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Univalence Conditions for a New Family of Integral Operators

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Abstract. For analytic functions in the open unit disk $\mathbb{U}$, we introduce a general family of integral operators. The main object of the this paper is to present a systematic study if this general family of integral operators and to determine the associated univalence conditions. Relevant connections of the results derived in this paper with those in several earlier works are also indicated.

1. Introduction, Definitions and Preliminaries

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (z \in \mathbb{U}), \tag{1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \qquad \text{and} \qquad |z| < 1\}$$

and satisfy the following normalization conditions:

$$f(0) = f'(0) - 1 = 0.$$

We denote by $\mathcal{S}$ the class of functions in $\mathcal{A}$ which are also univalent in $\mathbb{U}$ (see, for details, [4] and [11]).

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}^*(\kappa)$ of starlike functions of order κ $(0 \leqq \kappa < 1)$ in $\mathbb{U}$ if it satisfies the following inequality:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \kappa \qquad (z \in \mathbb{U};\ 0 \leqq \kappa < 1).$$

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We denote by $\mathcal{K}(\kappa)$ the class of convex functions of order κ $(0 \leqq \kappa < 1)$ in $\mathbb{U}$, that is, the class of functions in $\mathcal{A}$ which satisfy the following inequality:

$$\Re\left(\frac{zf''(z)}{f'(z)}+1\right) > \kappa \qquad (z \in \mathbb{U};\ 0 \leqq \kappa < 1).$$

A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{R}(\kappa)$ $(0 \leqq \beta < 1)$ if

$$\Re[f'(z)] > \kappa \qquad (z \in \mathbb{U};\ 0 \leqq \kappa < 1).$$

Recently, Frasin and Jahangiri [6] studied the class $\mathcal{B}(\mu, \kappa)$ $(\mu \geqq 0;\ 0 \leqq \kappa < 1)$, which consists of functions $f \in \mathcal{A}$ that satisfy the following condition:

$$\left| f'(z)\left(\frac{z}{f(z)}\right)^{\mu} - 1\right| < 1-\kappa \qquad (z \in \mathbb{U};\ 0 \leqq \kappa < 1;\ \mu \geqq 0). \tag{2}$$

This class $B(\mu, \kappa)$ is a comprehensive class of normalized analytic functions in $\mathbb{U}$ that contains several other classes of analytic and univalent functions in $\mathbb{U}$ such as

$$\mathcal{B}(1,\kappa) =: S^{*}_{\kappa}, \quad \mathcal{B}(0,\kappa) =: \mathcal{R}_{\kappa} \qquad \text{and} \qquad \mathcal{B}(2,\kappa) =: \mathcal{B}(\kappa).$$

In particular, the analytic and univalent function class $\mathcal{B}(\kappa)$ was studied by Frasin and Darus [5].

The problem of finding sufficient conditions for univalence of various integral operators has been investigated in many recent works (see, for example, [1–3, 9, 10, 12, 13]; see also the other relevant references cited in each of these earlier works). Here, in our present investigation, we study the univalence conditions for the function $I_{n,\beta}(z)$ given by the following integral operator:

$$I_{n,\beta}(z) := \left(\beta \int_0^z t^{\beta-1} \prod_{j=1}^{n} \left[\left(\frac{f_j(t)}{t}\right)\left(\frac{g_j(t)}{t}\right)^{\gamma_j}\right] dt\right)^{\frac{1}{\beta}} \tag{3}$$

when $\Re(\beta) > 0$ and the functions $f_1(z), \cdots, f_n(z)$ and $g_1(z), \cdots, g_n(z)$ are constrained suitably.

We note here that the following theorems on univalence conditions of certain given integral operators were proven recently by Pascu [9] and Pescar [10], respectively.

Theorem 1. (see Pascu [9]) *Let $f \in \mathcal{A}$ and $\beta \in \mathbb{C}$. If $\Re(\beta) > 0$ and*

$$\frac{1-|z|^{2\Re(\beta)}}{\Re(\beta)} \left|\frac{zf''(z)}{f'(z)}\right| \leqq 1 \qquad (z \in \mathbb{U}),$$

then the function $F_\beta(z)$ defined by

$$F_\beta(z) := \left(\beta \int_0^z t^{\beta-1} f'(t) dt\right)^{\frac{1}{\beta}} \qquad (z \in \mathbb{U}),$$

is in the class $\mathcal{S}$ of analytic and univalent functions in $\mathbb{U}$.

