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On Relation $\mathcal{Q}_\gamma$ in *le*-Γ-Semigroups

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Abstract. In this paper we introduce and study the relation $\mathcal{Q}_\gamma$ in *le*-Γ-semigroups. This relation in general turns out to have better properties than the relation $\mathcal{H}_\gamma$ studied in [10]. We give several properties that hold in every $\mathcal{Q}_\gamma$-class of an *le*-Γ-semigroup and especially in every $\mathcal{Q}_\gamma$-class satisfying the Green's condition. In particular, the γ-regularity and γ-intra-regularity of a $\mathcal{Q}_\gamma$-class is studied. We also consider a case a $\mathcal{Q}_\gamma$-class of an *le*-Γ-semigroup M forms a subsemigroup of $M_\gamma = (M, \circ)$.

1. Introduction and Preliminaries

In [10], it is studied the relation $\mathcal{H}_\gamma$ and investigated several properties that hold in every $\mathcal{H}_\gamma$-classes of an *le*-Γ-semigroup satisfying the so-called Green's condition and a necessary and sufficient condition when an $\mathcal{H}_\gamma$-class H of an *le*-Γ-semigroup M is a subgroup of $M_\gamma = (M, \circ)$ is provided. In [10], there are also provided several conditions that ensure that an $\mathcal{H}_\gamma$-class forms a subsemigroup of M_γ extending and generalizing those for *le*-semigroups studied in [7].

In [1], it is introduced and studied the relation $\mathcal{B}_\gamma$ which turns out to be finer that $\mathcal{H}_\gamma$. This means that each $\mathcal{H}_\gamma$-class can be partitioned into $\mathcal{B}_\gamma$-classes. An investigation of several properties that hold in every $\mathcal{B}_\gamma$-classes have been provided and also several results which shows that the relation $\mathcal{B}_\gamma$ may be a better candidate than $\mathcal{H}_\gamma$ for developing the structure theory for *le*-Γ-semigroups have been proved. It has been showed that the Green's condition is sufficient for a $\mathcal{B}_\gamma$-class to be γ-regular and γ-intra-regular. Also, in [1], several conditions were found ensuring that an $\mathcal{B}_\gamma$-class of an *le*-Γ-semigroup M forms a subsemigroup in $M_\gamma = (M, \circ)$.

The aim of this paper is to introduce and study the relation $\mathcal{Q}_\gamma$ in *le*-Γ-semigroups that mimics the relation $\mathcal{Q}$ in *le*-semigroups [4]. This relation in general turns out to have better properties than the relation $\mathcal{H}_\gamma$ studied in [10]. We give several properties that hold in every $\mathcal{Q}_\gamma$-class of an *le*-Γ-semigroup and especially in every $\mathcal{Q}_\gamma$-class satisfying the Green's condition. In particular, the γ-regularity and γ-intra-regularity of a $\mathcal{Q}_\gamma$-class is studied. We also consider a case a $\mathcal{Q}_\gamma$-class of an *le*-Γ-semigroup M forms a subsemigroup of $M_\gamma = (M, \circ)$ (cf. Theorem 5.3).

We introduce below necessary notions and present a few auxiliary results that will be used throughout the paper.

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In 1986, Sen and Saha [9] defined Γ-semigroup as a generalization of semigroup and ternary semigroup. We give the definition of Γ-semigroup in a different way as follows:

Definition 1.1. *Let R and Γ be two non-empty sets. Any map from $R \times \Gamma \times R \to R$ will be called a Γ-multiplication in R and denoted by $(\cdot)_\Gamma$. The result of this multiplication for $a, b \in R$ and $\alpha \in \Gamma$ is denoted by $a\alpha b$. A Γ-semigroup R is an ordered pair $(R, (\cdot)_\Gamma)$ where R and Γ are non-empty sets and $(\cdot)_\Gamma$ is a Γ-multiplication on R which satisfies the following property : $\forall (a, b, c, \alpha, \beta) \in R^3 \times \Gamma^2, (a\alpha b)\beta c = a\alpha(b\beta c)$.*

Example 1.2. *Let M be a semigroup and Γ be any non-empty set. Define a mapping $M \times \Gamma \times M \to M$ by $a\gamma b = ab$ for all $a, b \in M$ and $\gamma \in \Gamma$. Then M is a Γ-semigroup.*

Example 1.3. *Let M be a set of all negative rational numbers. Obviously M is not a semigroup under usual product of rational numbers. Let $\Gamma = \{-\frac{1}{p} | p \text{ is prime}\}$. Let $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. Now if $a\alpha b$ is equal to the usual product of rational numbers a, α, b, then $a\alpha b \in M$ and $(a\alpha b)\beta c = a\alpha(b\beta c)$. Hence M is a Γ-semigroup.*

Example 1.4. *Let $M = \{-i, 0, i\}$ and $\Gamma = M$. Then M is a Γ-semigroup under the multiplication over complex numbers while M is not a semigroup under complex number multiplication.*

These examples show that every semigroup is a Γ-semigroup. Therefore, Γ-semigroups are a generalization of semigroups.

