I₂–Uniform Convergence of Double Sequences of Functions

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Abstract. In this work, we discuss various kinds of I₂-uniform convergence for double sequences of functions and introduce the concepts of I₂ and I₂-uniform convergence, I₂-uniformly Cauchy sequences for double sequences of functions. Then, we show the relation between them.

1. Background and Introduction

The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [11] and Schoenberg [28]. This concept was extended to the double sequences by Mursaleen and Edely [21]. A lot of development have been made in this area after the works of Salát [27] and Fridy [13, 14]. Furthermore, Gökhan et al. [16] introduced the notion of pointwise and uniform statistical convergent of double sequences of real-valued functions. In general, statistically convergent sequences satisfy many of the properties of ordinary convergent sequences in metric spaces [11, 13, 14, 25]. Çakan and Altay [4] presented multidimensional analogues of the results presented by Fridy and Orhan [12].

Throughout the paper N denotes the set of all positive integers and R the set of all real numbers. The idea of I-convergence was introduced by Kostyrko et al. [18] as a generalization of statistical convergence which is based on the structure of the ideal I of subset of the set of natural numbers. Nuray and Ruckle [23] independently introduced the same with another name generalized statistical convergence. Das et al. [5] introduced the concept of I-convergence of double sequences in a metric space and studied some properties of this convergence. Balcerzak et al. [3] discussed various kinds of statistical convergence and I-convergence for sequences with values in R or in a metric space. Gezer and Karakuş [15] investigated I-pointwise and I-uniform convergence and I'-pointwise and I'-uniform convergence of function sequences and examined the relation between them. Dündar and Altay [8] investigated the relation between I₂-convergence and I₂'-convergence of double sequences of functions defined between linear metric spaces. Some results on I-convergence may be found in [2, 6, 19, 20, 22, 29].

In this work, we discuss various kinds of uniformly ideal convergence for double sequences of functions with values in R or in a metric space. We introduce the concepts of I₂, I₂-uniform convergence, I₂-uniformly Cauchy sequences for double sequences of functions and show the relation between them.

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2. Definitions and Notations

Now, we recall that the definitions of concepts of ideal convergence, ideal Cauchy sequences and basic concepts. (See [1, 5, 9, 11, 16, 18, 21, 24, 26]).

A double sequence \( x = (x_{mn})_{m,n \in \mathbb{N}} \) of real numbers is said to be convergent to \( L \in \mathbb{R} \) if for any \( \varepsilon > 0 \), there exists \( N_\varepsilon \in \mathbb{N} \) such that
\[
|x_{mn} - L| < \varepsilon,
\]
whenever \( m, n > N_\varepsilon \). In this case we write
\[
\lim_{m,n \to \infty} x_{mn} = L.
\]

A double sequence \( x = (x_{mn})_{m,n \in \mathbb{N}} \) of real numbers is said to be bounded if there exists a positive real number \( M \) such that \( |x_{mn}| < M \), for all \( m, n \in \mathbb{N} \). That is
\[
||x||_\infty = \sup_{m,n} |x_{mn}| < \infty.
\]

Let \( K \subset \mathbb{N} \times \mathbb{N} \). Let \( K_{mn} \) be the number of \((i,j) \in K\) such that \( j \leq m, \ k \leq n \). If the sequence \( \left\{ K_{mn} \right\} \) has a limit in Pringsheim’s sense then we say that \( K \) has double natural density and is denoted by
\[
d_2(K) = \lim_{m,n \to \infty} \frac{K_{mn}}{m \cdot n}.
\]

A double sequence \( x = (x_{mn})_{m,n \in \mathbb{N}} \) of real numbers is said to be statistically convergent to \( L \in \mathbb{R} \), if for any \( \varepsilon > 0 \) we have \( d_2(A(\varepsilon)) = 0 \), where \( A(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : |x_{mn} - L| \geq \varepsilon \} \).

