On Generalized Lorentz Sequence Space Defined by Modulus Functions

Oğuz Oğur, Cenap Duyar

Öndokuz Mayıs University, Art and Science Faculty, Department of Mathematics, Kurupelit campus, Samsun, TURKEY

Abstract. The object of this paper is to introduce generalized Lorentz sequence spaces \( L(f,v,p) \) defined by modulus function \( f \). Also we study some topologic properties of this space and obtain some inclusion relations.

1. Introduction

Throughout this work, \( \mathbb{N}, \mathbb{R} \) and \( \mathbb{C} \) denote the set of positive integers, real numbers and complex numbers, respectively. The concept of modulus function was introduced by Nakano [11]. We recall that a function \( f : [0, \infty) \to [0, \infty) \) is said to be a modulus function if it satisfies the following properties

1) \( f(x) = 0 \) if and only if \( x = 0 \);
2) \( f(x+y) \leq f(x) + f(y) \) for all \( x, y \in [0, \infty) \);
3) \( f \) is increasing;
4) \( f \) is continuous from right at 0.

It follows that \( f \) is continuous on \([0, \infty)\). The modulus function may be bounded or unbounded. For example, if we take \( f(x) = x/(x+1) \), then \( f(x) \) is bounded. But, for \( 0 < p < 1 \), \( f(x) = x^p \) is not bounded.

By the condition 2), we have \( f(nx) \leq nf(x) \) for all \( n \in \mathbb{N} \) and so \( f(x) = f\left(nx^\frac{1}{n}\right) \leq nf\left(\frac{x}{n}\right) \), and hence

\[
\frac{1}{n} f(x) \leq f\left(\frac{x}{n}\right)
\]

for all \( n \in \mathbb{N} \).

The FK-spaces \( L(f) \), introduced by Ruckle in [14], is in the form

\[
L(f) = \left\{ x \in \ell_\infty : \sum_{k=1}^{\infty} f(|x_k|) < \infty \right\}
\]

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Email addresses: oguz.ogur@omu.edu.tr (Oğuz Oğur), cenapd@omu.edu.tr (Cenap Duyar)
where \( f \) is a modulus function and \( w \) is the space of all complex sequences. This space is closely related to the space \( \ell_1 \), which is an \( L(f) \) space with \( f(x) = x \) for all \( x \geq 0 \). Later on, this space was investigated by many authors in [1], [4], [8], [9], [15].

The notion of paranorm is closely related to linear metric spaces. Let \( X \) be a linear space. A function \( p : X \to \mathbb{R} \) is called paranorm, if

1. \( p(0) = 0 \),
2. \( p(x) \geq 0 \) for all \( x \in X \),
3. \( p(-x) = p(x) \) for all \( x \in X \),
4. \( p(x + y) \leq p(x) + p(y) \) for all \( x, y \in X \),
5. \( (\lambda_n) \) be a sequence in \( \mathbb{C} \), \( \lambda \) be an element in \( \mathbb{C} \), \( \{x_n\} \) be a sequence in \( X \) and \( x \) be an element in \( X \). If \( |\lambda_n - \lambda| \to 0 \) as \( n \to \infty \) and \( p(x_n - x) \to 0 \) as \( n \to \infty \), then \( p(\lambda_n x_n - \lambda x) \to 0 \) as \( n \to \infty \) (continuity of multiplication by scalars).

A paranorm \( p \) for which \( p(x) = 0 \) implies \( x = 0 \) is called total [7].

The Lorentz space was introduced by G. G. Lorentz in [5], [6]. This space play an important role in the theory of Banach space. Many authors studied these spaces and explored their many properties.