An element a of a Γ-semigroup M is called a *γ-idempotent* if exists $\gamma \in \Gamma$, $a\gamma a = a$.

For non-empty subsets A and B of M and a non-empty subset Γ' of Γ, let $A\Gamma' B = \{a\gamma b | a \in A, b \in B$ and $\gamma \in \Gamma'\}$. If $A = \{a\}$, then we also write $a\Gamma' B$ instead of $\{a\}\Gamma' B$, and similarly if $B = \{b\}$ or $\Gamma' = \{\gamma\}$.

A Γ-semigroup M is called *commutative Γ-semigroup* if for all $a, b \in M$ and $\gamma \in \Gamma$, $a\gamma b = b\gamma a$. A non-empty subset K of a Γ-semigroup M is called a *sub-Γ-semigroup* of M if for all $a, b \in K$ and $\gamma \in \Gamma$, $a\gamma b \in K$.

Example 1.5. *Let $M = [0, 1]$ and $\Gamma = \{\frac{1}{n} | n \text{ is a positive integer}\}$. Then M is a Γ-semigroup under usual multiplication. Let $K = [0, \frac{1}{2}]$. We have that K is a nonemtpy subset of M and $a\gamma b \in K$ for all $a, b \in K$ and $\gamma \in \Gamma$. Then K is a sub-Γ-semigroup of M.*

Let M be a Γ-semigroup and γ be a fixed element of Γ. In [9] is defined $a \circ b$ in M by $a \circ b = a\gamma b$ for all $a, b \in M$ and is shown that $(M, \circ)$ is a semigroup and this semigroup is denoted by M_γ. Also, it is shown that if M_γ is a group for some $\gamma \in \Gamma$, then M_γ is a group for all $\gamma \in \Gamma$. A Γ-semigroup M is called a *Γ-group* if M_γ is a group for some (hence for all) $\gamma \in \Gamma$ [9].

Definition 1.6. *A po-Γ-semigroup is an ordered set M at the same time Γ-semigroup such that for all $c \in M$ and for all $\gamma \in \Gamma$*

$$a \leq b \Rightarrow a\gamma c \leq b\gamma c, c\gamma a \leq c\gamma b.$$

A *poe*-Γ-semigroup is a *po*-Γ-semigroup M with a greatest element "e" (i.e., for all $a \in M, e \geq a$).

In a *po*-Γ-semigroup M, for any $\gamma \in \Gamma$, the element a is called a *γ-right* (resp. *γ-left*) *ideal element* if for all $b \in M$, $a\gamma b \leq a$ (resp. $b\gamma a \leq a$). And a is called a *γ-ideal element* if it is both a γ- right and γ-left ideal element. In a *poe*-Γ-semigroup M, for any $\gamma \in \Gamma$, a is called a *γ-right* (resp. *γ-left*) *ideal element* if $a\gamma e \leq a$ (resp. $e\gamma a \leq a$).

For $A \subseteq M$, we denote

$$(A] = \{t \in M | t \leq a \text{ for some } a \in A\}.$$

An element a of a *poe*-Γ-semigroup is called a *γ-quasi-ideal element* if $e\gamma a \wedge a\gamma e$ exists for all $\gamma \in \Gamma$ and $a\gamma e \wedge e\gamma a \leq a$. The *$\gamma$-zero* of a *poe*-$\Gamma$-semigroup M is an element of M denoted by 0_γ such that for every $a \in M$, $e \neq 0_\gamma \leq a$ and $0_\gamma \gamma a = a\gamma 0_\gamma = 0_\gamma$ for all $\gamma \in \Gamma$. Let M be a *poe*-Γ-semigroup with 0_γ. A γ-quasi-ideal element a of M is called *minimal* if $a \neq 0_\gamma$ and there exists no γ-quasi-ideal element t of M such that $0_\gamma < t < a$. We say that $a \in M$ is a *γ-bi-ideal element* of M if and only if $a\gamma e\gamma a \leq a$.

Definition 1.7. *Let M be a semilattice under $\vee$ with a greatest element e and at the same time a po-Γ-semigroup such that for all $a, b, c \in M$ and for all $\gamma \in \Gamma$,*

$$a\gamma(b \vee c) = a\gamma b \vee a\gamma c$$

and

$$(a \vee b)\gamma c = a\gamma c \vee b\gamma c.$$

Then M is called a $\vee e - \Gamma$-semigroup.

