, G-sequential continuity and sequential connectedness. In this paper we give a further investigation of G-sequential continuity in topological groups.

Introduction

Connor and Grosse-Erdmann [20] have investigated the impact of changing the definition of the convergence of sequences on the structure of sequential continuity of real functions. Çakallı extended this concept to the topological group case and introduced the concept of G-sequential compactness in [15], investigated G-sequential continuity in [8] and gave the definition of sequential connectedness in [7]. Recently in [6], Çakallı and the first author of this paper have developed some further properties of sequential connectedness.

The aim of this paper is to give a further investigation of G-sequential continuity in topological groups and present some results which enrich the area.

1. Preliminaries

As a background, we give the following. Throughout this paper, $\mathbb{N}$ denotes the set of positive integers, X denotes a topological Hausdorff group, written additively, which satisfies the axiom of first countability. We will use boldface letters $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, ... for sequences $\mathbf{x} = (x_n)$, $\mathbf{y} = (y_n)$, $\mathbf{z} = (z_n)$, ... of terms of X. $s(X)$ and $c(X)$ respectively denote the set of all X-valued sequences, and the set of all X-valued convergent sequences of points in X.

Following the idea given in a 1946 American Mathematical Monthly problem [5], a number of authors Posner [28], Iwinski [26], Srinivasan [29], Antoni [1], Antoni and Salat [2], Spigel and Krupnik [30] have studied A-continuity defined by a regular summability matrix A. Some authors Öztürk [31], Savaş and

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Das [32], Savaş [33], Borsik and Salat [4] have studied A-continuity for methods of almost convergence and for related methods. See also [3] for an introduction to summability matrices and [19] for summability in topological groups.

The notion of statistical convergence has been introduced by Fast [22] and has been investigated by Fridy in [24]. In [34], Zygmund has called it *almost convergence* and has established a relation between it and strong summability. A sequence (x_k) of points in X is called *statistically convergent* to an element ℓ of X if for each neighborhood U of 0

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : x_k - \ell \notin U\}| = 0,$$

and this is written as $st - \lim_{n\to\infty} x_n = \ell$ ([13]). Statistical limit is an additive function on the group of statistically convergent sequences of points in X (see also [17], [27] and [9]).

Let $\theta = (k_r)$ be an increasing sequence of positive integers such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Such a sequence is called a *lacunary sequence*. A sequence (x_k) of points in a topological group is called *lacunary statistically convergent* to an element ℓ of X if

$$\lim_{r\to\infty} \frac{1}{h_r}|\{k \in I_r : x_k - \ell \notin U\}| = 0,$$

for every neighborhood U of 0 where $I_r = (k_{r-1}, k_r)$ [24].

For a constant lacunary sequence, $\theta = (k_r)$, the lacunary statistically convergent sequences in a topological group form a subgroup of the group of all X-valued sequences, and lacunary statistical limit is an additive function on this space (see [18] for topological group setting; [23], and [25] for the real case).

By a method of sequential convergence, or briefly a *method*, we mean an additive function G defined on a subgroup $c_G(X)$ of $s(X)$ into X [15]. A sequence $\mathbf{x} = (x_n)$ is said to be G-convergent to ℓ if $\mathbf{x} \in c_G(X)$ and $G(\mathbf{x}) = \ell$. In particular, lim denotes the limit function $\lim \mathbf{x} = \lim_n x_n$ on the group $c(X)$. A method G is called *regular* if every convergent sequence $\mathbf{x} = (x_n)$ is G-convergent with $G(\mathbf{x}) = \lim \mathbf{x}$. A map $f\colon X \to X$ is called *G-sequentially continuous* if $G(f(\mathbf{x})) = f(G(\mathbf{x}))$ for $\mathbf{x} \in c_G(X)$ [8]. Clearly, if f is a G-sequentially continuous on X, then it is G-sequentially continuous on every subset Z of X, but the converse is not necessarily true since in the latter case the sequences are restricted to Z. This was demonstrated by an example in [20] for a real function.

