Some Fundamental Properties of Fuzzy Linear Relations between Vector Spaces

Sorin Nădăban

Abstract. This paper aims at studying the fundamental properties of fuzzy linear relations between vector spaces. The sum of two fuzzy relations and the scalar multiplication are defined, in a natural way, and some properties of these operations are established. Fuzzy linear relations are investigated and among the results obtained, there should be underlined a characterization of fuzzy linear relations and the fact that the inverse of a fuzzy linear relation is also a fuzzy linear relation. Moreover, the paper shows that the composition of two fuzzy linear relations is a fuzzy linear relation as well. Finally, the article highlights that the family of all fuzzy linear relations is closed under addition and it is closed under scalar multiplication.

1. Introduction

Linear relations were introduced by R. Arens [1] in 1961. Since then, they have been a preoccupation for many mathematicians. It was only in 1998 that the theory of linear relations was systematized in a beautiful monograph by R. Cross [8]. The subject is not closed though. Thus, in 2003, A. Száz [15] investigated the possibility of extending linear relation. The development of spectral theory for linear relations was the aims of recent papers: in 2002 A.G. Baskakov and K.I. Chernyshov [2], in 2007 A.G. Baskakov and A.S. Zagorskii [3], in 2012 D. Gheorghe and F.-H. Vasilescu [11]. It has to be mentioned that D. Gheorghe and F.-H. Vasilescu study in paper [10] linear maps defined between spaces of the form $X/X_0$, where $X$ is a vector space and $X_0$ is a vector subspace of $X$. The motivation of this approach comes from the theory of linear relations.

On the other hand, the concept of fuzzy set introduced by L. Zadeh [19] in 1965, represented a natural frame for generalizing many of the concepts of mathematics. The introduction in 1977 by A.K. Katsaras and D.B. Liu [13] of the concept of fuzzy topological vector space resulted in what we can call today a new mathematical field “Fuzzy Functional Analysis”. The present paper is based on results refereing to fuzzy linear subspaces obtained by A.K. Katsaras and D.B. Liu. In 1985, N.S. Papageorgiou [14] introduced the notion of fuzzy multifunction and started the study of linear fuzzy multifunction. The investigation of fuzzy multifunctions was continued, through a series of papers by E. Tsiporkova, B. De Baets, E. Kerre [17], [18] and I. Beg [4], [5], [6], [7]. A brief survey, concerning weakly linear systems of fuzzy relation inequalities and their applications in fuzzy automata, the study of simulation and in the social network analysis, was made in paper [12] by J. Ignjatović and M. Ćirić. There are also some recent papers in this

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Email address: snadaban@gmail.com (Sorin Nădăban)
field (see [9]). In [16], B. Šešelja, A. Tepavčević and M. Udovič are dealing with fuzzy posets and their fuzzy substructures.

The aim of this paper is to study the fundamental properties of fuzzy linear relations between vector spaces. Some results in the present paper may look similar, at the first sight, to I. Beg’s results, but they are not. The differences between them are significant. Along this paper, we have identified fuzzy sets with their membership functions, while, I. Beg identified the fuzzy set with their support. Thus, in this paper, all equalities and inclusions are between fuzzy sets as compared to some of I. Beg’s results where the obtained equalities (or inclusions) are only between the support sets of some fuzzy sets.

In this paper the sum of two fuzzy relations and the scalar multiplication are defined, in a natural way, and some properties of these operations are established. Fuzzy linear relations are investigated and among the results obtained, there should be underlined a characterization of fuzzy linear relations and the fact that the inverse of a fuzzy linear relation is also a fuzzy linear relation. Moreover, the paper shows that the composition of two fuzzy linear relations is a fuzzy linear relation as well. Finally, the article highlights that the family of all fuzzy linear relations is closed under addition and it is closed under scalar multiplication.

2. Fuzzy Linear Subspaces

Let X be a nonempty set. A fuzzy set in X (see [19]) is a function \( \mu : X \rightarrow [0, 1] \). We denote by \( \mathcal{F}(X) \) the family of all fuzzy sets in X. The symbols \( \vee \) and \( \land \) are used for the supremum and infimum of a family of fuzzy sets. We write \( \mu_1 \preceq \mu_2 \) if \( \mu_1(x) \leq \mu_2(x), \forall x \in X \). Let \( f : X \rightarrow Y \). If \( \mu \in \mathcal{F}(Y) \), then \( f^{-1}(\mu) := \mu \circ f \). If \( \rho \in \mathcal{F}(X) \), then \( f(\rho) \in \mathcal{F}(Y) \) is defined by

\[
\text{if } f^{-1}(\mu) \neq \emptyset \quad \text{then} \quad f(\rho)(y) := \bigvee \{ \rho(x) : x \in f^{-1}(y) \},
\]

\[
\text{otherwise} \quad \text{then} \quad f(\rho)(y) := 0.
\]

If \( \mu \in \mathcal{F}(X) \), the support of \( \mu \) is supp \( \mu := \{ x \in X : \mu(x) > 0 \} \).

Let X be a vector space over \( \mathbb{K} \) (where \( \mathbb{K} \) is \( \mathbb{R} \) or \( \mathbb{C} \)). If \( \mu_1, \mu_2, \ldots, \mu_n \) are fuzzy sets in X, then \( \mu = \mu_1 \times \mu_2 \times \cdots \times \mu_n \) is a fuzzy set in \( X^n \) defined by (see [13])

\[
\mu(x_1, x_2, \ldots, x_n) = \mu_1(x_1) \land \mu_2(x_2) \land \cdots \land \mu_n(x_n).
\]

Let \( f : X^n \rightarrow X \), \( f(x_1, x_2, \ldots, x_n) = \sum_{k=1}^n x_k \). The fuzzy set \( f(\mu) \) is called the sum of fuzzy sets \( \mu_1, \mu_2, \ldots, \mu_n \) and it is denoted by \( \mu_1 + \mu_2 + \cdots + \mu_n \) (see [13]). In fact

\[
(\mu_1 + \mu_2 + \cdots + \mu_n)(x) = \bigvee \left\{ \mu_1(x_1) \land \mu_2(x_2) \land \cdots \land \mu_n(x_n) : x = \sum_{k=1}^n x_k \right\}.
\]

If \( \mu \in \mathcal{F}(X) \) and \( \lambda \in \mathbb{K} \), then the fuzzy set \( \lambda \mu \) is the image of \( \mu \) under the map \( g : X \rightarrow X, g(x) = \lambda x \). Thus (see [13])

\[
(\lambda \mu)(x) = \begin{cases} \mu \left( \frac{x}{\lambda} \right) & \text{if } \lambda \neq 0 \\ 0 & \text{if } \lambda = 0, x \neq 0 \\ \bigvee \{ \mu(y) : y \in X \} & \text{if } \lambda = 0, x = 0. \end{cases}
\]

**Proposition 2.1.** Let \( \mu_1, \mu_2 \in \mathcal{F}(X) \) and \( \lambda \in \mathbb{K} \). Then \( \lambda(\mu_1 + \mu_2) = \lambda \mu_1 + \lambda \mu_2 \).