A $\vee e$-Γ-semigroup which is also a lattice is called an *le*-Γ-*semigroup*.

Throught this paper M will stand for an *le*-Γ-semigroup. The usual order relation $\leq$ on M is defined in the following way

$$a \leq b \Leftrightarrow a \vee b = b.$$

Then we can show that for any $a, b, c \in M$ and $\gamma \in \Gamma$, $a \leq b$ implies $a\gamma c \leq b\gamma c$ and $c\gamma a \leq c\gamma b$.

Example 1.8. *[2] Let $(X, \leq)$ and $(Y, \leq)$ be two finite chains. Let M be the set of all isotone mappings from X into Y and Γ be the set of all isotone mappings from Y into X. Let $f, g \in M$ and $\alpha \in \Gamma$. We define $f\alpha g$ to denote the usual mapping composition of f, α and g. Then M is a Γ-semigroup. For $f, g \in M$, the mappings $f \vee g$ and $f \wedge g$ are defined by letting, for each $a \in X$,*

$$(f \vee g)(a) = max\{f(a), g(a)\}, (f \wedge g)(a) = min\{f(a), g(a)\}$$

(the maximum and minimum are considered with respect to the order $\leq$ in X and Y). The greatest element e is the mapping that sends every $a \in X$ to the greatest element of finite chains $(Y, \leq)$. Then M is an le-Γ-semigroup.

Example 1.9. *[2] Let M be a po-Γ-semigroup. Let M_1 be the set of all ideals of M. Then $(M_1, \subseteq, \cap, \cup)$ is an le-Γ-semigroup.*

Example 1.10. *[2] Let M be a po-Γ-semigroup. Let $M_1 = P(M)$ be the set of all subsets of M and $\Gamma_1 = P(\Gamma)$ the set of all subsets of Γ. Then M_1 is a po-Γ_1-semigroup if*

$$A\Lambda B = \begin{cases} (A\Lambda B] & \text{if } A, B \in M_1\backslash\{\emptyset\}, \Lambda \in \Gamma_1\backslash\{\emptyset\}, \\ \emptyset & \text{if } A = \emptyset \text{ or } B = \emptyset. \end{cases}$$

Then $(M_1, \subseteq, \cap, \cup)$ is an le-Γ_1-semigroup.

Example 1.11. *[3] Let G be a group, $I, \wedge$ two index sets and Γ the collection of some $\wedge \times I$ matrices over $G^o = G \cup \{0\}$, the group with zero. Let μ^o be the set of all elements $(a)_{i\lambda}$ where $i \in I, \lambda \in \wedge$ and $(a)_{i\lambda}$ the $I \times \wedge$ matrix over G^o having a in the i-th row and λ-th column, its remaining entries being zero. The expression $(0)_{i\lambda}$ will be used to denote the zero matrix. For any $(a)_{i\lambda}, (b)_{j\mu}, (c)_{k\nu} \in \mu^o$ and $\alpha = (p_{\lambda i}), \beta = (q_{\lambda i}) \in \Gamma$ we define $(a)_{i\lambda}\alpha(b)_{j\mu} = (ap_{\lambda j}b)_{i\mu}$. Then it is easy verified that $[(a)_{i\lambda}\alpha(b)_{j\mu}]\beta(c)_{k\nu} = (a)_{i\lambda}\alpha[(b)_{j\mu}\beta(c)_{k\nu}]$. Thus μ^o is a Γ-semigroup. We call Γ the sandwich matrix set and μ^o the Rees $I \times \wedge$ matrix Γ-semigroup over G^o with sandwich matrix set Γ and denote it by $\mu^o(G : I, \wedge, \Gamma)$. In [3], we deal with lattice-ordered Rees matrix Γ-semigroups.*

In [10], for any $\gamma \in \Gamma$, two mappings r_γ and l_γ are defined by for any $x \in M$ as follows:

$r_\gamma : M \to M, r_\gamma(x) = x\gamma e \vee x,$
$l_\gamma : M \to M, l_\gamma(x) = e\gamma x \vee x.$

In [1], we have defined in a $\vee e - \Gamma$-semigroup M for all $a \in M$ and for any $\gamma \in \Gamma$ the mappings q_γ and b_γ as follows:

$b_\gamma : M \to M, b_\gamma(x) = x \vee x\gamma e\gamma x$
$q_\gamma : M \to M, q_\gamma(x) = x \vee (e\gamma x \wedge x\gamma e)$