Theorem 2. (see Pescar [10]) *Let $c, \alpha \in \mathbb{C}$ with*

$$\Re(\alpha) > 0 \qquad \text{and} \qquad |c| \leqq 1 \quad (c \neq -1).$$

If the function $f(z)$, regular in $\mathbb{U}$, is given by (1) *and*

$$\left| c\,|z|^{2\alpha} + \left(1-|z|^{2\alpha}\right) \frac{zf''(z)}{\alpha f'(z)}\right| \leqq 1 \qquad (z \in \mathbb{U}), \tag{4}$$

then the function $F_\alpha(z)$ given by

$$F_\alpha(z) = \left(\alpha \int_0^z t^{\alpha-1} f'(t)dt\right)^{\frac{1}{\alpha}} = z + \sum_{k=2}^{\infty} a_k z^k \qquad (z \in \mathbb{U})$$

is in the class $\mathcal{S}$ of regular and univalent functions in $\mathbb{U}$.

In order to derive our main results, we recall here the General Schwarz Lemma as follows.

General Schwarz Lemma (see, for example, [7] and [8]). *Let the function f be regular in the disk*

$$\mathbb{U}_R = \{z : z \in \mathbb{C} \qquad \text{and} \qquad |z| < R\}$$

with

$$\left|f(z)\right| < M \qquad (|z| < \mathbb{U}_R;\ M > 0)$$

for fixed $M > 0$. If f has one zero at $z = 0$ with multiplicity $\geqq m$, then

$$\left|f(z)\right| \leqq \frac{M}{R^m} |z|^m \qquad (z \in \mathbb{U}_R). \tag{5}$$

The equality in (5) *holds true for $z \neq 0$ only if*

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m \qquad (z \in \mathbb{U}_R),$$

where θ is a constant.

2. Univalence Conditions on the Class $\mathcal{B}(\mu, \alpha)$

In this section, we first prove the univalence condition for the function $I_{n,\beta}(z)$ which is given in terms of the integral operator defined by (3).

Theorem 3. *Let the functions $f_j, g_j \in \mathcal{A}$ $(j = 1, \cdots, n)$. Suppose that*

$$\beta, \gamma_j \in \mathbb{C}, \quad \Re(\beta) > 0 \qquad \text{and} \qquad M_j, N_j \geqq 1 \qquad (j = 1, \cdots, n).$$

Also let

$$\Re(\beta) \geqq \sum_{j=1}^{n} \left[\left(\left(2 - \alpha_j\right) M_j^{\mu_j - 1} + 1\right) + |\gamma_j| \left(\left(2 - \alpha_j\right) N_j^{\mu_j - 1} + 1\right)\right]. \tag{6}$$

If

$$f_j, g_j \in \mathcal{B}(\mu_j, \alpha_j), \quad 0 \leqq \alpha_j < 1 \qquad \text{and} \qquad \mu_j \geqq 0 \qquad (j = 1, \cdots, n)$$

and

$$\left|f_j(z)\right| \leqq M_j \qquad \text{and} \qquad \left|g_j(z)\right| \leqq N_j \qquad (z \in \mathbb{U};\ j = 1, \cdots, n),$$

then the function $I_{n,\beta}(z)$ given by the integral operator (3) *is in the class $\mathcal{S}$ of analytic and univalent functions in $\mathbb{U}$.*

Proof. We begin by considering the function $h(z)$ defined by

$$h(z) := \int_0^z \prod_{j=1}^{n} \left[\left(\frac{f_j(t)}{t}\right)\left(\frac{g_j(t)}{t}\right)^{\gamma_j}\right] dt \qquad (z \in \mathbb{U}). \tag{7}$$

For this function $h(z)$, which is regular in $\mathbb{U}$, we calculate the first-order and the second-order derivatives as follows:

$$h'(z) = \prod_{j=1}^{n} \left[\left(\frac{f_j(z)}{z} \right) \left(\frac{g_j(z)}{z} \right)^{\gamma_j} \right] \tag{8}$$

and

$$\begin{aligned} h''(z) = &\sum_{j=1}^{n} \left[\left(\frac{z f_j'(z) - f_j(z)}{z^2} \right) \left(\frac{g_j(z)}{z} \right)^{\gamma_j} \right] \prod_{\substack{k=1 \\ (k \neq j)}}^{n} \left[\left(\frac{f_k(z)}{z} \right) \left(\frac{g_k(z)}{z} \right)^{\gamma_k} \right] \\ &+ \sum_{j=1}^{n} \left[\left(\frac{f_j(z)}{z} \right) \gamma_j \left(\frac{g_j(z)}{z} \right)^{\gamma_j - 1} \left(\frac{z g_j'(z) - g_j(z)}{z^2} \right) \right] \\ &\cdot \prod_{\substack{k=1 \\ (k \neq j}}^{n} \left[\left(\frac{f_k(z)}{z} \right) \left(\frac{g_k(z)}{z} \right)^{\gamma_k} \right]. \end{aligned} \tag{9}$$