We define the sum of two methods of sequential convergence G_1 and G_2 as

$$(G_1 + G_2)(\mathbf{x}) = G_1(\mathbf{x}) + G_2(\mathbf{x})$$

where $c_{G_1+G_2}(X) = c_{G_1}(X) \cap c_{G_2}(X)$ ([8]). The notion of regularity introduced above coincides with the classical notion of regularity for summability matrices. See [3] for an introduction to regular summability matrices and see [34] for a general view of sequences of reals or complex.

First of all, we recall the definition of *G-sequential closure* of a subset of X. Let $A \subseteq X$ and $\ell \in X$. Then ℓ is in the G-sequential closure of A (it is called *G-hull* of A in [20]) if there is a sequence $\mathbf{x} = (x_n)$ of points in A such that $G(\mathbf{x}) = \ell$. We denote G-sequential closure of a set A by $\overline{A}^G$. We say that a subset A is *G-sequentially closed* if it contains all the points in its G-sequential closure, i.e., if $\overline{A}^G \subseteq A$. It is clear that $\overline{\phi}^G = \phi$ and $\overline{X}^G = X$. If G is a regular method, then $A \subseteq \overline{A} \subseteq \overline{A}^G$, and hence A is G-sequentially closed if and only if $\overline{A}^G = A$. Even for regular methods, it is not always true that $\overline{(\overline{A}^G)}^G = \overline{A}^G$. Even for regular methods, the union of any two G-sequentially closed subsets of X need not be a G-sequentially closed subset of X as seen by considering Counterexample 1 given after Theorem 4 in [8].

Çakallı [15] has introduced the concept of G-sequential compactness and has proved that the G-sequentially continuous image of any G-sequentially compact subset of X is also G-sequentially compact [15, Theorem 7]. He investigated G-sequential continuity, and obtained further results in [8] (see also [16], [21], [10], [11] and [12] for some other types of continuities which can not be given by any sequential method). Among those results given there, the following is more useful for our investigation.

Theorem 1.1. [8, Theorem 5] *Let G be a regular method and $\{A_i : i \in I\}$ a collection of subsets of X. Then the following are satisfied:*

(i) $\bigcup_{i\in I} \overline{A_i}^G \subseteq \overline{\bigcup_{i\in I} A_i}^G$

(ii) $\overline{\bigcap_{i\in I} A_i}^G \subseteq \bigcap_{i\in I} \overline{A_i}^G$

(iii) $\sum_{i\in I} \overline{A_i}^G \subseteq \overline{\sum_{i\in I} A_i}^G$.

2. Results

In [8] and [15] the concept of G-sequential continuity has been investigated. We give further results on G-sequential continuity. First we prove the following theorem.

Theorem 2.1. *Let G be a method on X. The intersection of a collection $\{F_i : i \in I\}$ of G-sequentially closed subsets of X is G-sequentially closed.*

Proof: By Theorem 1.1 (ii) and the assumption we have that

$$\overline{\bigcap_{i\in I} F_i}^G \subseteq \bigcap_{i\in I} \overline{F_i}^G \subseteq \bigcap_{i\in I} F_i.$$

We note that contrary to what could be expected, the union of G-sequentially closed subsets of X need not be G-sequentially closed even for a regular method G. Define

$$G(\mathbf{x}) = \lim_{n\to\infty} \frac{x_n + x_{n+1}}{2}$$

where X is the real space with usual topology. The sets $\{0\}$ and $\{1\}$ are G-sequentially closed in X, but the union $A = \{0\} \cup \{1\} = \{0, 1\}$ is not G-sequentially closed. This example also shows that even for regular methods, not always $\overline{(\overline{A}^G)}^G = \overline{A}^G$; since $\overline{A}^G = \{0, \frac{1}{2}, 1\}$ while $\overline{(\overline{A}^G)}^G = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ (Counterexample 1 in [8]).

Definition 2.2. A subset A of X is *G-sequentially open* if its complement is G-sequentially closed, i.e., $\overline{X \setminus A}^G \subseteq X \setminus A$.

