Filomat 30:5 (2016), 1273–1281
DOI 10.2298/FIL1605273D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

$\mathcal{I}_2$–Uniform Convergence of Double Sequences of Functions

Erdinç Dündar[a], **Bilal Altay**[b]

[a] *Afyon Kocatepe Üniversitesi, Fen Edebiyat Fakültesi, Matematik Bölümü, Afyonkarahisar-03200/Türkiye*
[b] *İnönü Üniversitesi, Eğitim Fakültesi, Malatya-44280/Türkiye*

Abstract. In this work, we discuss various kinds of $\mathcal{I}_2$-uniform convergence for double sequences of functions and introduce the concepts of $\mathcal{I}_2$ and $\mathcal{I}_2^*$-uniform convergence, $\mathcal{I}_2$-uniformly Cauchy sequences for double sequences of functions. Then, we show the relation between them.

1. Background and Introduction

The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [11] and Schoenberg [28]. This concept was extended to the double sequences by Mursaleen and Edely [21]. A lot of development have been made in this area after the works of Šalát [27] and Fridy [13, 14]. Furthermore, Gökhan et al. [16] introduced the notion of pointwise and uniform statistical convergent of double sequences of real-valued functions. In general, statistically convergent sequences satisfy many of the properties of ordinary convergent sequences in metric spaces [11, 13, 14, 25]. Çakan and Altay [4] presented multidimensional analogues of the results presented by Fridy and Orhan [12].

Throughout the paper $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}$ the set of all real numbers. The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [18] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of the set of natural numbers. Nuray and Ruckle [23] indepedently introduced the same with another name generalized statistical convergence. Das et al. [5] introduced the concept of $\mathcal{I}$-convergence of double sequences in a metric space and studied some properties of this convergence. Balcerzak et al. [3] discussed various kinds of statistical convergence and $\mathcal{I}$-convergence for sequences of functions with values in $\mathbb{R}$ or in a metric space. Gezer and Karakuş [15] investigated $\mathcal{I}$-pointwise and $\mathcal{I}$-uniform convergence and $\mathcal{I}^*$-pointwise and $\mathcal{I}^*$-uniform convergence of function sequences and examined the relation between them. Dündar and Altay [8] investigated the relation between $\mathcal{I}_2$-convergence and $\mathcal{I}_2^*$-convergence of double sequences of functions defined between linear metric spaces. Some results on $\mathcal{I}$-convergence may be found in [2, 6, 19, 20, 22, 29].

In this work, we discuss various kinds of uniformly ideal convergence for double sequences of functions with values in $\mathbb{R}$ or in a metric space. We introduce the concepts of $\mathcal{I}_2$, $\mathcal{I}_2^*$-uniform convergence, $\mathcal{I}_2$-uniformly Cauchy sequences for double sequences of functions and show the relation between them.

2010 *Mathematics Subject Classification*. Primary 40A30; Secondary 40A35
Keywords. Ideal, Uniform Convergence, $\mathcal{I}$-Convergence, Double Sequences of Functions.
Received: 04 April 2014; Accepted: 27 April 2014
Communicated by Eberhard Malkowsky
Email addresses: erdincdundar79@gmail.com, edundar@aku.edu.tr (Erdinç Dündar), bilal.altay@inonu.edu.tr (Bilal Altay)

2. Definitions and Notations

Now, we recall that the definitions of concepts of ideal convergence, ideal Cauchy sequences and basic concepts. (See [1, 5, 9, 11, 16, 18, 21, 24, 26]).

A double sequence $x = (x_{mn})_{m,n\in\mathbb{N}}$ of real numbers is said to be convergent to $L \in \mathbb{R}$ if for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that

$$|x_{mn} - L| < \varepsilon,$$

whenever $m, n > N_\varepsilon$. In this case we write

$$\lim_{m,n\to\infty} x_{mn} = L.$$

A double sequence $x = (x_{mn})_{m,n\in\mathbb{N}}$ of real numbers is said to be bounded if there exists a positive real number M such that $|x_{mn}| < M$, for all $m, n \in \mathbb{N}$. That is

$$\|x\|_\infty = \sup_{m,n} |x_{mn}| < \infty.$$

Let $K \subset \mathbb{N} \times \mathbb{N}$. Let K_{mn} be the number of $(j, k) \in K$ such that $j \leq m$, $k \leq n$. If the sequence $\left\{\frac{K_{mn}}{m.n}\right\}$ has a limit in Pringsheim's sense then we say that K has double natural density and is denoted by

$$d_2(K) = \lim_{m,n\to\infty} \frac{K_{mn}}{m.n}.$$

A double sequence $x = (x_{mn})_{m,n\in\mathbb{N}}$ of real numbers is said to be statistically convergent to $L \in \mathbb{R}$, if for any $\varepsilon > 0$ we have $d_2(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : |x_{mn} - L| \geq \varepsilon\}$.