A double sequence of functions \( \{f_{mn}\} \) is said to be pointwise convergent to \( f \) on a set \( S \subset \mathbb{R} \), if for each point \( x \in S \) and for each \( \varepsilon > 0 \), there exists a positive integer \( N = N(x, \varepsilon) \) such that
\[
|f_{mn}(x) - f(x)| < \varepsilon,
\]
for all \( m, n > N \). In this case we write
\[
\lim_{m,n \to \infty} f_{mn}(x) = f(x) \text{ or } f_{mn} \to f
\]
on \( S \).

A double sequence of functions \( \{f_{mn}\} \) is said to be uniformly convergent to \( f \) on a set \( S \subset \mathbb{R} \), if for each \( \varepsilon > 0 \), there exists a positive integer \( N = N(\varepsilon) \) such that \( m, n > N \) implies
\[
|f_{mn}(x) - f(x)| < \varepsilon, \text{ for all } x \in S.
\]
In this case we write
\[
f_{mn} \Rightarrow f
\]
on \( S \).

A double sequence of functions \( \{f_{mn}\} \) is said to be pointwise statistically convergent to \( f \) on a set \( S \subset \mathbb{R} \), if for every \( \varepsilon > 0 \),
\[
\lim_{m,n \to \infty} \frac{1}{mn} \sum_{i \leq m, j \leq n} \mathbb{1}(|f_{ij}(x) - f(x)| \geq \varepsilon) = 0,
\]
for each (fixed) \( x \in S \), i.e., for each (fixed) \( x \in S \),
\[
|f_{ij}(x) - f(x)| < \varepsilon, \text{ a.a. } (i, j).
\]
In this case we write
\[
st - \lim_{m,n \to \infty} f_{mn}(x) = f(x) \text{ or } f_{mn} \to st f
\]
on $S$.

A double sequence of functions $\{f_{mn}\}$ is said to be uniformly statistically convergent to $f$ on a set $S \subset \mathbb{R}$, if for every $\varepsilon > 0$,

$$
\lim_{m,n \to \infty} \frac{1}{mn} \left| \left\{ (i, j), i \leq m \text{ and } j \leq n : |f_{ij}(x) - f(x)| \geq \varepsilon \right\} \right| = 0, \text{ for all } x \in S
$$

i.e., for all $x \in S$,

$$
|f_{ij}(x) - f(x)| < \varepsilon, \text{ a.a. } (i, j).
$$

In this case we write

$$
st - \lim_{m,n \to \infty} f_{mn}(x) = f(x) \text{ uniformly on } S \text{ or } f_{mn} \Rightarrow_{st} f
$$
on $S$.

Let $X \neq \emptyset$. A class $I$ of subsets of $X$ is said to be an ideal in $X$ provided:

i) $\emptyset \in I$, ii) $A, B \in I$ implies $A \cup B \in I$, iii) $A \in I$, $B \subseteq A$ implies $B \in I$.

$I$ is called a nontrivial ideal if $X \notin I$.

Let $X \neq \emptyset$. A non empty class $\mathcal{F}$ of subsets of $X$ is said to be a filter in $X$ provided:

i) $\emptyset \notin \mathcal{F}$, ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$, iii) $A \in \mathcal{F}$, $A \subseteq B$ implies $B \in \mathcal{F}$.

Lemma 2.1. [18] If $I$ is a nontrivial ideal in $X$, $X \neq \emptyset$, then the class

$$
\mathcal{F}(I) = \{ M \subset X : (\exists A \in I)(M = X \setminus A) \}
$$
is a filter on $X$, called the filter associated with $I$.

A nontrivial ideal $I$ in $X$ is called admissible if $[x] \in I$ for each $x \in X$.

Throughout the paper we take $I_2$ as a nontrivial admissible ideal in $\mathbb{N} \times \mathbb{N}$.

A nontrivial ideal $I_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $[i] \times \mathbb{N}$ and $\mathbb{N} \times [i]$ belong to $I_2$ for each $i \in \mathbb{N}$.

It is evident that a strongly admissible ideal is admissible also.