From the fact that for a regular method G, G-sequential closure of a subset of X includes the set itself, we see that a subset A is G-sequentially open if and only if $\overline{X \setminus A}^G = X \setminus A$.

By Theorem 2.1, the following theorem can be stated.

Theorem 2.3. *Let G be a method on X. Then the union of any G-sequentially open subsets of X is G-sequentially open.*

Proof: The proof follows easily from Theorem 2.1 and so it is omitted.

As we remarked above, since the union of G-sequentially closed subsets need not be G-sequentially closed, the finite intersection of G-sequentially open subsets of X need not be G-sequentially open. Therefore the set of G-sequentially open subsets of X does not always give a topology on X.

We now modify the ordinary concept of a neighborhood of a point to the G-sequential case.

Definition 2.4. Let G be a method, U a subset of X and $a \in U$. U is called a *G-sequential neighborhood* of a if there exists a G-sequential open subset A of X with $a \in A$ such that $A \subseteq U$.

It is immediate from the definition and Theorem 2.3 that a subset A of X is G-sequentially open if and only if it is a neighborhood of each point of A.

Theorem 2.5. *Let G be a method on X and $A \subseteq X$. Then A is G-sequentially open if and only if each $a \in A$ has a G-sequentially open neighborhood U_a such that $U_a \subseteq A$.*

Proof: The proof is clear since the union of G-sequentially open subsets is also G-sequentially open by Theorem 2.3.

Definition 2.6. Let G be a method on X and $A \subseteq X$. Then the set

$$\bigcup\{U \subseteq A \mid U \text{ is } G - \text{sequentially open}\}$$

is called *G-sequential interior* of A and denoted by A^{0G}.

Here we remark that $a \in A^{0G}$ if and only if there is a G-sequentially open neighborhood U of a such that $U \subseteq A$.

We now give the definition of a G-sequentially open function as follows.

Definition 2.7. A function f is said to be *G-sequentially open* if the image of any G-sequentially open subset of X is G-sequentially open.

In [8] Çakallı gave the following definition.

Definition 2.8. A function f is said to be *G-sequentially closed* if the image of any G-sequentially closed subset of X is G-sequentially closed.

Theorem 2.9. *Let G be a method on X. A function $f\colon X \to X$ is G-sequentially closed if $\overline{f(A)}^G \subseteq f(\overline{A}^G)$ for every subset A.*

Proof: Let $f\colon X \to X$ be a function such that $\overline{f(A)}^G \subseteq f(\overline{A}^G)$ for any subset A. Let K be a G-closed subset. By assumption $\overline{f(K)}^G \subseteq f(\overline{K}^G)$. Since G is regular $\overline{K}^G = K$ and so we have that $\overline{f(K)}^G \subseteq f(K)$; and therefore $f(K)$ is G-sequentially closed.

Theorem 2.10. *Let G be a method on X and $A, B \subseteq X$. Then we have the following properties.*

(i) *A^{0G} is G-sequentially open,*

(ii) *$A^{0G} \subseteq A$,*

(iii) *A is G-sequentially open if and only if $A = A^{0G}$,*

(iv) *If $A \subseteq B$, then $A^{0G} \subseteq B^{0G}$,*

(v) *$(A \cap B)^{0G} \subseteq A^{0G} \cap B^{0G}$,*

(vi) *$A^{0G} \cup B^{0G} \subseteq (A \cup B)^{0G}$.*

Proof: (i) A^{0G} is G-sequentially open as the union of G-sequentially open subsets included in A.

(ii), (iii) and (iv) are obvious by Definition 2.6.

(v) and (vi) follow immediately from (iv).

On the arbitrary intersections and unions of G-sequential interiors, we prove the following theorem.

Theorem 2.11. *Let G be a method and $\{A_i \mid i \in I\}$ a class of subsets of X. Then we have the following.*

(i) $(\bigcap_{i \in I} A_i)^{0G} \subseteq \bigcap_{i \in I} {A_i}^{0G}$

(ii) $\bigcup_{i \in I} A_i^{0G} \subseteq (\bigcup_{i \in I} A_i)^{0G}$

Proof: The assertions follow immediately from Theorem 2.10 (iv).