A double sequence of functions $\{f_{mn}\}$ is said to be pointwise convergent to f on a set $S \subset \mathbb{R}$, if for each point $x \in S$ and for each $\varepsilon > 0$, there exists a positive integer $N = N(x, \varepsilon)$ such that

$$|f_{mn}(x) - f(x)| < \varepsilon,$$

for all $m, n > N$. In this case we write

$$\lim_{m,n\to\infty} f_{mn}(x) = f(x) \ or \ f_{mn} \to f$$

on S.

A double sequence of functions $\{f_{mn}\}$ is said to be uniformly convergent to f on a set $S \subset \mathbb{R}$, if for each $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that $m, n > N$ implies

$$|f_{mn}(x) - f(x)| < \varepsilon, \text{ for all } x \in S.$$

In this case we write

$$f_{mn} \rightrightarrows f$$

on S.

A double sequence of functions $\{f_{mn}\}$ is said to be pointwise statistically convergent to f on a set $S \subset \mathbb{R}$, if for every $\varepsilon > 0$,

$$\lim_{m,n\to\infty} \frac{1}{mn} \Big| \{(i, j), i \leq m \ and \ j \leq n : |f_{ij}(x) - f(x)| \geq \varepsilon\} \Big| = 0,$$

for each (fixed) $x \in S$, i.e., for each (fixed) $x \in S$,

$$|f_{ij}(x) - f(x)| < \varepsilon, \ \ a.a. \ (i, j).$$

In this case we write

$$st - \lim_{m,n\to\infty} f_{mn}(x) = f(x) \ or \ f_{mn} \to_{st} f$$

on S.

A double sequence of functions $\{f_{mn}\}$ is said to be uniformly statistically convergent to f on a set $S \subset \mathbb{R}$, if for every $\varepsilon > 0$,

$$\lim_{m,n\to\infty} \frac{1}{mn}\Big|\{(i,j), i \leq m \ \ and \ \ j \leq n : |f_{ij}(x) - f(x)| \geq \varepsilon\}\Big| = 0, \text{ for all } x \in S$$

i.e., for all $x \in S$,

$$|f_{ij}(x) - f(x)| < \varepsilon, \ \ a.a. \ (i,j).$$

In this case we write

$$st - \lim_{m,n\to\infty} f_{mn}(x) = f(x) \ \text{ uniformly on S } \ or \ f_{mn} \rightrightarrows_{st} f$$

on S.

Let $X \neq \emptyset$. A class $\mathcal{I}$ of subsets of X is said to be an ideal in X provided:

i) $\emptyset \in \mathcal{I}$, ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$, iii) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$.

$\mathcal{I}$ is called a nontrivial ideal if $X \notin \mathcal{I}$.

Let $X \neq \emptyset$. A non empty class $\mathcal{F}$ of subsets of X is said to be a filter in X provided:

i) $\emptyset \notin \mathcal{F}$, ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$, iii) $A \in \mathcal{F}$, $A \subset B$ implies $B \in \mathcal{F}$.

Lemma 2.1. *[18] If $\mathcal{I}$ is a nontrivial ideal in X, $X \neq \emptyset$, then the class*

$$\mathcal{F}(\mathcal{I}) = \{M \subset X : (\exists A \in \mathcal{I})(M = X \backslash A)\}$$

is a filter on X, called the filter associated with $\mathcal{I}$.

A nontrivial ideal $\mathcal{I}$ in X is called admissible if $\{x\} \in \mathcal{I}$ for each $x \in X$.

Throughout the paper we take $\mathcal{I}_2$ as a nontrivial admissible ideal in $\mathbb{N} \times \mathbb{N}$.

A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in N$.

It is evident that a strongly admissible ideal is admissible also.