In an arbitrary *le*-Γ-semigroup M, the Green's relations are defined in [10] as follows:

$$\begin{aligned}
\mathcal{L}_\gamma &= \{(x,y)\in M^2 | e\gamma x \vee x = e\gamma y \vee y\} \\
&\textit{or} \\
\mathcal{L}_\gamma &= \{(x,y)\in M^2 | l_\gamma(x) = l_\gamma(y)\}, \\
\mathcal{R}_\gamma &= \{(x,y)\in M^2 | x\gamma e \vee x = y\gamma e \vee y\} \\
&\textit{or} \\
\mathcal{R}_\gamma &= \{(x,y)\in M | r_\gamma(x) = r_\gamma(y)\}, \\
\mathcal{H}_\gamma &= \mathcal{L}_\gamma \cap \mathcal{R}_\gamma.
\end{aligned}$$

It is clear that an element $x \in M$ is a γ-quasi [resp. bi, left, right] ideal element if $q_\gamma(x) = x$ [resp. $b_\gamma(x) = x, l_\gamma(x) = x, r_\gamma(x) = x$].

One can easily verify that for every $x \in M$, the elements $l_\gamma(x)$, $r_\gamma(x)$, $q_\gamma(x)$, $b_\gamma(x)$ are respectively the least γ-left, γ-right, γ-quasi and γ-bi-ideal elements above the x.

An element x of an *le*-Γ-semigroup M is called *γ-regular* [10] if and only if $x \le x\gamma l_\gamma(x)$ or equivalently, $x \le x\gamma e\gamma x$. An *le*-Γ-semigroup M is called *γ-regular* [10] if and only if every element of M is γ-regular. An element x of an *le*-Γ-semigroup M is called *γ-intra-regular* if and only if $x \le e\gamma x\gamma x\gamma e$. An *le*-Γ-semigroup M is called *γ-intra-regular* if and only if every element of M is γ-intra-regular.

An $\mathcal{H}_\gamma$-class H of Γ-semigroup M satisfy Green's condition if there exist elements x and y of H such that $x\gamma y \in H$ [10].

Lemma 1.12. *Let M be an le-Γ-semigroup. For each $x \in M$ and $\gamma \in \Gamma$, we have $q_\gamma(q_\gamma(x)) = q_\gamma(x)$.*

Proof. In fact,

$$\begin{aligned}
q_\gamma(q_\gamma(x)) &= q_\gamma(x \vee (x\gamma e \wedge e\gamma x)) = (x \vee (x\gamma e \wedge e\gamma x) \vee (e\gamma(x \vee (x\gamma e \wedge e\gamma x)) \\
&\quad \wedge(x \vee (x\gamma e \wedge e\gamma x))\gamma e = (x \vee (x\gamma e \wedge e\gamma x)) \vee ((e\gamma x \vee e\gamma(x\gamma e \wedge e\gamma x)) \\
&\quad \wedge(x\gamma e \vee (x\gamma e \wedge e\gamma x)\gamma e)) = (x \vee (x\gamma e \wedge e\gamma x)) \vee (e\gamma x \wedge x\gamma e) = \\
&\quad (x \vee (x\gamma e \wedge e\gamma x)) = q_\gamma(x).
\end{aligned}$$

If $a \in M$ is a γ-left ideal element and $b \in M$ is a γ-right ideal element, then as shown in Lemma 1.2 [10], $a \wedge b$ is a γ-quasi-ideal element of M.

If M is a distributive *le*-Γ-semigroup, then every quasi-ideal element is the intersection of a γ-right ideal element with a γ-left ideal element. Indeed: $q = q \vee (q\gamma e \wedge e\gamma q) = (q \vee e\gamma q) \wedge (q \vee q\gamma e) = l_\gamma(q) \wedge r_\gamma(q)$ and from Lemma 1.12, we have the desired result. □

Definition 1.13. *A γ-quasi-ideal element of an le-Γ-semigroup is said to have the intersection property if it is expressed as an intersection of a γ-left ideal element and a γ-right ideal element.*

Lemma 1.14. *The γ-quasi-ideal element q of an le-Γ-semigroup M, has the intersection property if and only if $q = l_\gamma(q) \wedge r_\gamma(q)$.*

Proof. If $q = a \wedge b$ where $a = l_\gamma(a)$ and $b = r_\gamma(b)$, then $l_\gamma(q) = l_\gamma(a \wedge b) \le l_\gamma(a)$ and $r_\gamma(q) = r_\gamma(a \wedge b) \le r_\gamma(b)$. Consequently, $q = l_\gamma(a) \wedge r_\gamma(b) \ge l_\gamma(q) \wedge r_\gamma(q)$. On the other hand, $q \le l_\gamma(q) \wedge r_\gamma(q)$ since $q = q \vee (q\gamma e \wedge e\gamma q) \le q \vee e\gamma q = l_\gamma(q)$ and $q = q \vee (q\gamma e \wedge e\gamma q) \le q \vee q\gamma e = r_\gamma(q)$.