From (8) and (9), we get

$$\frac{z h''(z)}{h'(z)} = \sum_{j=1}^{n} \left[\left(\frac{z f_j'(z)}{f_j(z)} - 1 \right) + \gamma_j \left(\frac{z g_j'(z)}{g_j(z)} - 1 \right) \right], \tag{10}$$

which readily yields

$$\left| \frac{z h''(z)}{h'(z)} \right| \leqq \sum_{j=1}^{n} \left(\left| \frac{z f_j'(z)}{f_j(z)} - 1 \right| + \left| \gamma_j \right| \left| \frac{z g_j'(z)}{g_j(z)} - 1 \right| \right). \tag{11}$$

Thus, clearly, we find from this last inequality (11) that

$$\begin{aligned} &\frac{1 - |z|^{2\Re(\beta)}}{\Re(\beta)} \left| \frac{z h''(z)}{h'(z)} \right| \\ &\quad \leqq \frac{1 - |z|^{2\Re(\beta)}}{\Re(\beta)} \sum_{j=1}^{n} \left[\left(\left| \frac{z f_j'(z)}{f_j(z)} \right| + 1 \right) + \left| \gamma_j \right| \left(\left| \frac{z g_j'(z)}{g_j(z)} \right| + 1 \right) \right] \\ &\quad \leqq \frac{1 - |z|^{2\Re(\beta)}}{\Re(\beta)} \sum_{j=1}^{n} \left(\left| f_j'(z) \left(\frac{z}{f_j(z)} \right)^{\mu_j} \right| \cdot \left| \frac{f_j(z)}{z} \right|^{\mu_j - 1} + 1 \right) \\ &\quad\quad + \frac{1 - |z|^{2\Re(\beta)}}{\Re(\beta)} \sum_{j=1}^{n} |\gamma_j| \left(\left| g_j'(z) \left(\frac{z}{g_j(z)} \right)^{\mu_j} \right| \cdot \left| \frac{g_j(z)}{z} \right|^{\mu_j - 1} + 1 \right). \end{aligned} \tag{12}$$

By the hypothesis of Theorem 3, we have

$$|f_j(z)| \leqq M_j \qquad \text{and} \qquad |g_j(z)| \leqq N_j \qquad (z \in \mathbb{U};\ j = 1, \cdots, n).$$

Therefore, by applying the General Schwarz Lemma to the functions $f_1, \cdots, f_n$ and $g_1, \cdots, g_n$, we obtain

$$|f_j(z)| \leqq M_j |z| \qquad \text{and} \qquad |g_j(z)| \leqq N_j |z| \qquad (z \in \mathbb{U};\ j = 1, \cdots, n). \tag{13}$$

Now, by using the inequalities (2) and (13), we get

$$\begin{aligned}
&\frac{1-|z|^{2\Re(\beta)}}{\Re(\beta)}\left|\frac{zh''(z)}{h'(z)}\right| \\
&\quad \leqq \frac{1-|z|^{2\Re(\beta)}}{\Re(\beta)}\sum_{j=1}^{n}\left[\left(\left|f_j'(z)\left(\frac{z}{f_j(z)}\right)^{\mu_j}-1\right|+1\right)M_j^{\mu_j-1}+1\right] \\
&\qquad + \frac{1-|z|^{2\Re(\beta)}}{\Re(\beta)}\sum_{j=1}^{n}|\gamma_j|\left[\left(\left|g_j'(z)\left(\frac{z}{g_j(z)}\right)^{\mu_j}-1\right|+1\right)N_j^{\mu_j-1}+1\right] \qquad (14) \\
&\qquad\qquad (z \in \mathbb{U}),
\end{aligned}$$

which can be rewritten as follows:

$$\begin{aligned}
&\frac{1-|z|^{2\Re(\beta)}}{\Re(\beta)}\left|\frac{zh''(z)}{h'(z)}\right| \\
&\quad \leqq \frac{1}{\Re(\beta)}\sum_{j=1}^{n}\left[\left(\left(2-\alpha_j\right)M_j^{\mu_j-1}+1\right)+|\gamma_j|\left(\left(2-\alpha_j\right)N_j^{\mu_j-1}+1\right)\right] \qquad (15) \\
&\qquad\qquad (z \in \mathbb{U}).
\end{aligned}$$