Theorem 2.12. *Let G be a method on X. Then a function $f\colon X \to X$ is G-sequentially open if and only if $f(A^{0G}) \subseteq f(A)^{0G}$ for any subset $A \subseteq X$.*

Proof: Let the function $f\colon X \to X$ be G-sequentially open and $A \subseteq X$. Since $A^{0G} \subseteq A$, we have that $f(A^{0G}) \subseteq f(A)$ and therefore $f(A^{0G})^{0G} \subseteq f(A)^{0G}$. Here since $f(A^{0G})$ is G-sequentially open, it follows that $f(A^{0G}) \subseteq f(A)^{0G}$.

Conversely suppose that $f(A^{0G}) \subseteq f(A)^{0G}$ for any subset $A \subseteq X$. So for any G-sequentially open subset U, we have that $f(U) \subseteq f(U)^{0G}$ and so $f(U)$ is G-sequentially open.

Theorem 2.13. *Let G be a method and $A \subseteq X$. Then*

$$\overline{A}^{G} \subseteq \bigcap\{K \mid A \subseteq K \text{ and } K \text{ is } G-\text{sequentially closed}\}.$$

Proof: If $A \subseteq K$ and K is G-sequentially closed then

$$\overline{A}^{G} \subseteq \overline{K}^{G} \subseteq K$$

which implies the result.

Theorem 2.14. *Let G be a regular method and $A \subseteq X$. If $x \in \overline{A}^{G}$, then for every G-sequentially open neighborhood U of x, we have that $A \cap U \neq \emptyset$.*

Proof: Let $x \in \overline{A}^{G}$. By Theorem 2.13 for every G-sequentially closed subset K containing A we have $x \in K$. If U is a G-sequentially open neighbourhood of x, then $U \cap A$ is non empty. Otherwise if $U \cap A$ is empty, then $A \subseteq X \setminus U$ and here $X \setminus U$ is a G-sequentially closed subset and $x \notin X \setminus U$. This is a contradiction.

We say that a subset of X is *G-sequentially dense* in X if $\overline{A}^{G} = X$.

Corollary 2.15. *If A is G-sequentially dense in X, then $A \cap U$ is non empty for each G-sequentially open subset U of X.*

Theorem 2.16. *For a subset A of X we have that $\overline{X \setminus A}^{G} \subseteq X \setminus A^{0G}$*

Proof: If $x \in A^{0G}$, then there is a G-sequentially open neighbourhood U of x such that $x \in U \subseteq A$. So $X \setminus U$ is a G-sequentially closed subset of X containing $X \setminus A$, but $x \notin X \setminus U$. Therefore by Theorem 2.3, $x \notin \overline{X \setminus A}^{G}$ which completes the proof.

In [8] the *sequential boundary* of a subset A is defined as the set of the points which lie in both the G-sequential closure of A and the G-sequential closure of the complement of A; and denoted by A^{bG}.

As a result of Theorem 2.16 we can give the following.

Corollary 2.17. *Let A be a subset of X and A^{bG} the G-sequentially boundary of A. Then $A^{bG} \subseteq \overline{A}^{G} \setminus A^{0G}$.*

Definition 2.18. ([20] and [8]) A method is called *subsequential* if whenever $\mathbf{x}$ is G-convergent with $G(\mathbf{x}) = \ell$, then there is a subsequence (x_{n_k}) of $\mathbf{x}$ with $\lim_k x_{n_k} = \ell$.

Lemma 2.19. [8] *Let G be a regular method. Then $\overline{A}^{G} = \overline{A}$ for every subset A of X if and only if G is a subsequential method, where $\overline{A}$ denotes the usual closure of A.*

We now prove that in the case where G is a regular and subsequential method, the G-sequentially open subsets and usual open subsets are same.