Let $\mathcal{I}_2^0 = \{A \subset \mathbb{N}\times\mathbb{N} : \big(\exists m(A) \in \mathbb{N}\big)\big(i, j \geq m(A) \Rightarrow (i, j) \notin A\big)\}$. Then $\mathcal{I}_2^0$ is a nontrivial strongly admissible ideal and clearly $\mathcal{I}_2$ is strongly admissible if and only if $\mathcal{I}_2^0 \subset \mathcal{I}_2$.

Let (X, ρ) be a linear metric space and $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence $x = (x_{mn})$ of elements of X is said to be $\mathcal{I}_2$-convergent to $L \in X$, if for any $\varepsilon > 0$ we have

$$A(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, L) \geq \varepsilon\} \in \mathcal{I}_2.$$

In this case we say that x is $\mathcal{I}_2$-convergent to $L \in X$ and we write

$$\mathcal{I}_2 - \lim_{m,n\to\infty} x_{mn} = L.$$

If $\mathcal{I}_2$ is a strongly admissible ideal on $\mathbb{N} \times \mathbb{N}$, then usual convergence implies $\mathcal{I}_2$-convergence.

Let (X, ρ) be a linear metric space and $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence $x = (x_{mn})$ of elements of X is said to be $\mathcal{I}_2^*$-convergent to $L \in X$, if and only if there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $\mathbb{N} \times \mathbb{N} \backslash M \in \mathcal{I}_2$) such that

$$\lim_{m,n\to\infty} x_{mn} = L,$$

for $(m, n) \in M$ and we write

$$\mathcal{I}_2^* - \lim_{m,n\to\infty} x_{mn} = L.$$

Let (X, ρ) be a linear metric space and $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence $x = (x_{mn})$ of elements of X is said to be $\mathcal{I}_2$-Cauchy if for every $\varepsilon > 0$, there exist $s = s(\varepsilon), t = t(\varepsilon) \in \mathbb{N}$ such that

$$A(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, x_{st}) \geq \varepsilon\} \in \mathcal{I}_2.$$

We say that an admissible ideal $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ satisfies the property (AP2) if for every countable family of mutually disjoint sets $\{A_1, A_2, ...\}$ belonging to $\mathcal{I}_2$, there exists a countable family of sets $\{B_1, B_2, ...\}$ such that $A_j \Delta B_j \in \mathcal{I}_2^0$, i.e., $A_j \Delta B_j$ is included in the finite union of rows and columns in $\mathbb{N}\times\mathbb{N}$ for each $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}_2$ (hence $B_j \in \mathcal{I}_2$ for each $j \in \mathbb{N}$).

Now we begin with quoting the lemmas due to Dündar and Altay [8, 9] which are needed throughout the paper.

Lemma 2.2. *[9] Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal, $\{f_{mn}\}$ is a double sequence of functions and f is a function on $S \subset \mathbb{R}$. Then*

$$\mathcal{I}_2^* - \lim_{m,n\to\infty} f_{mn}(x) = f(x) \text{ implies } \mathcal{I}_2 - \lim_{m,n\to\infty} f_{mn}(x) = f(x), \ (pointwise)$$

for each $x \in S$.

Lemma 2.3. *[9] Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. $\{f_{mn}\}$ is a double sequence of functions is pointwise $\mathcal{I}_2$-convergent to f on $S \subset \mathbb{R}$ if and only if it is pointwise $\mathcal{I}_2$-Cauchy sequences.*

Lemma 2.4. *[8] Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal having the property (AP2), (X, d_x) and (Y, d_y) two linear metric spaces, $f_{mn} : X \to Y$ a double sequence of functions and $f : X \to Y$. If $\{f_{mn}\}$ double sequence of functions is $\mathcal{I}_2$-convergent then it is $\mathcal{I}_2^*$-convergent.*

3. Main Results

First we prove the following theorem with an another way that it is given in [16].

Theorem 3.1. *Let f and f_{mn}, $m, n = 1, 2, ...,$ be continuous functions on $D = [a, b] \subset \mathbb{R}$. Then $f_{mn} \rightrightarrows f$ on $D = [a, b]$ if and only if*

$$\lim_{m,n\to\infty} c_{mn} = 0,$$

where $c_{mn} = max_{x\in D} |f_{mn}(x) - f(x)|$.