If we make use of the condition (6) from the hypothesis of Theorem 3, this last inequality (15) yields

$$\frac{1-|z|^{2\Re(\beta)}}{\Re(\beta)}\left|\frac{zh''(z)}{h'(z)}\right| \leqq 1 \qquad (z \in \mathbb{U}). \tag{16}$$

Finally, we apply Theorem 1 to the function $h(z)$ defined by (7). We thus conclude that the function $I_{n,\beta}(z)$ given by (3) is in the class $\mathcal{S}$ of analytic and univalent functions in $\mathbb{U}$. □

Corollary 1. *Let the functions $f_j(z)$ $(j = 1, \cdots, n)$ and $g_j(z)$ $(j = 1, \cdots, n)$ be in the class $\mathcal{A}$. Suppose also that $\beta, \gamma_j \in \mathbb{C}$ $(j = 1, \cdots, n)$ with $\Re(\beta) > 0$ and*

$$\Re(\beta) \geqq \sum_{j=1}^{n}\left[\left(3-\alpha_j\right)+|\gamma_j|\left(3-\alpha_j\right)\right]. \tag{17}$$

If $f_j, g_j \in \mathcal{S}^*(\alpha_j)$ $(j = 1, \cdots, n)$ for $0 \leqq \alpha_j \leqq 1$ $(j = 1, \cdots, n)$ and

$$|f_j(z)| \leqq 1 \qquad \text{and} \qquad |g_j(z)| \leqq 1 \qquad (z \in \mathbb{U};\ j = 1, \cdots, n),$$

then the function $I_{n,\beta}(z)$ given by (3) *under that above constraints is in the class $\mathcal{S}$ of analytic and univalent functions in $\mathbb{U}$.*

Proof. Corollary 1 follows readily by setting

$$\mu_j = M_j = N_j = 1 \qquad (j = 1, \cdots, n)$$

in Theorem 3. □

Corollary 2. *Let the functions $f(z)$ and $g(z)$ be in the class $\mathcal{A}$. Suppose also that $\beta, \gamma \in \mathbb{C}$ with $\Re(\beta) > 0$, $M \geqq 1$, $N \geqq 1$ and*

$$\Re(\beta) \geqq ((2-\alpha)M^{\mu-1}+1)+|\gamma|((2-\alpha)N^{\mu-1}+1). \tag{18}$$

If $f, g \in \mathcal{B}(\mu, \alpha)$ $(0 \leqq \alpha < 1;\ \mu \geqq 0)$ *and*

$$|f(z)| \leqq M \qquad \text{and} \qquad |g(z)| \leqq N \qquad (z \in \mathbb{U}),$$

then the function $J_\beta(z)$ *given by*

$$J_\beta(z) := \left[\beta \int_0^z t^{\beta-1} \left(\frac{f(t)}{t}\right)\left(\frac{g(t)}{t}\right)^\gamma dt\right]^{\frac{1}{\beta}} \tag{19}$$

is in the class $\mathcal{S}$ *of analytic and univalent functions in* $\mathbb{U}$.

Proof. Since

$$J_\beta(z) = I_{1,\beta}(z) \qquad \big(z \in \mathbb{U};\ \Re(\beta) > 0\big),$$

which is an immediate consequence of the definitions (3) and (19), Corollary 2 corresponds to the special case of Theorem 3 when $n = 1$. □

Next, by using Theorem 2 of Pescar [10], we get the following result.

Theorem 4. *Let the functions* $f_j, g_j \in \mathcal{A}$ $(j = 1, \cdots, n)$. *Suppose that*

$$c, \beta, \gamma_j \in \mathbb{C}, \quad \Re(\beta) > 0 \qquad \text{and} \qquad M_j, N_j \geqq 1 \qquad (j = 1, \cdots, n).$$

Also let

$$\Re(\beta) \geqq \sum_{j=1}^n \left[\left(\left(2-\alpha_j\right)M_j^{\mu_j-1} + 1\right) + |\gamma_j|\left(\left(2-\alpha_j\right)N_j^{\mu_j-1} + 1\right)\right] \tag{20}$$

and

$$|c| \leqq 1 - \frac{1}{\Re(\beta)} \sum_{j=1}^n \left[\left(\left(2-\alpha_j\right)M_j^{\mu_j-1} + 1\right) + |\gamma_j|\left(\left(2-\alpha_j\right)N_j^{\mu_j-1} + 1\right)\right]. \tag{21}$$