**Proof.** It is obvious.

**Definition 2.2.** [13] Let X be a vector space over \( \mathbb{K} \). \( \mu \in \mathcal{F}(X) \) is called fuzzy linear subspace of X if

1. \( \mu + \mu \subseteq \mu \);
2. \( \lambda \mu \subseteq \mu, (\forall) \lambda \in \mathbb{K} \).

We denote by \( \text{FLS}(X) \) the family of all fuzzy linear subspaces of X.
Proposition 2.3. [13] Let X be a vector space over ℜ and µ ∈ ℜ(X). The following statements are equivalent:
1. µ ∈ ℜS(X);
2. (∀)α, β ∈ ℜ, we have αµ + βµ ≤ µ;
3. (∀)x, y ∈ X, (∀)α, β ∈ ℜ, we have µ(αx + βy) ≥ µ(x) ∧ µ(y).

Proposition 2.4. [13] If µ, ρ ∈ ℜS(X) and λ ∈ ℜ, then:
1. µ + ρ ∈ ℜS(X);
2. λµ ∈ ℜS(X);
3. µ(x) ≤ µ(0), (∀)x ∈ X.

3. Fuzzy Relations

In this section we consider X, Y be two nonempty sets.

A fuzzy relation T (or fuzzy multilinear function, or fuzzy multivalued function) between X and Y is a fuzzy set in X × Y, i.e. a mapping T : X × Y → [0, 1]. For x ∈ X, we denote by T y the fuzzy set in Y defined by: T x : Y → [0, 1], T x(y) = T(x, y). Thus a fuzzy relation T can be seen as a mapping X ⊃ x ⊃ T x ∈ ℜ(Y) (see [14]). We denote by FR(X, Y) the family of all fuzzy relations between X and Y. If X = Y, then we set FR(X) = FR(X, Y).

The domain D(T) of T is a fuzzy set in X defined by D(T)(x) := sup y∈Y T(x, y) (see [18]). We note that

\[ \text{supp } D(T) = \{ x ∈ X : T_x \neq \emptyset \} = \{ x ∈ X : (∃) y ∈ Y \text{ such that } T(x, y) > 0 \}. \]

If for all x ∈ supp D(T) there exists an unique y ∈ Y such that T(x, y) > 0, then T is called fuzzy function (or single-valued fuzzy function). In this case, we denote this unique y by T(x).

If µ ∈ ℜ(Y), then T(µ) ∈ ℜ(Y) is defined by T(µ)(y) := sup x∈X [T(x, y) ∧ µ(x)] (see [4]). In particular, the range R(T) of T is a fuzzy set in Y defined by R(T)(y) := sup x∈X T(x, y) (see [18]).

Let T ∈ FR(X, Y), S ∈ FR(Y, Z). The composition S ◦ T ∈ FR(X, Z) (or simply ST) is defined by (see [19]):

\[ (S ◦ T)(x, z) := \sup_{y∈Y} [T(x, y) ∧ S(y, z)]. \]

Proposition 3.1. Let T ∈ FR(X, Y), S ∈ FR(Y, Z). Then (S ◦ T)x = S(Tx), (∀)x ∈ X.

Proof. It is obvious.

Proposition 3.2. The operation " ◦ " is associative.

Proof. It is obvious.

Definition 3.3. Let T ∈ FR(X, Y). If E ⊂ X, then the fuzzy relation T[E] defined by

\[ T[E] : E × Y → [0, 1], T[E](x, y) = T(x, y) \]

is called the restriction of T to E.

Moreover, the fuzzy relation T is called an extension to X of a fuzzy relation S ∈ FR(E, Y) if S = T[E].

The inverse (or reverse relation) T⁻¹ of a fuzzy relation T ∈ FR(X, Y) is a fuzzy set in Y × X defined by T⁻¹(y, x) = T(x, y). It is obvious that R(T) = D(T⁻¹) and R(T⁻¹) = D(T). We remark that, for µ ∈ ℜ(Y), we have T⁻¹(µ)(x) = sup y∈Y [T⁻¹(y, x) ∧ µ(y)] = sup y∈Y [T(x, y) ∧ µ(x)]. This type of inverse is usually called lower inverse (see [4]).

T ∈ FR(X, Y) is called surjective if supp R(T) = Y. A fuzzy relation T ∈ FR(X, Y) is called injective if, for x₁ ≠ x₂, we have T x₁ ∧ T x₂ = 0. This means that, for x₁ ≠ x₂, we have T(x₁, y) ∧ T(x₂, y) = 0, (∀)y ∈ Y.

A fuzzy relation I ∈ FR(X) is called identity relation if I(x, y) = 0 for all x, y ∈ X, x ≠ y. If we denote by E the support of D(I), the identity relation will be denoted I[E]. If µ ∈ ℜ(X), by identity relation on µ we understand ISupp µ(x, x) = µ(x).
Proposition 3.4. Let $T \in \text{FR}(X, Y)$. Then

1. $\text{supp } D(T) = X$ if and only if $I_X \subseteq T^{-1}T$;
2. $T$ is injective if and only if $T^{-1}T = I_{\text{supp } D(T)}$;
3. $T$ is a fuzzy function if and only if $TT^{-1} = I_{\text{supp } R(T)}$;
4. $T$ is injective if and only if $T^{-1}$ is a fuzzy function.