Proof. Suppose that $f_{mn} \rightrightarrows f$ on $D = [a, b]$. Since f and f_{mn} are continuous functions on $D = [a, b]$ so

$$|f_{mn} - f|$$

is continuous on $D = [a, b]$, for each $m, n \in \mathbb{N}$. Since $\lim_{m,n\to\infty} f_{mn}(x) = f(x)$ uniformly on $D = [a, b]$ then, for each $\varepsilon > 0$, there is a positive integer $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that $m, n > k_0$ implies

$$|f_{mn}(x) - f(x)| < \frac{\varepsilon}{2},$$

for all $x \in D$. Thus, when $m, n > k_0$ we have

$$c_{mn} = \max_{x\in D} |f_{mn}(x) - f(x)| \leq \frac{\varepsilon}{2} < \varepsilon.$$

This implies

$$\lim_{m,n\to\infty} c_{mn} = 0.$$

Now, suppose that $\lim_{m,n} c_{mn} = 0$. Then for each $\varepsilon > 0$, there is a positive integer $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that

$$0 \leq c_{mn} = \max_{x\in D} |f_{mn}(x) - f(x)| < \varepsilon,$$

for $m, n > k_0$. This implies that

$$|f_{mn}(x) - f(x)| < \varepsilon,$$

for all $x \in D$ and $m, n > k_0$. Hence, we have

$$\lim_{m,n\to\infty} f_{mn}(x) = f(x),$$

for all $x \in D$. $\square$

Definition 3.2. *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence of functions $\{f_{mn}\}$ is said to be $\mathcal{I}_2$-uniformly convergent to f on a set $S \subset \mathbb{R}$, if for every $\varepsilon > 0$*

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : |f_{mn}(x) - f(x)| \geq \varepsilon\} \in \mathcal{I}_2, \text{ for each } x \in S.$$

This can be written by the formula

$$(\forall \varepsilon > 0)\ (\exists H \in \mathcal{I}_2)\ (\forall (m,n) \notin H)\ (\forall x \in S)\ |f_{mn}(x) - f(x)| < \varepsilon.$$

This convergence can be showed by

$$f_{mn} \rightrightarrows_{\mathcal{I}_2} f.$$

Theorem 3.3. *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal, f and f_{mn} , $m,n = 1,2,...,$ be continuous functions on $D = [a,b] \subset \mathbb{R}$. Then*

$$f_{mn} \rightrightarrows_{\mathcal{I}_2} f$$

on $D = [a,b]$ if and only if

$$\mathcal{I}_2 - \lim_{m,n} c_{mn} = 0,$$

where $c_{mn} = max_{x\in D} |f_{mn}(x) - f(x)|$.

Proof. Suppose that $f_{mn} \rightrightarrows_{\mathcal{I}_2} f$ on $D = [a,b]$. Since f and f_{mn} be continuous functions on $D = [a,b]$, so

$$|f_{mn} - f|$$

is continuous on $D = [a,b]$ for each $m,n \in \mathbb{N}$. By $\mathcal{I}_2$-uniform convergence for $\varepsilon > 0$

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : |f_{mn}(x) - f(x)| \geq \frac{\varepsilon}{2}\right\} \in \mathcal{I}_2, \text{ for each } x \in D.$$

Hence, for $\varepsilon > 0$ it is clear that

$$c_{mn} = \max_{x\in D} |f_{mn}(x) - f(x)| \geq |f_{mn}(x) - f(x)| \geq \frac{\varepsilon}{2}, \text{ for each } x \in D.$$

Thus, we have

$$\mathcal{I}_2 - \lim_{m,n\to\infty} c_{mn} = 0.$$

Now, suppose that $\mathcal{I}_2 - \lim_{m,n} c_{mn} = 0$. Then, for $\varepsilon > 0$

$$A(\varepsilon) = \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \max_{x\in D} |f_{mn}(x) - f(x)| \geq \varepsilon\right\} \in \mathcal{I}_2.$$

Since, for $\varepsilon > 0$

$$\max_{x\in D} |f_{mn}(x) - f(x)| \geq |f_{mn}(x) - f(x)| \geq \varepsilon$$

we have

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : |f_{mn}(x) - f(x)| \geq \varepsilon\} \subset A(\varepsilon), \text{ for each } x \in D.$$

This proves the theorem. □

Definition 3.4. *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence of functions $\{f_{mn}\}$ is said to be $\mathcal{I}_2^*$-uniformly convergent to f on a set $S \subset \mathbb{R}$, if there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $\mathbb{N}\times\mathbb{N}\backslash M \in \mathcal{I}_2$) such that for every $\varepsilon > 0$*

$$\lim_{\substack{m,n\to\infty \\ (m,n)\in M}} f_{mn}(x) = f(x), \text{ for each } x \in S$$

and we write

$$f_{mn} \rightrightarrows_{\mathcal{I}_2^*} f.$$

Theorem 3.5. *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal, $\{f_{mn}\}$ be a double sequence of continuous functions and f be a function on S. If*

$$f_{mn} \rightrightarrows_{\mathcal{I}_2^*} f$$

then, f is continuous on S.