If

$$f_j, g_j \in \mathcal{B}(\mu_j, \alpha_j), \quad 0 \leqq \alpha_j < 1 \qquad \text{and} \qquad \mu_j \geqq 0 \qquad (j = 1, \cdots, n)$$

and

$$\left|f_j(z)\right| \leqq M_j \qquad \text{and} \qquad \left|g_j(z)\right| \leqq N_j \qquad (z \in \mathbb{U};\ j = 1, \cdots, n),$$

then the function $I_{n,\beta}(z)$ *given by the integral operator* (3) *is in the class* $\mathcal{S}$ *of analytic and univalent functions in* $\mathbb{U}$.

Proof. Just as in the proof of Theorem 3, we have

$$\frac{zh''(z)}{h'(z)} = \sum_{j=1}^n \left[\left(\frac{zf_j'(z)}{f_j(z)} - 1\right) + \gamma_j\left(\frac{zg_j'(z)}{g_j(z)} - 1\right)\right] \qquad (z \in \mathbb{U}), \tag{22}$$

which, for a given constant $c \in \mathbb{C}$, yields

$$\begin{aligned}
&\left| c|z|^{2\beta} + (1-|z|^{2\beta})\frac{zh''(z)}{\beta h'(z)} \right| \\
&\quad = \left| c|z|^{2\beta} + (1-|z|^{2\beta})\frac{1}{\beta}\sum_{j=1}^{n}\left[\left(\frac{zf_j'(z)}{f_j(z)} - 1\right) + \gamma_j\left(\frac{zg_j'(z)}{g_j(z)} - 1\right)\right] \right| \\
&\quad \leqq |c| + \frac{1}{|\beta|}\sum_{j=1}^{n}\left[\left(\left|\frac{zf_j'(z)}{f_j(z)}\right| + 1\right) + |\gamma_j|\left(\left|\frac{zg_j'(z)}{g_j(z)}\right| + 1\right)\right] \\
&\quad \leqq |c| + \frac{1}{|\beta|}\sum_{j=1}^{n}\left(\left|f_j'(z)\left(\frac{z}{f_j(z)}\right)^{\mu_j}\right| \cdot \left|\frac{f_j(z)}{z}\right|^{\mu_j-1} + 1\right) \\
&\qquad + \frac{1}{|\beta|}\sum_{j=1}^{n}\left(|\gamma_j|\left|g_j'(z)\left(\frac{z}{g_j(z)}\right)^{\mu_j}\right| \cdot \left|\frac{g_j(z)}{z}\right|^{\mu_j-1} + 1\right) \qquad (z \in \mathbb{U}).
\end{aligned} \tag{23}$$

Now, from the hypothesis of Theorem 4, we have

$$|f_j(z)| \leqq M_j \qquad \text{and} \qquad |g_j(z)| \leqq N_j \qquad (z \in \mathbb{U};\ j = 1, \cdots, n).$$

Applying the General Schwarz Lemma to the functions $f_1, \cdots, f_n$ and $g_1, \cdots, g_n$, we obtain

$$|f_j(z)| \leqq M_j|z| \qquad \text{and} \qquad |g_j(z)| \leqq N_j|z| \qquad (z \in \mathbb{U};\ j = 1, \cdots, n), \tag{24}$$

which, in conjunction with the inequality (2), leads us to following result:

$$\begin{aligned}
&\left| c|z|^{2\beta} + (1-|z|^{2\beta})\frac{zh''(z)}{\beta h'(z)} \right| \\
&\leqq |c| + \frac{1}{|\beta|}\sum_{j=1}^{n}\left[\left(\left|f_j'(z)\left(\frac{z}{f_j(z)}\right)^{\mu_j} - 1\right| + 1\right)M_j^{\mu_j-1} + 1\right] \\
&\qquad + \frac{1}{|\beta|}\sum_{j=1}^{n}|\gamma_j|\left[\left(\left|g_j'(z)\left(\frac{z}{g_j(z)}\right)^{\mu_j} - 1\right| + 1\right)N_j^{\mu_j-1} + 1\right] \\
&\leqq |c| + \frac{1}{\Re(\beta)}\sum_{j=1}^{n}\left[\left(\left(2-\alpha_j\right)M_j^{\mu_j-1} + 1\right) + |\gamma_j|\left(\left(2-\alpha_j\right)N_j^{\mu_j-1} + 1\right)\right] \\
&\qquad\qquad (z \in \mathbb{U}).
\end{aligned} \tag{25}$$