Proof. First we note that, for $x_1, x_2 \in X$, we have

$$T^{-1}(x_1, x_2) = \sup_{y \in Y} [T(x_1, y) \land T^{-1}(y, x_2)] = \sup_{y \in Y} [T(x_1, y) \land T(x_2, y)].$$

1) For $x \in X$, we have

$$T^{-1}(x, x) = \sup_{y \in Y} [T(x, y) \land T(x, y)] = \sup_{y \in Y} T(x, y) = D(T)(x).$$

Therefore

$$\text{supp } D(T) = X \iff D(T)(x) > 0, (\forall) x \in X \iff T^{-1}T(x, x) > 0, (\forall) x \in X \iff T^{-1}T \supseteq I_X.$$  

2) $\Rightarrow$ For $x_1 \neq x_2$ we have $T(x_1, y) \land T(x_2, y) = 0, (\forall) y \in Y$. Therefore $T^{-1}T(x_1, x_2) = 0$. For $x \in X$ we have $T^{-1}T(x, x) = \sup T(x, y) = D(T)(x)$. Thus $T^{-1}T = I_{\text{supp } D(T)}$.

$\Leftarrow$ Let $x_1 \neq x_2$. As $T^{-1}T = I_{\text{supp } D(T)}$, we have that $T^{-1}T(x_1, x_2) = 0$. Thus, for all $y \in Y$, we have $T(x_1, y) \land T(x_2, y) = 0$. Hence $T_{x_1} \land T_{x_2} = 0$.

3) $TT^{-1}(y_1, y_2) = \sup_{x \in X} [T^{-1}(y_1, x) \land T(x, y_2)] = \sup_{x \in X} T(x, y) = D(T)(y) > 0$. Thus $TT^{-1} = I_{\text{supp } R(T)}$.

$\Rightarrow$ For $y_1 \neq y_2$, we have that $T(x, y_1) = 0$ or $T(x, y_2) = 0$, for all $x \in X$. Hence $TT^{-1}(y_1, y_2) = 0$. On the other hand $TT^{-1}(y_1, y_2) = \sup T(x, y) = R(T)(y)$. Thus $TT^{-1} = I_{\text{supp } R(T)}$.

$\Leftarrow$ We suppose that $T$ is not a fuzzy function. Then $(\exists)x \in \text{supp } D(T), (\exists)y_1, y_2 \in Y, y_1 \neq y_2$ such that $T(x, y_1) > 0, T(x, y_2) > 0$. Therefore $TT^{-1}(y_1, y_2) > 0$, contradiction.

4) $T$ is injective $\iff T^{-1}T = I_{\text{supp } D(T)} \iff T^{-1}T = I_{\text{supp } R(T^{-1})} \iff T^{-1}$ is a fuzzy function.

4. Fuzzy Relations between Vector Spaces

In this section we consider $X, Y$ be two vector spaces.

Definition 4.1. Let $T, S \in \text{FR}(X, Y)$ and $\lambda \in \mathbb{K}$. We define the sum $T + S \in \text{FR}(X, Y)$ and scalar multiplication $\lambda T \in \text{FR}(X, Y)$ by

$$(T + S)(x, y) = \sup_{y_1 + y_2 = y} [T(x, y_1) \land S(x, y_2)]$$

and

$$(\lambda T)(x, y) = T \left( x, \frac{y}{\lambda} \right), \text{ if } \lambda \neq 0 \quad (0T)(x, y) = \begin{cases} D(T)x & \text{if } y = 0 \\ 0 & \text{if } y \neq 0 \end{cases}. $$

For $\lambda \in \mathbb{K}$ we define the fuzzy function $\lambda_T \in \text{FR}(Y)$ by

$$\lambda_T(x, y) = \begin{cases} 1 & \text{if } y = \lambda x \\ 0 & \text{if } y \neq \lambda x \end{cases}. $$

In fact this is an ordinary function, precisely the function $Y \ni x \mapsto \lambda x \in Y$.

Proposition 4.2. Let $T \in \text{FR}(X, Y)$ and $\lambda \in \mathbb{K}$. Then $\lambda T = \lambda_T \circ T$. 
Proof. Case 1. \( \lambda \neq 0 \).

\[
(\lambda_Y \circ T)(x, z) = \sup_{y \in Y} [T(x, y) \wedge \lambda_Y(y, z)] = T \left( x, \frac{z}{\lambda} \right) \wedge \lambda_Y \left( \frac{z}{\lambda}, z \right) = T \left( x, \frac{z}{\lambda} \right) = (\lambda T)(x, z).
\]

Case 2. \( \lambda = 0 \).

\[
(\lambda_Y \circ T)(x, z) = \sup_{y \in Y} [T(x, y) \wedge \lambda_Y(y, z)] = \begin{cases} 
\sup_{y \in Y} T(x, y) & \text{if } z = 0 \\
0 & \text{if } z \neq 0
\end{cases} = \begin{cases} 
D(T)x & \text{if } z = 0 \\
0 & \text{if } z \neq 0
\end{cases} = (\lambda T)(x, z).
\]

Proposition 4.3. Let \( T, S \in FR(X, Y) \) and \( \lambda \in \mathbb{K} \). Then

1. \( D(T + S) = D(T) \wedge D(S) \);
2. \( D(\lambda T) = D(T) \).

Proof. 1)

\[
D(T + S)(x) = \sup_{y \in Y} (T + S)(x, y) = \sup_{y \in Y} \sup_{y_1 + y_2 = y} [T(x, y_1) \wedge S(x, y_2)] = \sup_{y_1, y_2 \in Y} [T(x, y_1) \wedge S(x, y_2)] = D(T)x \wedge D(S)x = [D(T) \wedge D(S)](x).
\]

2) Case 1. \( \lambda \neq 0 \).

\[
D(\lambda T)(x) = \sup_{y \in Y} (\lambda T)(x, y) = \sup_{y \in Y} T \left( x, \frac{y}{\lambda} \right) = \sup_{y \in Y} T(x, y') = D(T)x.
\]

Case 2. \( \lambda = 0 \).

\[
D(0T)x = \sup_{y \in Y} (0T)(x, y) = (0T)(x, 0) = D(T)x.
\]

Proposition 4.4. Let \( T \in FR(X, Y) \) and \( \lambda \in \mathbb{K} \). Let \( \sup T := \sup \{T(x, y) : x \in X, y \in Y\} \). Then \( \sup \lambda T = \sup T \).