Let $I_2^0 = \{ A \subset \mathbb{N} \times \mathbb{N} : (\exists m(A) \in \mathbb{N})(i, j \geq m(A) \Rightarrow (i, j) \notin A) \}$. Then $I_2^0$ is a nontrivial strongly admissible ideal and clearly $I_2$ is strongly admissible if and only if $I_2^0 \subset I_2$.

Let $(X, \rho)$ be a linear metric space and $I_2 \subset 2^{\mathbb{N} \times \mathbb{N}}$ be a strongly admissible ideal. A double sequence $x = (x_{mn})$ of elements of $X$ is said to be $I_2$-convergent to $L \in X$, if for any $\varepsilon > 0$ we have

$$
A(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, L) \geq \varepsilon \} \in I_2.
$$

In this case we say that $x$ is $I_2$-convergent to $L \in X$ and we write

$$
I_2 - \lim_{m,n \to \infty} x_{mn} = L.
$$

If $I_2$ is a strongly admissible ideal on $\mathbb{N} \times \mathbb{N}$, then usual convergence implies $I_2$-convergence.

Let $(X, \rho)$ be a linear metric space and $I_2 \subset 2^{\mathbb{N} \times \mathbb{N}}$ be a strongly admissible ideal. A double sequence $x = (x_{mn})$ of elements of $X$ is said to be $I_2$-convergent to $L \in X$, if and only if there exists a set $M \in \mathcal{F}(I_2)$ (i.e., $\mathbb{N} \times \mathbb{N}, M \in I_2$) such that

$$
\lim_{m,n \to \infty} x_{mn} = L
$$
for $(m, n) \in M$ and we write

$$
I_2 - \lim_{m,n \to \infty} x_{mn} = L.
$$

Let $(X, \rho)$ be a linear metric space and $I_2 \subset 2^{\mathbb{N} \times \mathbb{N}}$ be a strongly admissible ideal. A double sequence $x = (x_{mn})$ of elements of $X$ is said to be $I_2$-Cauchy if for every $\varepsilon > 0$, there exist $s = s(\varepsilon), t = t(\varepsilon) \in \mathbb{N}$ such that

$$
A(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, x_{st}) \geq \varepsilon \} \in I_2.
$$
We say that an admissible ideal \( I_2 \subset 2^{\mathbb{N} \times \mathbb{N}} \) satisfies the property \((\text{AP}2)\) if for every countable family of mutually disjoint sets \( \{A_1, A_2, \ldots\} \) belonging to \( I_2 \), there exists a countable family of sets \( \{B_1, B_2, \ldots\} \) such that \( A_i \Delta B_j \in I_2 \), i.e., \( A_i \Delta B_j \) is included in the finite union of rows and columns in \( \mathbb{N} \times \mathbb{N} \) for each \( j \in \mathbb{N} \) and \( B = \bigcup_{j=1}^{\infty} B_j \) (hence \( B_i \in I_2 \) for each \( j \in \mathbb{N} \)).

Now we begin with quoting the lemmas due to Dündar and Altay [8, 9] which are needed throughout the paper.

**Lemma 2.2.** [9] Let \( I_2 \subset 2^{\mathbb{N} \times \mathbb{N}} \) be a strongly admissible ideal, \( \{f_{mn}\} \) is a double sequence of functions and \( f \) is a function on \( S \subset \mathbb{R} \). Then

\[
I_2 - \lim_{m,n \to \infty} f_{mn}(x) = f(x) \text{ implies } I_2 - \lim_{m,n \to \infty} f_{mn}(x) = f(x), \text{ (pointwise)}
\]

for each \( x \in S \).

**Lemma 2.3.** [9] Let \( I_2 \subset 2^{\mathbb{N} \times \mathbb{N}} \) be a strongly admissible ideal. \( \{f_{mn}\} \) is a double sequence of functions is pointwise \( I_2 \)-convergent to \( f \) on \( S \subset \mathbb{R} \) if and only if it is pointwise \( I_2 \)-Cauchy sequences.