The converse is evident (cf. Lemma 1.2 [10]). □

We observe here that if $q = q_\gamma(a) = a \vee (a\gamma e \wedge e\gamma a)$, then

$$\begin{aligned}
l_\gamma(q) &= l_\gamma(a \vee (a\gamma e \wedge e\gamma a)) = a \vee (a\gamma e \wedge e\gamma a) \vee e\gamma(a \vee (a\gamma e \wedge e\gamma a)) = l_\gamma(a), \\
r_\gamma(q) &= l_\gamma(a \vee (a\gamma e \wedge e\gamma a)) = a \vee (a\gamma e \wedge e\gamma a) \vee (a \vee (a\gamma e \wedge e\gamma a))\gamma e = r_\gamma(a).
\end{aligned}$$

Whence $q = r_\gamma(a) \wedge l_\gamma(a)$ in case $q = q_\gamma(a)$.

2. The Relation $\mathcal{Q}_\gamma$ in *le*-Γ-semigroups

We define now the following equivalence relation $\mathcal{Q}_\gamma$ in *le*-Γ-semigroup M:

$$\begin{aligned}\mathcal{Q}_\gamma &= \{(x,y)\in M^2 | x\vee(x\gamma e\wedge e\gamma x) = y\vee(y\gamma e\wedge e\gamma y)\},\\ &\mathit{or}\\ \mathcal{Q}_\gamma &= \{(x,y)\in M^2 | q_\gamma(x) = q_\gamma(y)\}.\end{aligned}$$

It can be easily proved the following lemma.

Lemma 2.1. *Let M be an le-Γ-semigroup. Then $\mathcal{Q}_\gamma \subseteq \mathcal{H}_\gamma$.*

Remark 1. Concerning the above lemma we notice that : *In distributive le-Γ-semigroups, $\mathcal{Q}_\gamma = \mathcal{H}_\gamma$.* In fact, let M be a distributive *le*-Γ-semigroup and $(x,y)\in\mathcal{Q}_\gamma$. Since $q_\gamma(x) = q_\gamma(y)$, we have for all $\gamma\in\Gamma$,

$$x\vee(x\gamma e\wedge e\gamma x) = y\vee(y\gamma e\wedge e\gamma y).$$

Thus

$$x\le y\vee(y\gamma e\wedge e\gamma y) \text{ and } y\le x\vee(x\gamma e\wedge e\gamma x).$$

Then, for all $\gamma\in\Gamma$,

$$\begin{aligned}x\gamma e\vee x &\le (y\vee(y\gamma e\wedge e\gamma y))\gamma e\vee y\vee(y\gamma e\wedge e\gamma y)\\ &= y\gamma e\vee(y\gamma e\wedge e\gamma y)\gamma e\vee y\vee(y\gamma e\wedge e\gamma y)\\ &= y\gamma e\vee y,\end{aligned}$$

similarly, we have $e\gamma x\vee x\le e\gamma y\vee y$. From $y\le x\vee(x\gamma e\wedge e\gamma x)$, by symmetry, we have $y\gamma e\vee y\le x\gamma e\vee x$ and $e\gamma y\vee y\le e\gamma x\vee x$. Hence, $(x,y)\in\mathcal{R}_\gamma\cap\mathcal{L}_\gamma = \mathcal{H}_\gamma$. Let now M be a distributive *le*-Γ-semigroup and $(x,y)\in\mathcal{H}_\gamma$. Since $(x,y)\in\mathcal{R}_\gamma$ and $(x,y)\in\mathcal{L}_\gamma$, we get $x\gamma e\vee x = y\gamma e\vee y$ and $e\gamma x\vee x = e\gamma y\vee y$. Then

$$(x\gamma e\vee x)\wedge(e\gamma x\vee x) = (y\gamma e\vee y)\wedge(e\gamma y\vee y).$$

Since M is distributive, we have

$$x\vee(x\gamma e\wedge e\gamma x) = y\vee(y\gamma e\wedge e\gamma y),$$

that is $(x,y)\in\mathcal{Q}_\gamma$.

Lemma 2.2. *Let M be an le-Γ-semigroup. Each $\mathcal{Q}_\gamma$-class Q of M contains a unique γ-quasi-ideal element which is the greatest element of the class.*

Proof. For every element $x\in Q$, by Lemma 1.12 and the definition of relation $\mathcal{Q}_\gamma$, we have $q_\gamma(x)\in Q$. If z is a γ-quasi-ideal element belonging to Q, then $q_\gamma(x) = q_\gamma(z) = z$, which shows that $q_\gamma(x)$ is the only γ-quasi-ideal element of the class. Since $x\le q_\gamma(x)$, we see that $q_\gamma(x)$ is the greatest element of Q. □