Proof. Case 1. \( \lambda \neq 0 \).

\[
\sup \lambda T := \sup \{(\lambda T)(x, y) : x \in X, y \in Y\} = \sup T \left( x, \frac{y}{\lambda} \right) : x \in X, y \in Y = \sup T.
\]

Case 2. \( \lambda = 0 \).

\[
\sup \lambda T := \sup \{(0T)(x, y) : x \in X, y \in Y\} = \sup \sup X \left( x \in X \right) \sup T(x, y) = \sup T.
\]

Theorem 4.5. Let \( T, S, R \in FR(X, Y) \) and \( \alpha, \beta \in \mathbb{K} \). Then

1. \( (T + S) + R = T + (S + R) \);
2. \( T + S = S + T \);
3. \( T + 0 = T \), where 0 \( \in FR(X, Y) \) is defined by \( 0(x, y) = \begin{cases} 
1 & \text{if } y = 0 \\
0 & \text{if } y \neq 0
\end{cases} \);
4. \( \alpha(T + S) = \alpha T + \alpha S \);
5. \( \alpha(\beta T) = (\alpha \beta)T \);
6. \( 1T = T \).
Proof. 1) Let \((x, y) \in X \times Y\). Then

\[
[(T + S) + R](x, y) = \sup_{z+t=y} [(T + S)(x, z) \land R(x, t)] = \sup_{z+t=y} \left[ \sup_{x \in X} T(x, h) \land S(x, u) \land R(x, t) \right].
\]

\[
= \sup_{h \in H, t \in T} \left[ T(x, h) \land S(x, u) \land R(x, t) \right].
\]

\[
[T + (S + R)](x, y) = \sup_{h \in H, t \in T} \left[ T(x, h) \land (S + R)(x, z) \right] = \sup_{h \in H, t \in T} \left[ T(x, h) \land S(x, u) \land R(x, t) \right].
\]

2) and 3) are obvious.

4) Case 1. \(\alpha \neq 0\).

\[
[a(T + S)](x, y) = (T + S) \left( x, \frac{y}{\alpha} \right) = \sup_{\frac{y_1 + y_2}{\beta}} \left[ T \left( x, \frac{y_1}{\alpha} \right) \land S \left( x, \frac{y_2}{\alpha} \right) \right] = \sup_{y_1 + y_2 = y} \left[ (\alpha T)(x, y_1) \land (\alpha S)(x, y_2) \right] = (\alpha T + \alpha S)(x, y).
\]

Case 2. \(\alpha = 0\).

\[
(0T + 0S)(x, y) = \sup_{y_1 + y_2 = y} \left[ (0T)(x, y_1) \land (0S)(x, y_2) \right].
\]

For \(y \neq 0\), we have \(y_1 \neq 0\) or \(y_2 \neq 0\). Thus \((0T)(x, y_1) \land (0S)(x, y_2) = 0\). Hence \((0T + 0S)(x, y) = 0\). For \(y = 0\), we have

\[
\sup_{y_1 + y_2 = y} \left[ (0T)(x, y_1) \land (0S)(x, y_2) \right] = (0T)(x, 0) \land (0S)(x, 0) = D(T)x \land D(S)x = (D(T) \land D(S))(x) = D(T + S)x.
\]

Therefore

\[
(0T + 0S)(x, y) = \left\{ \begin{array}{ll}
D(T + S)x & \text{if } y = 0 \\
0 & \text{if } y \neq 0
\end{array} \right. = \left[ 0(T + S) \right](x, y).
\]

5) Case 1. \(\alpha \neq 0, \beta \neq 0\).

\[
[a(\beta T)](x, y) = (\beta T) \left( x, \frac{y}{\alpha} \right) = T \left( x, \frac{y}{\alpha \beta} \right) = [(\alpha \beta)T](x, y).
\]

Case 2. \(\alpha = 0\).

\[
[a(\beta T)](x, y) = \left\{ \begin{array}{ll}
D(\beta T)x & \text{if } y = 0 \\
0 & \text{if } y \neq 0
\end{array} \right. =
\]

\[
\left\{ \begin{array}{ll}
D(T)x & \text{if } y = 0 \\
0 & \text{if } y \neq 0
\end{array} \right. = (0T)(x, y) = [(\alpha \beta)T](x, y).
\]

Case 3. \(\alpha \neq 0, \beta = 0\).

\[
[a(\beta T)](x, y) = (\beta T) \left( x, \frac{y}{\alpha} \right) = \left\{ \begin{array}{ll}
D(T)x & \text{if } y = 0 \\
0 & \text{if } y \neq 0
\end{array} \right. = (0T)(x, y) = [(\alpha \beta)T](x, y).
\]

6) It is obvious.
5. Fuzzy Linear Relations

Definition 5.1. Let $X, Y$ be two vector spaces over $\mathbb{K}$. A fuzzy linear relation (or fuzzy linear multivalued operator) between $X$ and $Y$ is a fuzzy linear subspace in $X \times Y$.

We denote by $\text{FLR}(X,Y)$ the family of all fuzzy linear relations between $X$ and $Y$. If $X = Y$, then we set $\text{FLR}(X,X)$.

If $T \in \text{FLR}(X,Y)$ is a single-valued fuzzy function, then $T$ will be called fuzzy linear operator. We denote by $\text{FLO}(X,Y)$ the family of all fuzzy linear operators between $X$ and $Y$. In the case $X = Y$ the family $\text{FLO}(X,X)$ will be denoted $\text{FLO}(X)$.

Remark 5.2. Using Proposition 2.3, the linearity of $T \in \text{FLR}(X,Y)$ can be written as

$$T(\alpha(x_1, y_1) + \beta(x_2, y_2)) \supseteq T(x_1, y_1) \land T(x_2, y_2), \text{ } (\forall) \alpha, \beta \in \mathbb{K}, (\forall)(x_1, y_1), (x_2, y_2) \in X \times Y.$$ 

We also note that, for $\beta = 0$, we obtain $T(\alpha(x_1, y_1)) \supseteq T(x_1, y_1)$.

Remark 5.3. Proposition 2.4 implies that $T(0,0) = 0$.

Proposition 5.4. Let $T \in \text{FR}(X,Y)$. Then $T \in \text{FLR}(X,Y)$ if and only if

1. $T_{x_1} + T_{x_2} \subseteq T_{x_1+x_2}, (\forall)x_1, x_2 \in X$;
2. $\alpha T_x \subseteq T_{\alpha x}, (\forall)x \in X, (\forall)\alpha \in \mathbb{K}$.

Proof. $\Rightarrow$ 1) Let $y \in Y$. Then

$$T_{x_1+x_2}(y) = T(x_1 + x_2, y) = T(x_1, y_1) \land T(x_2, y - y_1).$$

Therefore

$$T_{x_1+x_2}(y) \supseteq \sup_{y_1 \in Y}[T(x_1, y_1) \land T(x_2, y - y_1)] = (T_{x_1} + T_{x_2})(y).$$

Hence $T_{x_1+x_2} \supseteq T_{x_1} + T_{x_2}$.