**Lemma 2.4.** [8] Let \( I_2 \subset 2^{\mathbb{N} \times \mathbb{N}} \) be a strongly admissible ideal having the property \((\text{AP}2)\), \((\mathcal{X}, d_x)\) and \((\mathcal{Y}, d_y)\) two linear metric spaces, \( f_{mn} : \mathcal{X} \to \mathcal{Y} \) a double sequence of functions and \( f : \mathcal{X} \to \mathcal{Y} \). If \( \{f_{mn}\} \) double sequence of functions is \( I_2 \)-convergent then it is \( I_2 \)-convergent.

3. Main Results

First we prove the following theorem with an another way that it is given in [16].

**Theorem 3.1.** Let \( f \) and \( f_{mn} \), \( m, n = 1, 2, \ldots \), be continuous functions on \( D = [a, b] \subset \mathbb{R} \). Then \( f_{mn} \Rightarrow f \) on \( D = [a, b] \) if and only if

\[
\lim_{m,n \to \infty} c_{mn} = 0,
\]

where \( c_{mn} = \max_{x \in D} |f_{mn}(x) - f(x)| \).

**Proof.** Suppose that \( f_{mn} \Rightarrow f \) on \( D = [a, b] \). Since \( f \) and \( f_{mn} \) are continuous functions on \( D = [a, b] \) so

\[
|f_{mn} - f| < \varepsilon
\]

is continuous on \( D = [a, b] \), for each \( m, n \in \mathbb{N} \). Since \( \lim_{m,n \to \infty} f_{mn}(x) = f(x) \) uniformly on \( D = [a, b] \) then, for each \( \varepsilon > 0 \), there is a positive integer \( k_0 = k_0(\varepsilon) \in \mathbb{N} \) such that \( m, n > k_0 \) implies

\[
|f_{mn}(x) - f(x)| < \varepsilon/2,
\]

for all \( x \in D \). Thus, when \( m, n > k_0 \) we have

\[
c_{mn} = \max_{x \in D} |f_{mn}(x) - f(x)| \leq \varepsilon/2 < \varepsilon.
\]

This implies

\[
\lim_{m,n \to \infty} c_{mn} = 0.
\]

Now, suppose that \( \lim_{m,n} c_{mn} = 0 \). Then for each \( \varepsilon > 0 \), there is a positive integer \( k_0 = k_0(\varepsilon) \in \mathbb{N} \) such that

\[
0 \leq c_{mn} = \max_{x \in D} |f_{mn}(x) - f(x)| < \varepsilon,
\]

for \( m, n > k_0 \). This implies that

\[
|f_{mn}(x) - f(x)| < \varepsilon,
\]

for all \( x \in D \) and \( m, n > k_0 \). Hence, we have

\[
\lim_{m,n \to \infty} f_{mn}(x) = f(x),
\]

for all \( x \in D \). \( \square \)
**Definition 3.2.** Let \( I_2 \subset 2^{\mathbb{N} \times \mathbb{N}} \) be a strongly admissible ideal. A double sequence of functions \( \{f_{mn}\} \) is said to be \( I_2 \)-uniformly convergent to \( f \) on a set \( S \subset \mathbb{R} \), if for every \( \varepsilon > 0 \)

\[
(m, n) \in \mathbb{N} \times \mathbb{N} : |f_{mn}(x) - f(x)| \geq \varepsilon \}\in I_2, \text{ for each } x \in S.
\]

This convergence can be showed by the formula

\[
(\forall \varepsilon > 0) \ (\exists H \in I_2) \ (\forall (m, n) \notin H) \ (\forall x \in S) \ |f_{mn}(x) - f(x)| < \varepsilon.
\]

This convergence can be showed by

\[
f_{mn} \Rightarrow_{I_2} f.
\]

**Theorem 3.3.** Let \( I_2 \subset 2^{\mathbb{N} \times \mathbb{N}} \) be a strongly admissible ideal, \( f \) and \( f_{mn} \), \( m, n = 1, 2, \ldots \), be continuous functions on \( D = [a, b] \subset \mathbb{R} \). Then

\[
f_{mn} \Rightarrow_{I_2} f
\]

on \( D = [a, b] \) if and only if

\[
I_2 - \lim_{m,n} c_{mn} = 0,
\]

where \( c_{mn} = \max_{x \in D} |f_{mn}(x) - f(x)| \).