Lemma 2.2 implies that for each $x\in M$, the γ-quasi-ideal element $q_\gamma(x)$ depends on the $\mathcal{Q}_\gamma$-class Q of x rather than on x itself. We call the γ-quasi-ideal element $q_\gamma(x)$ the *representative γ-quasi-ideal element* of the $\mathcal{Q}_\gamma$-class Q and denote it by q_Q. So, we have two kind of quasi-ideal elements: the representative quasi-ideal elements of the $\mathcal{H}_\gamma$-classes defined in [10] and the above. Since each quasi-ideal element is included in a $\mathcal{Q}_\gamma$-class and since Q_γ contains only one quasi-ideal element, we obtain that the set of quasi-ideal elements of an *le*-Γ-semigroup coincide with the set of the representative quasi-ideal elements of the $\mathcal{Q}_\gamma$-classes.

The following proposition gives a sufficient and neccessary condition for an *le*-Γ-semigroup under which the relations $\mathcal{H}_\gamma$ and $\mathcal{Q}_\gamma$ coincide.

Proposition 2.3. *Let M be an le-Γ-semigroup. The relations $\mathcal{H}_\gamma$ and $\mathcal{Q}_\gamma$ coincide if and only if the set of all quasi-ideal elements have the intersection property.*

Proof. If $a\mathcal{H}b$, then $l_\gamma(a) = l_\gamma(b)$ and $r_\gamma(a) = r_\gamma(b)$. Since the quasi-ideal $l_\gamma(a) \wedge r_\gamma(a) = l_\gamma(b) \wedge r_\gamma(b)$ has the intersection property and by Lemma 1.14, we have

$$q_\gamma(a) = l_\gamma(a) \wedge r_\gamma(a) = l_\gamma(b) \wedge r_\gamma(b) = q_\gamma(b),$$

which means that $a\mathcal{Q}_\gamma b$, whence $\mathcal{H}_\gamma \subseteq \mathcal{Q}_\gamma$. By Lemma 2.1, we get $\mathcal{H}_\gamma = \mathcal{Q}_\gamma$.

Conversely, let q be a quasi-ideal element of M. It is certainly the representative quasi-ideal element of a Q_a for a certain $a \in M$. By the assumption, we have $Q_a = H_a$, then by Lemma 1.4 [10], we may write $q = l_\gamma(a) \wedge r_\gamma(a)$ which shows that q has the intersection property. □

The following proposition gives a sufficient condition for an *le*-Γ-semigroup M in order that the relations $\mathcal{H}_\gamma$ and $\mathcal{Q}_\gamma$ coincide in M.

Proposition 2.4. *If M is a γ-regular le-Γ-semigroup, then $\mathcal{H}_\gamma = \mathcal{Q}_\gamma$.*

Proof. For every γ-quasi-ideal element q due to γ-regularity, we have $q \leq q\gamma e\gamma q \leq q\gamma e \wedge e\gamma q$, hence $l_\gamma(q) = q \vee e\gamma q = e\gamma q$ and $r_\gamma(q) = q \vee q\gamma e = q\gamma e$. It follows that $q\gamma e \wedge e\gamma q \leq q \leq q\gamma e \wedge e\gamma q$, therefore $q = q\gamma e \wedge e\gamma q = l_\gamma(q) \wedge r_\gamma(q)$ which means that q has the intersection property. Proposition 2.3 implies $H_\gamma = Q_\gamma$. □

3. $\mathcal{Q}_\gamma$-classes Satisfying Green's Condition

We say that a $\mathcal{Q}_\gamma$-class Q of an *le*-Γ-semigroup M satisfies the Green's condition if there exist elements $a, b \in Q$ such that $a\gamma b \in Q$.

Lemma 3.1. *If the $\mathcal{Q}_\gamma$-class Q_a of an le-Γ-semigroup M satisfies the Green's condition, then $Q_a = H_a$.*

Proof. Since $Q_a \subseteq H_a$, we have that H_a satisfies the Green's condition. Theorem 2.1 [10] implies that H_a contains the quasi-ideal $q = q_H = l_\gamma(a) \wedge r_\gamma(a)$ which is the only quasi-ideal element of H_a. On the other hand, since $q_\gamma(q_\gamma(a)) = q_\gamma(a)$, the quasi-ideal element $q_\gamma(a)$ belongs to the $\mathcal{Q}_\gamma$-class Q_a. Hence $q_H = l_\gamma(a) \wedge r_\gamma(a) = q_\gamma(a)$. For each $x \in H_a$, we have $x \in Q_x \subseteq H_a$ and consequently $q_\gamma(x) = l_\gamma(a) \wedge r_\gamma(a) = q_\gamma(a)$, which means that $x \in Q_a$. Therefore $H_a \subseteq Q_a$. Thus $Q_a = H_a$. □

Using the above lemma, we obtain the following analogue of Theorem 2.1 [10].