2) First we note that

$$(\alpha T_x)(y) = \begin{cases} T_x \left(\frac{y}{\alpha}\right) & \text{ if } \alpha \neq 0 \\ 0 & \text{ if } \alpha = 0, y \neq 0 \\ \forall \{T_x(z) : z \in Y\} & \text{ if } \alpha = 0, y = 0 \end{cases}.$$

Case 1. $\alpha \neq 0$.

$$(T_{\alpha x})(y) = T(ax, y) = T(0, y) \supseteq T \left(\frac{x, \frac{y}{\alpha}}{\alpha}\right) = T_x \left(\frac{y}{\alpha}\right) = (\alpha T_x)(y).$$

Case 2. $\alpha = 0$. If $y \neq 0$, as $(\alpha T_x)(y) = 0$, we have that $(\alpha T_x)(y) \leq T_{\alpha x}(y)$. For $y = 0$ we have

$$(\alpha T_x)(y) = \forall \{T_x(z) : z \in Y\} \leq T(0,0) = T(ax, y) = T_{\alpha x}(y).$$

Hence $(\alpha T_x)(y) \leq T_{\alpha x}(y), (\forall)y \in Y$. Thus $\alpha T_x \subseteq T_{\alpha x}$.

$\Leftarrow$ Let $(x_1 + x_2, y) \in X \times Y$. Then

$$T(x_1 + x_2, y) = T_{x_1+x_2}(y) \supseteq (T_{x_1} + T_{x_2})(y) = \sup_{y_1 + y_2 = y} [T_{x_1}(y_1) \land T_{x_2}(y_2)].$$

Therefore

$$T(x_1 + x_2, y_1 + y_2) \supseteq T_{x_1}(y_1) \land T_{x_2}(y_2).$$

On the other hand, for $\alpha \neq 0$, we have $(\alpha T_x)(y) = T_x(y) = T(x, y)$. As $\alpha T_x \subseteq T_{\alpha x}$, we obtain that $(\alpha T_x)(y) \leq T_{\alpha x}(y)$, namely $T(x, y) \leq T(\alpha(x, y))$. If $\alpha = 0$, we have that

$$(\alpha T_x)(y) = \forall \{T_x(z) : z \in Y\} = \forall \{T(x, z) : z \in Y\}.$$
and
\[ T_{ax}(ay) = T(ax, ay) = T(0, 0) . \]
Therefore \( \alpha T_x \subseteq T_{ax} \) implies that \( \forall \{T(x, z) : z \in Y\} \leq T(0, 0) \), namely \( T(x, z) \leq T(0, 0) \), for all \( x \in X, z \in Y \). Now, we will prove that
\[ T(\alpha(x_1, y_1) + \beta(x_2, y_2)) \geq T(x_1, y_1) \land T(x_2, y_2) . \]

Case 1. \( \alpha \neq 0, \beta \neq 0 \).
\[ T(\alpha(x_1, y_1) + \beta(x_2, y_2)) \geq T(x_1, y_1) \land T(\beta x_2, \beta y_2) \geq T(x_1, y_1) \land T(x_2, y_2) . \]

Case 2. \( \alpha = 0, \beta \neq 0 \).
\[ T(\alpha(x_1, y_1) + \beta(x_2, y_2)) = T(\beta(x_2, y_2)) \geq T(x_1, y_1) \land T(x_2, y_2) . \]

Case 3. \( \alpha \neq 0, \beta = 0 \). Similarly to the previous case.

Case 4. \( \alpha = 0, \beta = 0 \).
\[ T(\alpha(x_1, y_1) + \beta(x_2, y_2)) = T(0, 0) \geq T(x_1, y_1) \land T(x_2, y_2) . \]

Corollary 5.5. If \( T \in \text{FLR}(X, Y) \), then \( T_0 \) is a fuzzy linear subspace of \( Y \).

Proof. By previous theorem we have that \( T_0 + T_0 \leq T_0, \alpha T_0 \leq T_0 \). This means that \( T_0 \) is a fuzzy linear subspace of \( Y \).

Theorem 5.6. Let \( T \in \text{FR}(X, Y) \). Then \( T \in \text{FLR}(X, Y) \) if and only if \( T^{-1} \in \text{FLR}(Y, X) \).

Proof. Let \( \alpha, \beta \in \mathbb{K}, (y_1, x_1), (y_2, x_2) \in Y \times X \). Then
\[ T^{-1}(\alpha(y_1, x_1) + \beta(y_2, x_2)) = T^{-1}(\alpha y_1 + \beta y_2, \alpha x_1 + \beta x_2) = T(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2) = T(\alpha(x_1, y_1) + \beta(x_2, y_2)) \geq T(x_1, y_1) \land T(x_2, y_2) = T^{-1}(y_1, x_1) \land T^{-1}(y_2, x_2) . \]

Similarly, we can prove that if \( T^{-1} \in \text{FLR}(Y, X) \), then \( T \in \text{FLR}(X, Y) \).

Corollary 5.7. If \( T \in \text{FLR}(X, Y) \), then \( T^{-1} \) is a fuzzy linear subspace of \( X \).

Definition 5.8. If \( T \in \text{FLR}(X, Y) \), then \( T^{-1} \) is called the kernel of \( T \) and it is denoted \( \text{Ker}(T) \).

Proposition 5.9. Let \( T \in \text{FLO}(X, Y) \) nonempty. Then
1. \( T(x_1) + T(x_2) = T(x_1 + x_2) \), \((\forall) x_1, x_2 \in D(T)\);
2. \( \alpha T(x) = T(\alpha x), (\forall) x \in D(T), (\forall) \alpha \in \mathbb{K} \).

Proof. 1)
\[ (T_{x_1} + T_{x_2})(T(x_1) + T(x_2)) = \sup\{T_{x_1}(y_1) \land T_{x_2}(y_2) : y_1 + y_2 = T(x_1) + T(x_2)\} = T_{x_1}(T(x_1)) \land T_{x_2}(T(x_2)) > 0 . \]
Therefore \( T_{x_1} + T_{x_2}(T(x_1) + T(x_2)) > 0 \), namely \( T(x_1) + T(x_2) = T(x_1 + x_2) \).
2) For \( \alpha \neq 0 \), we have \( \alpha T_x(\alpha T(x)) = T_x(T(x)) > 0 \). Thus \( T_{ax}(\alpha T(x)) > 0 \). Hence \( \alpha T(x) = T(\alpha x) \). If \( \alpha = 0 \), we must prove that \( T(0) = 0 \), namely \( T(0, 0) > 0 \). But
\[ T(0, 0) \geq T(x, y), (\forall) x \in X, (\forall) y \in Y . \]
Therefore, if \( T(0, 0) = 0 \), we obtain that \( T(x, y), (\forall) x \in X, (\forall) y \in Y \), i.e. \( T \) is the empty set.