Lemma 2.20. *If G is a regular and subsequential method, then $A^{0G} = A^0$ for any subset A of X.*

Proof: If G is a regular and subsequential method, by Lemma 2.19, $\overline{A}^G = \overline{A}$ for a subset A of X. So if A is G-sequentially open, $X\setminus A$ is G-sequentially closed and therefore by the regularity of G, $X\setminus A = \overline{X \setminus A}^G = \overline{X \setminus A}$. Hence A is open in the usual sense. Conversely if A is a usual open subset, then $X \setminus A = \overline{X \setminus A} = \overline{X \setminus A}^G$ and so A is G-sequentially open. Hence A^{0G} is a usual open subset such that $A^{0G} \subseteq A$ and so $A^{0G} \subseteq A^0$. Similarly A^0 is a G-sequentially open subset contained in A and so $A^0 \subseteq A^{0G}$. Therefore $A^{0G} = A^0$.

Lemma 2.21. [8] *Let G be a regular method. If a function f is G-sequentially continuous, then $f(\overline{A}^G) \subseteq \overline{f(A)}^G$ for every subset A of X.*

Lemma 2.22. [8] *Let G be a regular subsequential method. Then every G-sequentially continuous function is continuous in the ordinary sense.*

Lemma 2.23. [15] *Let G be a regular method. If a function f is G-sequentially continuous on X, then the inverse image $f^{-1}(K)$ of any G-sequentially closed subset K of X is G-sequentially closed.*

We now prove the following Theorem.

Theorem 2.24. *Let G be a regular method. If a function f is G-sequentially continuous on X, then the inverse image $f^{-1}(U)$ of any G-sequentially open subset U of X is G-sequentially open.*

Proof: Let $f\colon X \to X$ be any G-sequentially continuous function and A be any G-sequentially open subset of X. Then $X \setminus A$ is G-sequentially closed. By Lemma 2.23, $f^{-1}(X \setminus A)$ is G-sequentially closed. On the other hand

$$f^{-1}(X \setminus A) = f^{-1}(X) \setminus f^{-1}(A) = X \setminus f^{-1}(A)$$

and so it follows that $f^{-1}(A)$ is G-sequentially open. This completes the proof of the theorem.

Theorem 2.25. *Let G be a regular method. Let $f\colon X \to X$ be a bijection. If f is G-sequentially continuous, then $f(A)^{0G} \subseteq f(A^{0G})$ for any subset A of X.*

Proof: By Theorem 2.24, f^{-1} is sequentially open. Hence by Theorem 2.12, $f^{-1}(B^{0G}) \subseteq f^{-1}(B)^{0G}$ for any subset B of X. Replacing B by $f(A)$ we have that $f(A)^{0G} \subseteq f(A^{0G})$, as was claimed.

Definition 2.26. ([15]) A function $f\colon X \to X$ is *G-sequentially continuous* at a point u if for any sequence $\mathbf{x} = (x_n)$ of points in X, $G(\mathbf{x}) = u$ implies that $G(f(\mathbf{x})) = f(u)$.

We now prove that the G-sequential continuity of an additive function at the origin implies the G-sequential continuity of the function at any point in X, i.e., an additive function defined on X to X is G-sequentially continuous at the origin if and only if it is G-sequentially continuous at any point $a \in X$.

Theorem 2.27. *Suppose that G is a regular method. Let $f\colon X \to X$ be an additive function on X into X. Then f is G-sequentially continuous at the origin if and only if f is G-sequentially continuous at any point $a \in X$.*

Proof: Let the additive function $f\colon X \to X$ be G-sequentially continuous at the origin. So $G(f(\mathbf{x})) = 0$ whenever $G(\mathbf{x}) = 0$. Let $\mathbf{x}$ be a sequence in X with G-lim$\mathbf{x} = a$ and $\mathbf{a}$ the constant sequence $\mathbf{a} = (a, a, \dots)$. Since G is regular $G(\mathbf{a}) = a$. Therefore the sequence $\mathbf{x} - \mathbf{a}$ is G-convergent to the origin 0. So by assumption $G(f(\mathbf{x} - \mathbf{a})) = 0$. Since f and G are additive $G(f(\mathbf{x})) - G(f(\mathbf{a})) = 0$. Here since the constant sequence $f(\mathbf{a})$ tends to $f(a)$ and G is regular, $G(f(\mathbf{a})) = f(a)$. Therefore we have that $G(f(\mathbf{x}))) = f(a)$.