**Proof.** Suppose that \( f_{mn} \Rightarrow_{I_2} f \) on \( D = [a, b] \). Since \( f \) and \( f_{mn} \) be continuous functions on \( D = [a, b] \), so

\[
|f_{mn} - f|
\]

is continuous on \( D = [a, b] \) for each \( m, n \in \mathbb{N} \). By \( I_2 \)-uniform convergence for \( \varepsilon > 0 \)

\[
\left\{(m, n) \in \mathbb{N} \times \mathbb{N} : |f_{mn}(x) - f(x)| \geq \varepsilon \right\} \in I_2, \text{ for each } x \in D.
\]

Hence, for \( \varepsilon > 0 \) it is clear that

\[
c_{mn} = \max_{x \in D} |f_{mn}(x) - f(x)| \geq |f_{mn}(x) - f(x)| \geq \varepsilon \text{ for each } x \in D.
\]

Thus, we have

\[
I_2 - \lim_{m,n \to \infty} c_{mn} = 0.
\]

Now, suppose that \( I_2 - \lim_{m,n} c_{mn} = 0 \). Then, for \( \varepsilon > 0 \)

\[
A(\varepsilon) = \left\{(m, n) \in \mathbb{N} \times \mathbb{N} : \max_{x \in D} |f_{mn}(x) - f(x)| \geq \varepsilon \right\} \in I_2.
\]

Since, for \( \varepsilon > 0 \)

\[
\max_{x \in D} |f_{mn}(x) - f(x)| \geq |f_{mn}(x) - f(x)| \geq \varepsilon
\]

we have

\[
\left\{(m, n) \in \mathbb{N} \times \mathbb{N} : |f_{mn}(x) - f(x)| \geq \varepsilon \right\} \subset A(\varepsilon), \text{ for each } x \in D.
\]

This proves the theorem. \( \square \)

**Definition 3.4.** Let \( I_2 \subset 2^{\mathbb{N} \times \mathbb{N}} \) be a strongly admissible ideal. A double sequence of functions \( \{f_{mn}\} \) is said to be \( I_2 \)-uniformly convergent to \( f \) on a set \( S \subset \mathbb{R} \), if there exists a set \( M \in \mathcal{F}(I_2) \) (i.e., \( \mathbb{N} \times \mathbb{N} \setminus M \in I_2 \)) such that for every \( \varepsilon > 0 \)

\[
\lim_{m,n \to \infty} f_{mn}(x) = f(x), \text{ for each } x \in S
\]

and we write

\[
f_{mn} \Rightarrow_{I_2} f.
\]
**Theorem 3.5.** Let $I_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal, $(f_{mn})$ be a double sequence of continuous functions and $f$ be a function on $S$. If

$$f_{mn} \Rightarrow I_2 f$$

then, $f$ is continuous on $S$.

**Proof.** Assume $f_{mn} \Rightarrow I_2 f$ on $S$. Then for every $\varepsilon > 0$, there exists a set $M \in \mathcal{F}(I_2)$ (i.e., $H = \mathbb{N} \times \mathbb{N} \setminus M \in I_2$) and $k_0 = k_0(\varepsilon)$, $l_0 = l_0(\varepsilon) \in \mathbb{N}$ such that

$$|f_{mn}(x) - f(x)| < \frac{\varepsilon}{3}, \quad (m, n) \in M$$

for each $x \in S$ and for all $m > k_0$, $n > l_0$. Now, let $x_0 \in S$ is arbitrary. Since $(f_{klb})$ is continuous at $x_0 \in S$, there is a $\delta > 0$ such that $|x - x_0| < \delta$ implies