Theorem 5.10. If \( T \in \text{FLR}(X, Y) \), \( \mu, \mu_1, \mu_2 \in \mathcal{F}(X) \) and \( \lambda \in \mathbb{K} \), then
1. \( T(\mu_1) + T(\mu_2) \subseteq T(\mu_1 + \mu_2) \);
2. $\mu_1 \subseteq \mu_2 \Rightarrow T(\mu_1) \subseteq T(\mu_2)$;
3. $AT(\mu) \subseteq T(\lambda \mu)$ and $\lambda T(\mu) = T(\lambda \mu)$, for $\lambda \neq 0$.

Proof. 1) Let $y \in Y$ fixed. We will prove that

$$[T(\mu_1) + T(\mu_2)](y) \leq T(\mu_1 + \mu_2)(y).$$

If $[T(\mu_1) + T(\mu_2)](y) = 0$, then the previous inequality is obvious. We suppose that

$$A = [T(\mu_1) + T(\mu_2)](y) > 0.$$

As

$$A = [T(\mu_1) + T(\mu_2)](y) = \sup_{y_1 + y_2 = y} [T(\mu_1)(y_1) + T(\mu_2)(y_2)],$$

for $\varepsilon \in (0, A)$ arbitrary, $(\exists) y_1, y_2 \in Y : y_1 + y_2 = y$, such that

$$T(\mu_1)(y_1) + T(\mu_2)(y_2) > A - \varepsilon.$$

But $T(\mu_1)(y_1) = \sup_{x \in X} [T(x, y_1) \wedge \mu_1(x)] > A - \varepsilon$ implies that there exists $x_1 \in X$ such that

$$T(x_1, y_1) > A - \varepsilon, \mu_1(x_1) > A - \varepsilon.$$

On the other hand $T(\mu_2)(y_2) = \sup_{x \in X} [T(x, y_2) \wedge \mu_2(x)] > A - \varepsilon$ implies that there exists $x_2 \in X$ such that $T(x_2, y_2) > A - \varepsilon$, $\mu_2(x_2) > A - \varepsilon$. Then

$$T(x_1 + x_2, y_1 + y_2) \geq T(x_1, y_1) \wedge T(x_2, y_2) > A - \varepsilon$$

and

$$(\mu_1 + \mu_2)(x_1 + x_2) = \sup_{x_1, x_2 : x_1 + x_2} [\mu_1(x_1') \wedge \mu_2(x_2')] \geq \mu_1(x_1) \wedge \mu_2(x_2) > A - \varepsilon.$$

Thus

$$B = T(\mu_1 + \mu_2)(y) = \sup_{x \in X} [T(x, y) \wedge (\mu_1 + \mu_2)(x)] \geq T(x_1 + x_2, y) \wedge (\mu_1 + \mu_2)(x_1 + x_2) > A - \varepsilon.$$

Hence $B > A - \varepsilon$. As $\varepsilon$ is arbitrary, we obtain that $B \geq A$, i.e. the desired inequality.

2) Let $y \in Y$. Then

$$T(\mu_1)(y) = \sup_{x \in X} [T(x, y) \wedge \mu_1(x)] \leq \sup_{x \in X} [T(x, y) \wedge \mu_2(x)] = T(\mu_2)(y).$$

3) Case 1. $\lambda \neq 0$. First, we note that

$$T \left( \frac{x}{\lambda}, \frac{y}{\lambda} \right) = T \left( \frac{1}{\lambda} (x, y) \right) \geq T(x, y).$$

On the other hand

$$T(x, y) = T \left( \lambda \left( \frac{x}{\lambda}, \frac{y}{\lambda} \right) \right) \geq T \left( \frac{x}{\lambda}, \frac{y}{\lambda} \right).$$

Thus

$$T \left( \frac{x}{\lambda}, \frac{y}{\lambda} \right) = T(x, y).$$

Therefore

$$[\lambda T(\mu)](y) = T(\mu) \left( \frac{y}{\lambda} \right) = \sup_{x \in X} \left[ T \left( \frac{x}{\lambda}, \frac{y}{\lambda} \right) \wedge \mu(x) \right] = \sup_{x \in X} \left[ T \left( \frac{x}{\lambda} \right) \wedge (\lambda \mu)(\lambda x) \right].$$
\begin{align*}
= \sup_{x \in X} \left[ T \left( \frac{X}{\lambda}, \frac{Y}{\lambda} \right) \right] = \sup_{x \in X} [T(x, y) \land (\lambda \mu)(x)] = [T(\lambda \mu)](y).
\end{align*}

Case 2. Let \( \lambda = 0 \). If \( y \neq 0 \), then \( [\lambda T(\mu)](y) = 0 \). Thus \( [\lambda T(\mu)](y) \leq [T(\lambda \mu)](y) \). If \( y = 0 \), we have \( [\lambda T(\mu)](y) = \sup_{z \in Y} T(\mu)(z) = \sup_{z \in Y} [T(x, z) \land \mu(x)] \).

As \( T(x, z) \leq T(0, 0) \), \( \forall x \in X, \forall z \in Y \), we obtain that \( [\lambda T(\mu)](y) \leq T(0, 0) \land \sup_{x \in X} \mu(x) \).

But \( T(\lambda \mu)(y) = [T(\lambda \mu)](0) = \sup_{x \in X} [T(x, 0) \land (\lambda \mu)(x)] \).

As \( \lambda = 0 \), for \( x \neq 0 \), we have that \( (\lambda \mu)(x) = 0 \). Therefore \( [T(\lambda \mu)](y) = T(0, 0) \land (\lambda \mu)(0) = T(0, 0) \land \sup_{x \in X} \mu(x) \).

Hence \( [\lambda T(\mu)](y) \leq [T(\lambda \mu)](y), \forall y \in Y \).