Corollary 2.28. *Let G a regular method. Then for any $a \in X$, the function $f_a\colon X \to X, x \mapsto a + x$ is G-sequentially continuous, G-sequentially closed and G-sequentially open.*

Proof: Let $\mathbf{x}$ be a G-sequentially convergent sequence with $G(\mathbf{x}) = u \in X$ and let $\mathbf{a}$ be the constant sequence $\mathbf{a} = (a, a, \ldots,)$. Then the sequence $\mathbf{a} + \mathbf{x}$ is G-sequentially convergent to $a + u$, since the constant sequence $\mathbf{a} = (a, a, \ldots,)$ is G-sequentially convergent to a. Here G is additive and regular; and hence $G(\mathbf{a} + \mathbf{x}) = a + u$. Therefore

$$G(f_a(\mathbf{x})) = G(\mathbf{a} + \mathbf{x}) = a + u = f_a(u)$$

and f_a is G-sequentially continuous.

Since the inverse of f_a is f_{-a}, by Lemma 2.23 the function f_a is G-sequentially closed and by Theorem 2.24, f_a is G-sequentially open.

Theorem 2.29. *Let G be a regular method. If one of the sets A and B is G-sequentially open, then so also is the sum $A + B$.*

Proof: Suppose that B is a G-sequentially open subset and A is any subset. By Corollary 2.28, $a + B$ is G-sequentially open for any $a \in A$. Since

$$A + B = \bigcup_{a \in A} a + B$$

by Theorem 2.3, $A + B$ is G-sequentially open.

Theorem 2.30. *Let G be a regular method on X and $f, g \colon X \to X$ be functions on X. Then the following are satisfied.*

(i) *If f and g are G-sequentially continuous, then so also is gf.*

(ii) *If f and g are G-sequentially open (closed), then so also is gf.*

(iii) *If f and g are G-sequentially continuous, then so also is $f + g$.*

(iv) *If gf is G-sequentially open (closed) and f is onto, then g is G-sequentially open (closed).*

(v) *If gf is G-sequentially open (closed) and g is one to one, then f is G-sequentially open. (closed).*

Proof: (i) Let $\mathbf{x}$ be a sequence in X such that $G(\mathbf{x}) = u \in X$. Since f is G-sequentially continuous at u, we get $G(f(\mathbf{x})) = f(u)$ and since g is G-sequentially continuous at $f(u)$ we have that $G(g(f(\mathbf{x}))) = g(f(u))$. Therefore the function gf is G-sequentially continuous.

(ii) is obvious.

(iii) Let $\mathbf{x}$ be a sequence in X with $G(\mathbf{x}) = u \in X$. Since the functions f and g are G-sequentially continuous, $G(f(\mathbf{x})) = f(u)$ and $G(g(\mathbf{x})) = g(u)$. Therefore by the additivity of G

$$G((f + g)(\mathbf{x})) = G(f(\mathbf{x}) + g(\mathbf{x})) = G(f(\mathbf{x})) + G(g(\mathbf{x})) = f(u) + g(u) = (f + g)(u)$$

i.e., $f + g$ is G-sequentially continuous.

(iv) Let A be a G-sequentially open subset of X. Since f is G-sequentially continuous $f^{-1}(A)$ is G-sequentially open. Since gf is G-sequentially open and f is onto we have that $(gf)(f^{-1}(A)) = g(A)$ is G-sequentially open. For the case where A is closed, the proof is similar.