Proof. Assume $f_{mn} \rightrightarrows_{\mathcal{I}_2^*} f$ on S. Then for every $\varepsilon > 0$, there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $H = \mathbb{N}\times\mathbb{N}\backslash M \in \mathcal{I}_2$) and $k_0 = k_0(\varepsilon)$, $l_0 = l_0(\varepsilon) \in \mathbb{N}$ such that

$$|f_{mn}(x) - f(x)| < \frac{\varepsilon}{3},\ (m,n) \in M$$

for each $x \in S$ and for all $m > k_0$, $n > l_0$. Now, let $x_0 \in S$ is arbitrary. Since $\{f_{k_0l_0}\}$ is continuous at $x_0 \in S$, there is a $\delta > 0$ such that $|x - x_0| < \delta$ implies

$$|f_{k_0l_0}(x) - f_{k_0l_0}(x_0)| < \frac{\varepsilon}{3}.$$

Then, for all $x \in S$ for which $|x - x_0| < \delta$, we have

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_{k_0l_0}(x)| + |f_{k_0l_0}(x) - f_{k_0l_0}(x_0)| + |f_{k_0l_0}(x_0) - f(x_0)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Since $x_0 \in S$ is arbitrary, f is continuous on S. □

Theorem 3.6. *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal with the property (AP2), S be a compact subset of $\mathbb{R}$ and $\{f_{mn}\}$ be a double sequence of continuous functions on S. Assume that $\{f_{mn}\}$ be monotonic decreasing on S i.e.,*

$$f_{(m+1),(n+1)}(x) \leq f_{mn}(x),\ (m,n = 1,2,...)$$

for every $x \in S$, f is continuous and

$$\mathcal{I}_2 - \lim_{m,n\to\infty} f_{mn}(x) = f(x)$$

on S. Then

$$f_{mn} \rightrightarrows_{\mathcal{I}_2} f$$

on S.

Proof. Let

$$g_{mn} = f_{mn} - f \tag{1}$$

a sequence of functions on S. Since $\{f_{mn}\}$ is continuous and monotonic decreasing and f is continuous on S, then $\{g_{mn}\}$ is continuous and monotonic decreasing on S. Since

$$\mathcal{I}_2 - \lim_{m,n\to\infty} f_{mn}(x) = f(x),$$

then by (1)

$$\mathcal{I}_2 - \lim_{m,n\to\infty} g_{mn}(x) = 0$$

on S and since $\mathcal{I}_2$ satisfy the condition (AP2) then we have

$$\mathcal{I}_2^* - \lim_{m,n\to\infty} g_{mn}(x) = 0$$

on S. Hence, for every $\varepsilon > 0$ and each $x \in S$ there exists $K_x \in \mathcal{F}(\mathcal{I}_2)$ such that

$$0 \leq g_{mn}(x) < \frac{\varepsilon}{2},\ \Big((m,n),\ \big(m(x) = m(x,\varepsilon),\ n(x) = n(x,\varepsilon)\big) \in K_x\Big)$$

for $m \geq m(x)$ and $n \geq n(x)$. Since $\{g_{mn}\}$ is continuous at $x \in S$, for every $\varepsilon > 0$ there is an open set $A(x)$ which contains x such that

$$|g_{m(x)n(x)}(t) - g_{m(x)n(x)}(x)| < \frac{\varepsilon}{2},$$

for all $t \in A(x)$. Then for $\varepsilon > 0$ by monotonicity, we have

$$\begin{aligned} 0 \leq g_{mn}(t) &\leq g_{m(x)n(x)}(t) \\ &= g_{m(x)n(x)}(t) - g_{m(x)n(x)}(x) + g_{m(x)n(x)}(x) \\ &\leq |g_{m(x)n(x)}(t) - g_{m(x)n(x)}(x)| + g_{m(x)n(x)}(x), \ \ ((m,n) \in K_x) \end{aligned}$$