$$|f_{klb}(x) - f_{klb}(x_0)| < \frac{\varepsilon}{3}.$$

Then, for all $x \in S$ for which $|x - x_0| < \delta$, we have

$$|f(x) - f(x_0)| \leq |f(x) - f_{klb}(x)| + |f_{klb}(x) - f_{klb}(x_0)| + |f_{klb}(x_0) - f(x_0)|< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Since $x_0 \in S$ is arbitrary, $f$ is continuous on $S$. \qed

**Theorem 3.6.** Let $I_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal with the property (AP2), $S$ be a compact subset of $\mathbb{R}$ and $(f_{mn})$ be a double sequence of continuous functions on $S$. Assume that $(f_{mn})$ be monotonic decreasing on $S$ i.e.,

$$f_{(m+1),n}(x) \leq f_{mn}(x), \quad (m, n = 1, 2, ...)$$

for every $x \in S$, $f$ is continuous and

$$I_2 - \lim_{m,n \to \infty} f_{mn}(x) = f(x)$$

on $S$. Then

$$f_{mn} \Rightarrow I_2 f$$

on $S$.

**Proof.** Let

$$g_{mn} = f_{mn} - f$$

a sequence of functions on $S$. Since $(f_{mn})$ is continuous and monotonic decreasing and $f$ is continuous on $S$, then $g_{mn}$ is continuous and monotonic decreasing on $S$. Since

$$I_2 - \lim_{m,n \to \infty} f_{mn}(x) = f(x),$$

then by (1)

$$I_2 - \lim_{m,n \to \infty} g_{mn}(x) = 0$$

on $S$ and since $I_2$ satisfy the condition (AP2) then we have

$$I_2' - \lim_{m,n \to \infty} g_{mn}(x) = 0$$

on $S$. Hence, for every $\varepsilon > 0$ and each $x \in S$ there exists $K_x \in \mathcal{F}(I_2)$ such that

$$0 \leq g_{mn}(x) < \frac{\varepsilon}{2}, \quad \left( (m, n), (m(x) = m(x, \varepsilon), n(x) = n(x, \varepsilon)) \in K_x \right)$$
for \( m \geq m(x) \) and \( n \geq n(x) \). Since \( \{g_{m,n}\} \) is continuous at \( x \in S \), for every \( \varepsilon > 0 \) there is an open set \( A(x) \) which contains \( x \) such that

\[
|g_{m(x),n(x)}(t) - g_{m(x),n(x)}(x)| < \frac{\varepsilon}{2},
\]

for all \( t \in A(x) \). Then for \( \varepsilon > 0 \) by monotonicity, we have

\[
0 \leq g_{m,n}(t) \leq g_{m(x),n(x)}(t) = g_{m(x),n(x)}(t) - g_{m(x),n(x)}(x) + g_{m(x),n(x)}(x) \leq |g_{m(x),n(x)}(t) - g_{m(x),n(x)}(x)| + g_{m(x),n(x)}(x), \quad ((m, n) \in K_x)
\]

for every \( t \in A(x) \) and for all \( m \geq m(x), n \geq n(x) \) and for each \( x \in S \). Since \( S \subset \bigcup_{x \in S} A(x) \) and \( S \) is a compact set, by the Heine-Borel theorem \( S \) has a finite open covering such that

\[
S \subset A(x_1) \cup A(x_2) \cup A(x_3) \cup \ldots \cup A(x_i).
\]

Now, let

\[
K = K_{x_1} \cap K_{x_2} \cap K_{x_3} \cap \ldots \cap K_{x_i}
\]

and define

\[
M = \max \{m(x_1), m(x_2), m(x_3), \ldots, m(x_i)\},
\]

\[
N = \max \{n(x_1), n(x_2), n(x_3), \ldots, n(x_i)\}.
\]