**Theorem 5.11.** If \( T \in FLR(X, Y), S \in FLR(Y, Z) \), then \( S \circ T \in FLR(X, Z) \).

**Proof.** Let \( x_1, x_2 \in X \). Then
\[
(S \circ T)_{x_1} + (S \circ T)_{x_2} = S(T_{x_1}) + S(T_{x_2}) \subseteq S(T_{x_1} + T_{x_2}) \subseteq S(T_{x_1} + T_{x_2}) = (S \circ T)_{x_1 + x_2}.
\]

Let \( x \in X \) and \( \lambda \in K \). Then
\[
\lambda(S \circ T)_x = \lambda S(T_x) \subseteq S(\lambda T_x) \subseteq S(T_{\lambda x}) = (S \circ T)_{\lambda x}.
\]

**Proposition 5.12.** Let \( T \in FLR(X, Y) \) nonempty. Then \( T \) is single valued fuzzy function if and only if \( \text{supp } T_0 = \{0\} \).

First, we note that \( 0 \in \text{supp } T_0 \), i.e. \( T(0, 0) > 0 \), contrary \( T \) is empty set.

\( \Rightarrow \) If there exists \( x \neq 0, x \in \text{supp } T_0 \), we obtain that \( T_0(x) > 0 \), namely \( T(0, x) > 0 \), contradiction with the fact that \( T \) is single-valued.

\( \Leftarrow \) Let \( x \in D(T) \) fixed. Then there exists \( y \in Y : T(x, y) > 0 \). We suppose that \( T \) is not a single-valued fuzzy function. Then there exists \( y' \in Y, y' \neq y : T(x, y') > 0 \). As
\[
T(x, y) \land T(x, y') \leq T((x, y) - (x, y')) = T(0, y - y') = 0,
\]
we have that \( T(x, y) \land T(x, y') = 0 \), contradiction.

**Proposition 5.13.** Let \( T \in FLR(X, Y) \). Then
1. \( T(T_0^{-1}) = T_0 \);
2. \( T^{-1}(T_0) = T_0^{-1} \).

**Proof.**
1) \( T(T_0^{-1})(y) = \sup_{x \in X} [T(x, y) \land T_0^{-1}(x)] = \sup_{x \in X} [T(x, y) \land T(x, 0)] \).

First we note that
\[
T(T_0^{-1})(y) = \sup_{x \in X} [T(x, y) \land T(x, 0)] \geq T(0, y) \land T(0, 0) = T(0, y).
\]

On the other hand, as \( T(x, y) \land T(x, 0) \leq T((x, y) - (x, 0)) = T(0, y) \), we have
\[
T(T_0^{-1})(y) = \sup_{x \in X} [T(x, y) \land T(x, 0)] \leq \sup_{x \in X} T(0, y) = T(0, y).
\]

Therefore
\[
T(T_0^{-1})(y) = T(0, y) = T_0(y), \forall y \in Y.
\]

2) The proof is similar.
Proposition 5.14. Let $T \in \text{FLR}(X, Y)$ nonempty. Then $T$ is injective if and only if $\text{supp Ker}(T) = \emptyset$.

Proof. As $T(x, y) \leq T(0, 0)$, $(\forall) x \in X, (\forall) y \in Y$, we have that $T(0, 0) > 0$, contrary $T$ is empty set. This implies that $0 \in \text{supp Ker}(T)$.

$\Rightarrow$ We suppose that there exists $x \neq 0 : x \in \text{supp Ker}(T)$, i.e. $T(x, 0) > 0$. Therefore $T(x, y) \neq 0$ for $y \in Y$. Particular, for $y = 0$, we have $T(x, 0) = T(0, 0) = 0$, contradiction.

$\Leftarrow$ Let $x_1 \neq x_2$. Then $x_1 - x_2 \neq 0$. Therefore $x_1 - x_2 \notin \text{supp Ker}(T)$, namely $T(x_1 - x_2, 0) = 0$. We will prove that $T_{x_1} \cap T_{x_2} = \emptyset$. Indeed, for $y \in Y$, we have

$$T_{x_1}(y) \cap T_{x_2}(y) = T(x_1, y) \cap T(x_2, y) \leq T(x_1, y) - (x_2, y) = T(x_1 - x_2, 0) = 0.$$ 

Thus $T_{x_1}(y) \cap T_{x_2}(y) = 0$, $(\forall) y \in Y$, i.e. $T_{x_1} \cap T_{x_2} = \emptyset$. Therefore $T$ is injective.

Proposition 5.15. If $T \in \text{FLR}(X, Y)$, then $T_x = T_x + T_0$, $x \in X$.

Proof. Let $y \in Y$. Then $(T_x + T_0)(y) = \sup_{y_1 + y_2 = y} [T_x(y_1) \wedge T_0(y_2)] \geq T_x(y) \wedge T_0(0) = T(x, y)$. On the other hand

$$T_x(y_1) \wedge T_0(y_2) = T(x, y_1) \wedge T(0, y_2) \leq T((x, y_1) + (0, y_2)) = T(x, y).$$

Hence

$$(T_x + T_0)(y) = \sup_{y_1 + y_2 = y} [T_x(y_1) \wedge T_0(y_2)] \leq T(x, y).$$

Therefore $(T_x + T_0)(y) = T(x, y) = T_x(y)$, i.e. $T_x + T_0 = T_0$.

Theorem 5.16. $\text{FLR}(X, Y)$ is closed under addition.

Proof. Let $T, S \in \text{FLR}(X, Y)$. We will prove that $T + S \in \text{FLR}(X, Y)$. First, we note that

$$(T + S)_x(y) = \sup_{y_1 + y_2 = y} [T_x(y_1) \wedge S_x(y_2)] = (T_x + S_x)(y).$$

Hence $(T + S)_x = T_x + S_x$. Therefore

$$(T + S)_{x_1}(y_1 + y_2) + (T + S)_{x_2}(y_1 + y_2) = (T_{x_1} + S_{x_1}) + (T_{x_2} + S_{x_2}) \leq (T_{x_1} + S_{x_1}) + (T_{x_2} + S_{x_2}) = (T + S)_{x_1 + x_2}.$$ 

Using Proposition 2.1, we obtain

$$a(T + S)_x = a(T_x + S_x) = aT_x + aS_x \leq T_{ax} + S_{ax} = (T + S)_{ax}.$$ 

Theorem 5.17. $\text{FLR}(X, Y)$ is closed under scalar multiplication.

Proof. Let $T \in \text{FLR}(X, Y)$ and $\lambda \in \mathbb{K}$. We will prove that $\lambda T \in \text{FLR}(X, Y)$.