for every $t \in A(x)$ and for all $m \geq m(x)$, $n \geq n(x)$ and for each $x \in S$. Since $S \subset \bigcup_{x \in S} A(x)$ and S is a compact set, by the Heine-Borel theorem S has a finite open covering such that

$$S \subset A(x_1) \cup A(x_2) \cup A(x_3) \cup ... \cup A(x_i).$$

Now, let

$$K = K_{x_1} \cap K_{x_2} \cap K_{x_3} \cap ... \cap K_{x_i}$$

and define

$$\begin{aligned} M &= \text{maks}\left\{m(x_1), m(x_2), m(x_3), ..., m(x_i)\right\}, \\ N &= \text{maks}\left\{n(x_1), n(x_2), n(x_3), ..., n(x_i)\right\}. \end{aligned}$$

Since for every K_{x_i} belong to $\mathcal{F}(\mathcal{I}_2)$, we have $K \in \mathcal{F}(\mathcal{I}_2)$. Then, when all $(m,n) \geq (M,N)$

$$0 \leq g_{mn}(t) < \varepsilon, \ \ (m,n) \in K,$$

for every $t \in A(x)$. So

$$g_{mn} \rightrightarrows_{\mathcal{I}_2^*} 0$$

on S. Since $\mathcal{I}_2$ is a strongly admissible ideal,

$$g_{mn} \rightrightarrows_{\mathcal{I}_2} 0$$

on S and by (1) we have

$$f_{mn} \rightrightarrows_{\mathcal{I}_2} f$$

on S. $\square$

Theorem 3.7. *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal, (X, d_x) and (Y, d_y) be two metric spaces, $f_{mn} : X \to Y$, $(m, n \in \mathbb{N})$, are equi-continuous and $f : X \to Y$. Assume that*

$$f_{mn} \to_{\mathcal{I}_2} f$$

on X. Then, f is continuous on X. Also, if X is compact then we have

$$f_{mn} \rightrightarrows_{\mathcal{I}_2} f$$

on X.

Proof. First we will prove that f is continuous on X. Let $x_0 \in X$ and $\varepsilon > 0$. By the equi-continuity of f_{mn}'s, there exists $\delta > 0$ such that

$$d_y\Big(f_{mn}(x), f_{mn}(x_0)\Big) < \frac{\varepsilon}{3},$$

for every $m, n \in \mathbb{N}$ and $x \in B_\delta(x_0)$ ($B_\delta(x_0)$ stands for an open ball in X with center x_0 and radius δ). Let $x \in B_\delta(x_0)$ be fixed. Since $f_{mn} \to_{\mathcal{I}_2} f$, the set

$$\Big\{(m,n) \in \mathbb{N}\times\mathbb{N} : d_y(f_{mn}(x_0), f(x_0)) \geq \frac{\varepsilon}{3}\Big\} \cup \Big\{(m,n) \in \mathbb{N}\times\mathbb{N} : d_y(f_{mn}(x), f(x)) \geq \frac{\varepsilon}{3}\Big\}$$

is in $\mathcal{I}_2$ and is different from $\mathbb{N}\times\mathbb{N}$. Hence, there exists $(m,n) \in \mathbb{N}\times\mathbb{N}$ such that

$$d_y\Big(f_{mn}(x_0), f(x_0)\Big) < \frac{\varepsilon}{3} \text{ and } d_y\Big(f_{mn}(x), f(x)\Big) < \frac{\varepsilon}{3}.$$

Thus, we have

$$\begin{aligned} d_y\Big(f(x_0), f(x)\Big) &\leq d_y\Big(f(x_0), f_{mn}(x_0)\Big) + d_y\Big(f_{mn}(x_0), f_{mn}(x)\Big) + d_y\Big(f_{mn}(x), f(x)\Big) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

so f is continuous on X.