Since for every \( K_x \) belong to \( \mathcal{F}(I_2) \), we have \( K \in \mathcal{F}(I_2) \). Then, when all \( (m, n) \geq (M, N) \)

\[
0 \leq g_{m,n}(t) < \varepsilon, \quad (m, n) \in K,
\]

for every \( t \in A(x) \). So

\[
g_{m,n} \Rightarrow I_2 0
\]

on \( S \). Since \( I_2 \) is a strongly admissible ideal,

\[
g_{m,n} \Rightarrow I_2 0
\]

on \( S \) and by (1) we have

\[
f_{m,n} \Rightarrow I_2 f
\]

on \( S \). \( \square \)

**Theorem 3.7.** Let \( I_2 \subset 2^{\mathbb{N} \times \mathbb{N}} \) be a strongly admissible ideal, \( (X, d_x) \) and \( (Y, d_y) \) be two metric spaces, \( f_{m,n} : X \rightarrow Y \), \( (m, n) \in \mathbb{N} \), are equi-continuous and \( f : X \rightarrow Y \). Assume that

\[
f_{m,n} \Rightarrow I_2 f
\]

on \( X \). Then, \( f \) is continuous on \( X \). Also, if \( X \) is compact then we have

\[
f_{m,n} \Rightarrow I_2 f
\]

on \( X \).
Proof. First we will prove that \( f \) is continuous on \( X \). Let \( x_0 \in X \) and \( \varepsilon > 0 \). By the equi-continuity of \( f_{mn} \)'s, there exists \( \delta > 0 \) such that
\[
d_y(f_{mn}(x), f_{mn}(x_0)) < \frac{\varepsilon}{3},
\]
for every \( m, n \in \mathbb{N} \) and \( x \in B_\delta(x_0) \) (\( B_\delta(x_0) \) stands for an open ball in \( X \) with center \( x_0 \) and radius \( \delta \)). Let \( x \in B_\delta(x_0) \) be fixed. Since \( f_{mn} \rightarrow_{I_2} f \), the set
\[
\left\{(m, n) \in \mathbb{N} \times \mathbb{N} : d_y(f_{mn}(x_0), f(x)) \geq \frac{\varepsilon}{3}\right\} \cup \left\{(m, n) \in \mathbb{N} \times \mathbb{N} : d_y(f_{mn}(x), f(x)) \geq \frac{\varepsilon}{3}\right\}
\]
is in \( I_2 \) and is different from \( \mathbb{N} \times \mathbb{N} \). Hence, there exists \( (m, n) \in \mathbb{N} \times \mathbb{N} \) such that
\[
d_y(f_{mn}(x_0), f(x_0)) < \frac{\varepsilon}{3} \quad \text{and} \quad d_y(f_{mn}(x), f(x)) < \frac{\varepsilon}{3}.
\]
Thus, we have
\[
d_y(f(x_0), f(x)) \leq d_y(f(x_0), f_{mn}(x_0)) + d_y(f_{mn}(x_0), f_{mn}(x)) + d_y(f_{mn}(x), f(x))
\]
\[
< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\]
so \( f \) is continuous on \( X \).

Now, assume that \( X \) is compact. Let \( \varepsilon > 0 \). Since \( X \) is compact, it follows that \( f \) is uniformly continuous and \( f_{mn} \)'s are equi-uniformly continuous on \( X \). So, pick \( \delta > 0 \) such that for any \( x, x' \in X \) with
\[
d_x(x, x') < \delta,
\]
then, by equi-uniformly and uniformly continuous we have
\[
d_y(f_{mn}(x), f_{mn}(x')) < \frac{\varepsilon}{3} \quad \text{and} \quad d_y(f(x), f(x')) < \frac{\varepsilon}{3}.
\]
By the compactness of \( X \), we can choose a finite subcover
\[
B_{\gamma_1}(\delta), B_{\gamma_2}(\delta), ..., B_{\gamma_k}(\delta)
\]
from the cover \( \{B_x(\delta)\}_{x \in X} \) of \( X \). Using \( f_{mn} \rightarrow_{I_2} f \) pick a set \( M \in I_2 \) such that
\[
d_y(f_{mn}(x_i), f(x_i)) < \frac{\varepsilon}{3}, \quad i \in \{1, 2, ..., k\},
\]
for all \( (m, n) \notin M \). Let \( (m, n) \notin M \) and \( x \in X \). Thus, \( x \in B_{\gamma_i}(\delta) \) for some \( i \in \{1, 2, ..., k\} \). Hence, we have
\[
d_y(f_{mn}(x), f(x)) \leq d_y(f_{mn}(x), f_{mn}(x_i)) + d_y(f_{mn}(x_i), f(x_i)) + d_y(f(x_i), f(x))
\]
\[
< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\]
and so
\[
f_{mn} \Rightarrow_{I_2} f
\]
on \( X \). \( \square \)