Case 1. $\lambda \neq 0$. As $(\lambda T)_x(y) = T \left( x, \frac{y}{\lambda} \right)$, we obtain that $(\lambda T)_x(y) = T_x \left( x, \frac{y}{\lambda} \right)$. Thus

$$[(\lambda T)_{x_1} + (\lambda T)_{x_2}](y) = \sup_{y_1 + y_2 = y} [(\lambda T)_{x_1}(y_1) \wedge (\lambda T)_{x_2}(y_2)] = \sup_{y_1 + y_2 = y} \left[ T_{x_1} \left( \frac{y_1}{\lambda} \right) \wedge T_{x_2} \left( \frac{y_2}{\lambda} \right) \right] = \sup_{\frac{y_1}{\lambda} + \frac{y_2}{\lambda} = y} \left[ T_{x_1} \left( \frac{y_1}{\lambda} \right) \wedge T_{x_2} \left( \frac{y_2}{\lambda} \right) \right] = (T_{x_1} + T_{x_2}) \left( \frac{y}{\lambda} \right) \leq T_{x_1 + x_2}(y) = (\lambda T)_{x_1 + x_2}(y).$$

Hence $(\lambda T)_{x_1} + (\lambda T)_{x_2} \leq (\lambda T)_{x_1 + x_2}$. Now we prove that $a(\lambda T)_x \leq (\lambda T)_{ax}$.

Case 1a. $\alpha \neq 0$. Let $y \in Y$. Then

$$[a(\lambda T)_x(y)] = (\lambda T)_x \left( \frac{y}{\alpha \lambda} \right) = (\lambda T)_x \left( \frac{y}{\alpha} \right) = T \left( x, \frac{y}{\alpha \lambda} \right) = T \left( x, \frac{y}{\alpha} \right) = T_x \left( x, \frac{y}{\alpha \lambda} \right).$$
Case 1b. \( \alpha = 0 \). For \( y \neq 0 \), we have that \( \{0(\lambda T)_{\alpha}, y\} = 0 \). Thus \( \{0(\lambda T)_{\alpha}, y\} \leq (\lambda T)_{\alpha\alpha}(y) \). If \( y = 0 \), we have

\[
\{0(\lambda T)_{\alpha}, y\} = \sup_{z \in Y} (\lambda T)(z) = \sup_{z \in Y} (\lambda T)(x, z) = D(\lambda T)(x) = D(T)(x) = \sup T(x, 0) \leq T(0, 0) = (\lambda T)(0, 0) = (\lambda T)_{\alpha\alpha}(0).
\]

Hence \( \{0(\lambda T)_{\alpha}, y\} \leq (\lambda T)_{\alpha\alpha}(y), (\forall) y \in Y \). Thus \( 0(\lambda T)_{\alpha} \subseteq (\lambda T)_{\alpha\alpha} \).

Case 2. \( \lambda = 0 \). Then

\[
T(x, y) = \begin{cases} 
D(T)x & \text{if } y = 0 \\
0 & \text{if } y \neq 0.
\end{cases}
\]

Therefore

\[
\{0(\lambda T)_{\alpha}, y\} = \begin{cases} 
\{0(\lambda T)_{\alpha}, y\} & \text{if } y = 0 \\
0 & \text{if } y \neq 0.
\end{cases}
\]

On the other hand

\[
(0T)(x_1 + x_2, y) = \begin{cases} 
D(T)(x_1 + x_2) & \text{if } y = 0 \\
0 & \text{if } y \neq 0.
\end{cases}
\]

We remark that \( D(T)(x_1) \wedge D(T)(x_2) \leq D(T)(x_1 + x_2) \) and thus \( \{0(\lambda T)_{\alpha}, y\} \subseteq (\lambda T)_{\alpha\alpha} \). Indeed,

\[
D(T)(x_1) \wedge D(T)(x_2) = \sup_{y \in Y} T(x_1, y) \wedge \sup_{y \in Y} T(x_2, y) = \sup_{y_1, y_2 \in Y} T(x_1 + x_2, y_1 + y_2) = D(T)(x_1 + x_2).
\]

It rest to show that \( \alpha(0T)_{\alpha} \subseteq (\lambda T)_{\alpha\alpha} \).

Case 2a. \( \alpha \neq 0 \). Then

\[
\alpha(0T)_{\alpha}(y) = (0T)_{\alpha} \begin{bmatrix} y \\ \alpha \end{bmatrix} = (0T) \begin{bmatrix} x & y \\ \alpha \end{bmatrix} = \begin{cases} D(T)x & \text{if } y = 0 \\
0 & \text{if } y \neq 0.\end{cases} = (0T)(x, y).
\]

On the other hand

\[
(\lambda T)_{\alpha\alpha}(y) = (\lambda T)(\alpha x, y) = \begin{cases} D(T)(\alpha x) & \text{if } y = 0 \\
0 & \text{if } y \neq 0.
\end{cases}
\]

In order to establish the inclusion \( \alpha(0T)_{\alpha} \subseteq (\lambda T)_{\alpha\alpha} \), we must show that \( D(T)x \leq D(T)(\alpha x) \). But

\[
D(T)(\alpha x) = \sup_{y \in Y} T(x, y) = \sup_{y \in Y} T \left( x, \frac{y}{\alpha} \right) = \sup_{y \in Y} \left( x, \frac{y}{\alpha} \right) \sup T(x, y') = D(T)x.
\]

Case 2b. \( \alpha = 0 \). For \( y \neq 0 \), we have \( \{0(\lambda T)_{\alpha}, y\} = 0 \). Thus \( \{0(\lambda T)_{\alpha}, y\} \leq (\lambda T)_{\alpha\alpha}(y) \). If \( y = 0 \), then

\[
\{0(\lambda T)_{\alpha}, y\} = \sup_{z \in Z} (\lambda T)(z) = \sup_{z \in Z} (\lambda T)(x, z) = (\lambda T)(x, 0) = D(T)(x)
\]

and

\[
(\lambda T)_{\alpha\alpha}(0) = (\lambda T)(0, 0) = D(T)(0) = \sup_{y \in Y} T(0, y) = T(0, 0).
\]

As \( D(T)(x) = \sup_{y \in Y} T(x, y) \leq T(0, 0) \), we obtain that \( \{0(\lambda T)_{\alpha}, y\} \leq (\lambda T)_{\alpha\alpha}(y) \). Thus

\[
\{0(\lambda T)_{\alpha}, y\} = (\lambda T)_{\alpha\alpha}(y), (\forall) y \in Y.
\]

Hence \( 0(\lambda T)_{\alpha} \subseteq (\lambda T)_{\alpha\alpha} \).
References