Now, assume that X is compact. Let $\varepsilon > 0$. Since X is compact, it follows that f is uniformly continuous and f_{mn}'s are equi-uniformly continuous on X. So, pick $\delta > 0$ such that for any $x, x' \in X$ with

$$d_x(x, x') < \delta,$$

then, by equi-uniformly and uniformly continuous we have

$$d_y\Big(f_{mn}(x), f_{mn}(x')\Big) < \frac{\varepsilon}{3} \text{ ve } d_y\Big(f(x), f(x')\Big) < \frac{\varepsilon}{3}.$$

By the compactness of X, we can choose a finite subcover

$$B_{x_1}(\delta), B_{x_2}(\delta), ..., B_{x_k}(\delta)$$

from the cover $\{B_x(\delta)\}_{x\in X}$ of X. Using $f_{mn} \to_{\mathcal{I}_2} f$ pick a set $M \in \mathcal{I}_2$ such that

$$d_y\Big(f_{mn}(x_i), f(x_i)\Big) < \frac{\varepsilon}{3}, \ \ i \in \{1, 2, ..., k\},$$

for all $(m,n) \notin M$. Let $(m,n) \notin M$ and $x \in X$. Thus, $x \in B_{x_i}(\delta)$ for some $i \in \{1, 2, ..., k\}$. Hence, we have

$$\begin{aligned} d_y\Big(f_{mn}(x), f(x)\Big) &\leq d_y\Big(f_{mn}(x), f_{mn}(x_i)\Big) + d_y\Big(f_{mn}(x_i), f(x_i)\Big) + d_y\Big(f(x_i), f(x)\Big) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

and so

$$f_{mn} \rightrightarrows_{\mathcal{I}_2} f$$

on X. □

Definition 3.8. *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal and $\{f_{mn}\}$ be a double sequence of functions on $S \subset \mathbb{R}$. $\{f_{mn}\}$ is said to be $\mathcal{I}_2$-uniformly Cauchy if for every $\varepsilon > 0$ there exist $s = s(\varepsilon)$, $t = t(\varepsilon) \in \mathbb{N}$ such that*

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : |f_{mn}(x) - f_{st}(x)| \geq \varepsilon\} \in \mathcal{I}_2, \text{ for each } x \in S. \tag{2}$$

Now, we give $\mathcal{I}_2$-Cauchy criteria for $\mathcal{I}_2$-uniform convergence.

Theorem 3.9. *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal with the property (AP2) and let $\{f_{mn}\}$ be a sequence of bounded functions on $S \subset \mathbb{R}$. Then $\{f_{mn}\}$ is $\mathcal{I}_2$-uniformly convergent if and only if it is $\mathcal{I}_2$-uniformly Cauchy on S.*

Proof. Necessity of Theorem is similar to that of Lemma 2.3.

Conversely, assume that $\{f_{mn}\}$ is $\mathcal{I}_2$-uniformly Cauchy on S. Let $x \in S$ be fixed. By (2), for every $\varepsilon > 0$ there exist $s = s(\varepsilon)$ and $t = t(\varepsilon) \in \mathbb{N}$ such that

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : |f_{mn}(x) - f_{st}(x)| < \varepsilon\} \notin \mathcal{I}_2.$$

Hence, $\{f_{mn}\}$ is $\mathcal{I}_2$-Cauchy, so by Lemma 2.3 we have that $\{f_{mn}\}$ is $\mathcal{I}_2$-convergent to $f(x)$. Then, $\mathcal{I}_2 - \lim_{m,n\to\infty} f_{mn}(x) = f(x)$ on S. Note that since $\mathcal{I}_2$ satisfy the property (AP2), by (2) there is a $M \notin \mathcal{I}_2$ such that

$$|f_{mn}(x) - f_{st}(x)| < \varepsilon, \ \Big((m,n),(s,t) \in M\Big) \tag{3}$$

for all $m,n,s,t \geq N$ and $N = N(\varepsilon) \in \mathbb{N}$ and for each $x \in S$. By (3), for $s,t \to \infty$ we have

$$|f_{mn}(x) - f(x)| < \varepsilon, \ \Big((m,n) \in M\Big),$$

for all $m,n > N$ and for each $x \in S$. This shows that

$$f_{mn} \rightrightarrows_{\mathcal{I}_2^*} f$$

on S. Since $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ is a strongly admissible ideal we have