**Definition 3.8.** Let \( I_2 \subset 2^{\mathbb{N} \times \mathbb{N}} \) be a strongly admissible ideal and \( \{f_{mn}\} \) be a double sequence of functions on \( S \subset \mathbb{R} \). \( \{f_{mn}\} \) is said to be \( I_2 \)-uniformly Cauchy if for every \( \varepsilon > 0 \) there exist \( s = s(\varepsilon), t = t(\varepsilon) \in \mathbb{N} \) such that
\[
\{(m, n) \in \mathbb{N} \times \mathbb{N} : |f_{mn}(x) - f_{nt}(x)| \geq \varepsilon\} \in I_2, \quad \text{for each} \ x \in S.
\]
(2)

Now, we give \( I_2 \)-Cauchy criteria for \( I_2 \)-uniform convergence.

**Theorem 3.9.** Let \( I_2 \subset 2^{\mathbb{N} \times \mathbb{N}} \) be a strongly admissible ideal with the property (AP2) and let \( \{f_{mn}\} \) be a sequence of bounded functions on \( S \subset \mathbb{R} \). Then \( \{f_{mn}\} \) is \( I_2 \)-uniformly convergent if and only if it is \( I_2 \)-uniformly Cauchy on \( S \).
Proof. Necessity of Theorem is similar to that of Lemma 2.3.

Conversely, assume that \( \{f_{mn}\} \) is \( I_2 \)-uniformly Cauchy on \( S \). Let \( x \in S \) be fixed. By (2), for every \( \varepsilon > 0 \) there exist \( s = s(\varepsilon) \) and \( t = t(\varepsilon) \in \mathbb{N} \) such that
\[
(\{m, n\} \in \mathbb{N} \times \mathbb{N} : |f_{mn}(x) - f_{st}(x)| < \varepsilon) \notin I_2.
\]
Hence, \( \{f_{mn}\} \) is \( I_2 \)-Cauchy, so by Lemma 2.3 we have that \( \{f_{mn}\} \) is \( I_2 \)-convergent to \( f(x) \). Then, \( I_2 \lim_{m,n\to \infty} f_{mn}(x) = f(x) \) on \( S \). Note that since \( I_2 \) satisfy the property (AP2), by (2) there is a \( M \notin I_2 \) such that
\[
|f_{mn}(x) - f_{st}(x)| < \varepsilon, \quad \left( (m, n), (s, t) \in M \right)
\]
for all \( m, n, s, t \geq N \) and \( N = N(\varepsilon) \in \mathbb{N} \) and for each \( x \in S \). By (3), for \( s, t \to \infty \) we have
\[
|f_{mn}(x) - f(x)| < \varepsilon, \quad \left( (m, n) \in M \right),
\]
for all \( m, n > N \) and for each \( x \in S \). This shows that
\[
f_{mn} \Rightarrow I_2 f
\]
on \( S \). Since \( I_2 \subset 2^{\mathbb{N} \times \mathbb{N}} \) is a strongly admissible ideal we have
\[
f_{mn} \Rightarrow I_2 f.
\]
\( \square \)

References