(v) Let A be a G-sequentially open subset of X. Since gf is G-sequentially open $gf(A)$ is G-sequentially open. Since g is G-sequentially continuous and one to one we have that $g^{-1}gf(A) = f(A)$ is G-sequentially open. In the case where A is closed, the proof is similar.

The proofs of the following are straightforward but we write the details to check the conditions.

Theorem 2.31. *Let G be a method. Then we have the following.*

(i) *If $f \colon X \to X$ is a G-sequentially continuous, then so also is a restriction $f \colon A \to X$ to a subset A.*

(ii) *The identity map $f \colon X \to X$ is G-sequentially continuous.*

(iii) *For a subset $A \subseteq X$, the inclusion map $f : A \to X$ is G-sequentially continuous.*

(iv) *If G is regular, then the constant map $f : X \to X$ is G-sequentially continuous.*

(v) *If f is G-sequentially continuous, then so also is $-f$.*

(vi) *The inverse function $f : X \to X, f(x) = -x$ is G-sequentially continuous.*

(vii) *The inverse function $f : X \to X, f(x) = -x$ is G-sequentially closed.*

Proof: (i) Let $\mathbf{x}$ be a sequence of the terms in A with $G(\mathbf{x}) = u$. Since f is G-sequentially continuous, $G(f(\mathbf{x})) = f(u)$.

(ii) Let $G(\mathbf{x}) = u$ for a sequence $\mathbf{x}$ in X. Then $G(f(\mathbf{x})) = G(\mathbf{x}) = u = f(u)$ and so f is G-sequentially continuous.

(iii) follows immediately from (i) and (ii).

(iv) Let $f : X \to X$ be a constant map with $f(x) = x_0$ and let $\mathbf{x}$ be a sequence in X with $G(\mathbf{x}) = u$. Then $f(\mathbf{x}) = (x_0, x_0, ..., x_0, ...)$ which converges to x_0. Since G is regular $G(f(\mathbf{x})) = x_0 = f(u)$. Therefore f is G-sequentially continuous.

(v) Let $\mathbf{x}$ be a sequence in X such that $G(\mathbf{x}) = u$. Since f is G-sequentially continuous $G(f(\mathbf{x})) = f(u)$. Therefore $G(-f(\mathbf{x})) = -G(f(\mathbf{x})) = -f(u)$ and hence $-f$ is sequentially continuous.

(vi) follows immediately from (ii) and (v).

(vii) follows immediately from (vi) and Lemma 2.23.

From Theorem 2.30 and Theorem 2.31, we have the following corollary.

Corollary 2.32. *Let G be a regular method and $C^G(X)$ the class of G-sequentially continuous functions. Then $C^G(X)$ becomes a group with the sum of functions.*

Theorem 2.33. *Let G be a regular method and $f : X \to X$ a G-sequentially continuous map. Then $A = \{x \in X \mid f(x) = 0\}$, the kernel of f is a G-sequentially closed subset of X.*

Proof: Since $A = f^{-1}(\{0\})$ and $\{0\}$ is G-sequentially closed, the claim follows immediately from Lemma 2.23.

Theorem 2.34. *Let G be a regular method and $f, g : X \to X$ G-sequentially continuous functions. Then $A = \{x \in X \mid f(x) = g(x)\}$ is a G-sequentially closed subset of X.*

Proof: The proof follows immediately from Theorem 2.33 applied to the function $f - g : X \to X$.

3. Conclusion

The present work improves not only the work of Connor and Grosse-Erdmann [20] as we have presented it in a more general setting, i.e., in a topological group which is more general than the real space, but also the papers [15] and [8] of Çakallı, which are wholly new. So that one may expect it to be more useful tool in the field of topology in modeling various problems occurring in many areas of science, computer science, information theory, and biological science. It seems that an investigation of the present work taking nets instead of sequences could be possible using the properties of nets instead of using the properties of sequences. For further study, we also suggest to investigate the present work for the fuzzy case. However, due to the change in settings, the definitions and methods of proofs will not always be analogous to those of the present work (see [14] for the definitions in the fuzzy setting).

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