Theorem 3.2. *Let M be an le-Γ-semigroup. If Q is a $\mathcal{Q}_\gamma$-class of M satisfying the Green's condition and let $q = l_\gamma(a) \wedge r_\gamma(a)$ where $a \in Q$. Then:*

1. *$q\gamma q \in Q$ and $q = q\gamma e \wedge e\gamma q$;*
2. *q is the only γ-quasi-ideal element of Q;*
3. *if $x, y \in Q$, then $y \leq x\gamma e$ and $y \leq e\gamma x$;*
4. *$q\gamma q = q\gamma e\gamma q = (q\gamma)^{n-1}q$ for all integers $n \geq 2$; in particular, $q\gamma q$ is γ-idempotent;*
5. *every element of Q is γ-intra-regular;*
6. *$q = q\gamma q$ if and only if q is γ-regular in which case every element of $Q = H_q$ is γ-regular.*

An immediate corollary of the Theorem 3.2 is the following.

Corollary 3.3. *A $\mathcal{Q}_\gamma$-class Q satisfies the Green's condition if and only if it contains a γ-idempotent element.*

Theorem 3.4. *A $\mathcal{Q}_\gamma$-class of an le-Γ-semigroup M is a subgroup of M_γ if and only if it consists of a single γ-idempotent element.*

Proof. The "if" part is obvious. Assume that Q is a subgroup of M_γ. It satisfies the Green's condition and as a result it coincides with the $\mathcal{H}_\gamma$-class of any of its elements. The result follows by Theorem 2.3 [10]. □

4. γ-Regularity and γ-intra-regularity of $\boldsymbol{Q}_\gamma$-classes

In this section we give some necessary and sufficient conditions for a $\boldsymbol{Q}_\gamma$-class to be γ-regular or γ-intra-regular.

Proposition 4.1. *Let M be an le-Γ-semigroup. A $\boldsymbol{Q}_\gamma$-class Q_a of M is γ-regular if and only if the representative γ-quasi-ideal element $q_\gamma(a)$ of Q_a is γ-regular element.*

Proof. It is clear that in general, a γ-quasi-ideal element $q \in M$ is γ-regular if and only if $q = q\gamma e\gamma q$. Thus the γ-regularity of $q_\gamma(a)$ implies that

$$\begin{aligned} q_\gamma(a) &= q_\gamma(a)\gamma e\gamma q_\gamma(a) = (a \vee (a\gamma e \wedge e\gamma a))\gamma e\gamma(a \vee (a\gamma e \wedge e\gamma a)) = \\ &\quad a\gamma e\gamma a \vee a\gamma e(a\gamma e \wedge e\gamma a) \vee (a\gamma e \wedge e\gamma a)\gamma e\gamma a \vee \\ &\quad \vee(a\gamma e \wedge e\gamma a)\gamma e\gamma(a\gamma e \wedge e\gamma a) = a\gamma e\gamma a. \end{aligned}$$

Since $a \le q_\gamma(a)$, we have $a \le a\gamma e\gamma a$ which means that a is γ-regular.
The converse is obvious. □

Proposition 4.2. *The $\boldsymbol{Q}_\gamma$-class Q_a of an le-Γ-semigroup is γ-intra-regular if and only if the representative γ-quasi-ideal element $q_\gamma(a)$ of Q_a is γ-intra-regular.*

Proof. The inequalities

$$\begin{aligned} a &\le q_\gamma(a) \le e\gamma q_\gamma(a)\gamma q_\gamma(a)\gamma e = e\gamma(a \vee (a\gamma e \wedge e\gamma a)\gamma(a \vee (a\gamma e \wedge e\gamma a)\gamma e = \\ &\quad e\gamma(a\gamma a \vee a\gamma(a\gamma e \wedge e\gamma a) \vee (a\gamma e \wedge e\gamma a)\gamma a \vee (a\gamma e \wedge e\gamma a)\gamma(a\gamma e \wedge e\gamma a))\gamma e = \\ &\quad e\gamma a\gamma a\gamma e \end{aligned}$$

show that a is γ-intra-regular as desired.
The converse is obvious. □

Proposition 4.3. *Let M be an le-Γ-semigroup. If B_x and B_y are two γ-regular $\mathcal{B}_\gamma$-classes contained in the same $\boldsymbol{Q}_\gamma$-class of M, then they coincide.*

Proof. From the γ-regularity of both x and y, we have $b_\gamma(x) = x\gamma e\gamma x$ and $b_\gamma(y) = y\gamma e\gamma y$. Since x and y are in the same $\boldsymbol{Q}_\gamma$-class, Lemma 1.10 [1] yields $x\gamma e\gamma x = y\gamma e\gamma y$. Hence we have $b_\gamma(x) = b_\gamma(y)$ and consequently $(x, y) \in \mathcal{B}_\gamma$. □