Thus, from the condition (21) of Theorem 4, we find that

$$\left| c|z|^{2\beta} + (1-|z|^{2\beta})\frac{zh''(z)}{\beta h'(z)} \right| \leqq 1 \qquad (z \in \mathbb{U}). \tag{26}$$

Finally, by applying Theorem 2 to the function $h(z)$ given by (7), we deduce the desired assertion that the function $I_{n,\beta}(z)$ given by the integral operator (3) is in the class $\mathcal{S}$ of analytic and univalent functions in $\mathbb{U}$. $\square$

Corollary 3. *Let the functions $f_j(z)$ $(j = 1, \cdots, n)$ and $g_j(z)$ $(j = 1, \cdots, n)$ be in the class $\mathcal{A}$. Suppose also that $c, \beta, \gamma_j \in \mathbb{C}$ $(j = 1, \cdots, n)$ with $\Re(\beta) > 0$ and*

$$\Re(\beta) \geqq \sum_{j=1}^{n}\left[\left(3-\alpha_j\right) + |\gamma_j|\left(3-\alpha_j\right)\right]. \tag{27}$$

If $f_j, g_j \in \mathcal{S}^*(\alpha_j)$ $(j = 1, \cdots, n)$ *for* $0 \leqq \alpha_j \leqq 1$ $(j = 1, \cdots, n)$,

$$|f_j(z)| \leqq 1 \qquad \text{and} \qquad |g_j(z)| \leqq 1 \qquad (z \in \mathbb{U};\ j = 1, \cdots, n)$$

and

$$|c| \leqq 1 - \frac{1}{\Re(\beta)} \sum_{j=1}^{n} \left[\left(3 - \alpha_j\right) + |\gamma_j| \left(3 - \alpha_j\right)\right], \tag{28}$$

then the function $I_{n,\beta}(z)$ *given by* (3) *under the above constraints is in the class* $\mathcal{S}$ *of analytic and univalent functions in* $\mathbb{U}$.

Proof. Corollary 3 follows easily upon setting

$$\mu_j = M_j = N_j = 1 \qquad (j = 1, \cdots, n)$$

in Theorem 4. □

Corollary 4. *Let the functions* $f(z)$ *and* $g(z)$ *be in the class* $\mathcal{A}$. *Suppose also that* $c, \beta, \gamma \in \mathbb{C}$ *with* $\Re(\beta) > 0$, $M \geqq 1$, $N \geqq 1$ *and*

$$\Re(\beta) \geqq ((2 - \alpha)M^{\mu-1} + 1) + |\gamma|((2 - \alpha)N^{\mu-1} + 1). \tag{29}$$

If $f, g \in \mathcal{B}(\mu, \alpha)$ $(0 \leqq \alpha < 1;\ \mu \geqq 0)$,

$$|f(z)| \leqq M \qquad \text{and} \qquad |g(z)| \leqq N \qquad (z \in \mathbb{U})$$

and

$$|c| \leqq 1 - \frac{1}{\Re(\beta)} \left[\left(2 - \alpha)M^{\mu-1} + 1\right) + |\gamma|(2 - \alpha)N^{\mu-1} + 1\right)\right], \tag{30}$$

then the function $J_\beta(z)$ *given by* (19) *is in the class* $\mathcal{S}$ *of analytic and univalent functions in* $\mathbb{U}$.

Proof. In its special case when $n = 1$, Theorem 4 would obviously correspond to Corollary 4. □

3. Concluding Remarks and Observations

Our present investigation was motivated essentially by several recent works dealing with the interesting problem of finding sufficient conditions for univalence of normalized analytic functions which are defined in terms of various families integral operators (see, for example, [1–3, 9, 10, 12, 13]; see also the other relevant references cited in each of these earlier works). In our study here, we have successfully determined the univalence conditions for the function $I_{n,\beta}(z)$ given by the general family of integral operators in (3).

Our main results (Theorems 3 and 4 in this paper) are shown to yield several corollaries and consequences. Some of these applications of our main results are stated here as Corollaries 1, 2, 3 and 4.

Derivations of further corollaries and consequences of the results presented in this paper, including also their connections with known results given in several earlier works, are being left here as exercises for the interested reader.

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