$$f_{mn} \rightrightarrows_{\mathcal{I}_2} f.$$

□

References

[1] B. Altay, F. Başar, Some new spaces of double sequences, J. Math. Anal. Appl. 309 (1)(2005) 70–90.
[2] V. Baláz, J. Červeňanský, P. Kostyrko, T. Šalát, $\mathcal{I}$-convergence and $\mathcal{I}$-continuity of real functions, Acta Mathematica, Faculty of Natural Sciences, Constantine the Philosopher University, Nitra, 5 (2004) 43–50.
[3] M. Balcerzak, K. Dems, A. Komisarski, Statistical convergence and ideal convergence for sequences of functions, J. Math. Anal. Appl. 328 (2007) 715–729.
[4] C. Çakan, B. Altay, Statistically boundedness and statistical core of double sequences, J. Math. Anal. Appl. 317 (2006) 690–697.
[5] P. Das, P. Kostyrko, W. Wilczyński, P. Malik, $\mathcal{I}$ and $\mathcal{I}^*$-convergence of double sequences, Math. Slovaca, 58 (2008), No. 5, 605–620.
[6] P. Das, P. Malik, On extremal $\mathcal{I}$-limit points of double sequences, Tatra Mt. Math. Publ. 40 (2008) 91–102.
[7] K. Demirci, $\mathcal{I}$-limit superior and limit inferior, Math. Commun. 6 (2001) 165–172.
[8] E. Dündar, B. Altay, On some properties of $\mathcal{I}_2$-convergence and $\mathcal{I}_2$-Cauchy of double sequences, Gen. Math. Notes, 7 (1)(2011) 1–12.
[9] E. Dündar, B. Altay, $\mathcal{I}_2$-convergence of double sequences of functions, Electronic Journal of Mathematical Analysis and Applications, Vol. 3(1) Jan. 2015, pp. 111–121.
[10] E. Dündar, B. Altay, $\mathcal{I}_2$-convergence and $\mathcal{I}_2$-Cauchy of double sequences, Acta Math. Sci. Ser. B Engl. Ed, 34B (2)(2014) 343–353.
[11] H. Fast, Sur la convergenc statistique, Colloq. Math. 2 (1951) 241–244.
[12] J. A. Fridy, C. Orhan, Statistical limit superior and inferior, Proc. Amer. Math. Soc. 125 (1997) 3625–3631.
[13] J. A. Fridy, On statistical convergence, Analysis, 5 (1985) 301–313.
[14] J. A. Fridy, Statistical limit points, Proc. Amer. Math. Soc. 118 (1993) 1187–1192.
[15] F. Gezer, S. Karakuş, $\mathcal{I}$ and $\mathcal{I}^*$ convergent function sequences, Math. Commun. 10 (2005) 71–80.
[16] A. Gökhan, M. Güngör and M. Et, Statistical convergence of double sequences of real-valued functions, Int. Math. Forum, 2 (8)(2007) 365–374.
[17] M. Gürdal, A. Şahiner, Extremal $\mathcal{I}_2$-limit points of double sequences, Appl. Math. E-Notes, 2 (2008) 131–137.
[18] P. Kostyrko, T. Salat and W. Wilczyński, $\mathcal{I}$-convergence, Real Anal. Exchange, 26 (2)(2000) 669–686.
[19] P. Kostyrko, M. Macaj, T. Salat and M. Sleziak, $\mathcal{I}$-convergence and extremal $\mathcal{I}$-limit points, Math. Slovaca, 55 (2005) 443–464.
[20] V. Kumar, On I and I^*-convergence of double sequences, Math. Commun. 12 (2007) 171–181.
[21] M. Mursaleen, O.H.H. Edely, Statistical convergence of double sequences, J. Math. Anal. Appl. 288 (2003) 223–231.
[22] A. Nabiev, S. Pehlivan and M. Gürdal, On $\mathcal{I}$-Cauchy sequence, Taiwanese J. Math. 11 (2)(2007) 569–576.
[23] F. Nuray, W. H. Ruckle, Generalized statistical convergence and convergence free spaces, J. Math. Anal. Appl. 245 (2000) 513–527.
[24] A. Pringsheim, Zur theorie der zweifach unendlichen Zahlenfolgen, Math. Ann. 53 (1900) 289–321.
[25] D. Rath, B. C. Tripaty, On statistically convergence and statistically Cauchy sequences, Indian J. Pure Appl. Math. 25 (4)(1994) 381–386.
[26] T. Šalát, B. C. Tripathy and M. Ziman,On $\mathcal{I}$-convergence field, Ital. J. Pure Appl. Math. 17 (2005) 45–54.
[27] T. Šalát, On statistically convergent sequences of real numbers, Math. Slovaca, 30 (1980) 139–150.
[28] I. J. Schoenberg, The integrability of certain functions and related summability methods, Amer. Math. Monthly 66 (1959) 361–375.
[29] B. Tripathy, B. C. Tripathy, On $\mathcal{I}$-convergent double sequences, Soochow J. Math. 31 (2005) 549–560.