In [8], Theorem 2 shows a nice situation in Γ-semigroups concerning the transmission of regularity from elements to subsets, that is, if an element is regular, then the whole $\mathcal{D}_\gamma$-class containing it is γ-regular too. In contrast with the Γ-semigroup case, the Proposition 4.3 shows that in *le*-Γ-semigroups, the γ-regularity of a $\boldsymbol{Q}_\gamma$-class Q is "localized" in a unique $\mathcal{B}_\gamma$-class B contained in Q, that is, an element x of M is γ-regular together with its own $\mathcal{B}_\gamma$-class B_x and none of the other $\mathcal{B}_\gamma$-classes included in Q_x (if there is any) is γ-regular. The following problem arises:

Problem 1 *Does γ-regularity of an element x imply γ-regularity of Q_x, or equivalently, does it imply $B_x = Q_x$?*

An approach to find a non-γ-regular $\boldsymbol{Q}_\gamma$-class containing a γ-regular element would be to construct an *le*-Γ-semigroup with a non-γ-regular $\boldsymbol{Q}_\gamma$-class satisfying the Green's condition.

Problem 2 *Is there an le-Γ-semigroup containing a $\boldsymbol{Q}_\gamma$-class that satisfies the Green's condition but is not γ-regular?*

The following Proposition [1, Proposition 2.9] has been proved and it gives us a sufficient condition under which γ-bi-ideal elements and γ-quasi-ideal elements coincide.

Proposition 4.4. *Let M be an le-Γ-semigroup. If for each $\mathcal{H}_\gamma$-class H of M the $\mathcal{B}_\gamma$-class B_q of the representative γ-quasi-ideal element $q = q_H$ satisfies the Green's condition, then the γ-quasi-ideal elements and the γ-bi-ideal elements of M coincide.*

5. Minimal γ-quasi-ideal and γ-bi-ideal Elements in *le*-Γ-semigroups

In [10], it is proved the following result.

Proposition 5.1. *[10, Proposition 2.9] Let H be an $\mathcal{H}_\gamma$-class of M such that its representative γ-quasi-ideal element $q = q_H$ is minimal in the set of all γ-quasi-ideal elements of M. Then $H = (q] = \{a \in M | a \leq q\}$ and H is a subsemigroup of M_γ.*

The following Theorem proved in [1] gives a sufficient condition, under which a $\mathcal{B}_\gamma$-class or an $\mathcal{H}_\gamma$-class of an *le*-Γ-semigroup M is a subsemigroup of M_γ.

Theorem 5.2. *[1, Theorem 3.10] Let M be an le-Γ-semigroup. If $b \in M$ is minimal in the set of all γ-bi-ideal elements of M, then*

1. *$B_b = (b] = \{x \in M | x \leq b\}$ and B_b is a subsemigroup of M_γ.*
2. *$H_b = \{x \in M | x \leq b\gamma e \wedge e\gamma b\}$ and H_b is a subsemigroup of M_γ.*

Now we prove the following theorem.

Theorem 5.3. *Let $q \in M$ be a γ-quasi-ideal element. If q is minimal, then $H_q = Q_q = (q] = \{a \in M | a \leq q\}$ and $H_q = Q_q$ is a subsemigroup of M_γ. Conversely, if $H_q = Q_q = (q] = \{a \in M | a \leq q\}$, then q is minimal.*

Proof. Theorem 6 [5] implies that $Q_q = (q] = \{a \in M | a \leq q\}$. This and the inequalities $q\gamma q \leq q\gamma e \wedge e\gamma q \leq q$, imply that Q_q and hence H_q satisfies the Green's condition. By Theorem 2.1 [10] and Theorem 2.9 [10] it follows that $H_q = \{a \in M | a \leq q\}$ and that H_q is a subsemigroup of M_γ. Thus $Q_q = H_q = (q] = \{a \in M | a \leq q\}$ is a subsemigroup of M_γ.

Conversely, since $q\gamma q \leq q$, we have that H_q satisfies the Green's condition and by Theorem 3.2(2) we have that q is the only γ-quasi-ideal element of the class. Indeed: If $q' < q$, then $q' \in (q] = Q_q = H_q$. But H_q satisfies Green's condition, so H_q contains a single γ-quasi-ideal element. This implies $q' = q$. This means that q is a minimal γ-quasi-ideal element in M. □

Remark 5.4. *In particular, since le-semigroups are a special case of $le-\Gamma$-semigroups, all the results of this paper hold true for le-semigroups by simply applying them for Γ a singleton.*

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