Let \( (E, \|\|) \) be a Banach space. The Lorentz sequence space \( l(p, q, E) \) (or \( l_{p,q}(E) \)) for \( 1 \leq p, q \leq \infty \) is the collection of all sequences \( \{a_n\} \in c_0(E) \) such that

\[
\|\{a_n\}\|_{p,q} = \left\{ \sum_{i=1}^{\infty} \frac{\|a_{x_i}\|^q}{\sup_j 1/j^q \|a_{x_i}\|^q} \right\}^{1/q} \text{ for } 1 \leq p \leq \infty, \quad 1 \leq q < \infty
\]

\[
\|\{a_n\}\|_{p,q} = \sup_{i} \left( \sum_{n=1}^{\infty} \|a_{x_i}\|^q \|v(i)\|^q \right)^{1/p} \text{ for } 1 \leq p < \infty, \quad q = \infty
\]

is finite, where \( \|\{a_{x_0}\}\| \) is non-increasing rearrangement of \( \|\{a_i\}\| \) (We can interpret that the decreasing rearrangement \( \|\{a_{x_0}\}\| \) is obtained by rearranging \( \|\{a_i\}\| \) in decreasing order). This space was introduced by Miyazaki in [10] and examined comprehensively by Kato in [3].

A weight sequence \( v = \{v(i)\} \) is a positive decreasing sequence such that \( v(1) = 1 \), \( \lim_{i \to \infty} v(i) = 0 \) and \( \lim_{i \to \infty} V(i) = \infty \), where \( V(i) = \sum_{n=1}^{i} v(n) \) for every \( i \in \mathbb{N} \). Popa [13] defined the generalized Lorentz sequence space \( d(v, p) \) for \( 0 < p < \infty \) as follows

\[
d(v, p) = \left\{ x = \{x_i\} \in \mathcal{W} : \|x\|_{v,p} = \sup_{\pi} \left( \sum_{n=1}^{\infty} |x_{\pi(i)}|^p v(i) \right)^{1/p} < \infty \right\}
\]

where \( \pi \) ranges over all permutations of the positive integers and \( v = \{v(i)\} \) is a weight sequence. It is known that \( d(v, p) \subset c_0 \) and hence for each \( x \in d(v, p) \) there exists a non-increasing rearrangement \( \{x'\} = \{x'_i\} \) of \( x \) and

\[
\|x\|_{v,p} = \left( \sum_{n=1}^{\infty} |x'_n|^p v(n) \right)^{1/p}
\]

(see [12], [13]).

Let \( (X, \|\|) \) be a Banach space, \( f \) be a modulus function and \( v = \{v(n)\} \) be a weight sequence. We introduce the generalized Lorentz sequence space \( L(f, v, p) \) for \( 0 < p < \infty \) using a modulus function \( f \). The space \( L(f, v, p) \) is the collection of all \( X \)-valued \( 0 \)-sequences \( \{x_n\} \) \( (\{x_n\} \in c_0(X)) \) such that

\[
g(x) = \left( \sum_{n=1}^{\infty} f(\|x_{\pi(n)}\|) v(n) \right)^{1/p}
\]
is finite, where \( \{\|x_{\psi(\alpha)}\|\} \) is non-increasing rearrangement of \( \{\|x_n\|\} \). If we take \( f(x) = x \), then \( L(f, v, p) = d(v, p) \) ([13]).

We shall need the following lemmas.

**Lemma 1.1.** (Hardy, Littlewood and Polya [2]) Let \( \{c_i^+\} \) and \( \{c_i^-\} \) be the non-increasing and non-decreasing rearrangements of a finite sequence \( \{c_i\}_{1 \leq i \leq n} \) of positive numbers, respectively. Then for two sequences \( \{a_i\}_{1 \leq i \leq n} \) and \( \{b_i\}_{1 \leq i \leq n} \) of positive numbers we have

\[
\sum_i a_i^+ \cdot b_i \leq \sum_i a_i^- \cdot b_i \leq \sum_i a_i^- \cdot b_i^+ .
\]

**Lemma 1.2.** (Kato [3]) Let \( \{x_i^{(o)}\} \) be an \( X \)-valued double sequence such that \( \lim_{n \to \infty} x_i^{(o)} = 0 \) for each \( \mu \in \mathbb{N} \) and let \( \{x_i\} \) be an \( X \)-valued sequence such that \( \lim_{n \to \infty} x_i^{(o)} = x_i \) (uniformly in \( i \)). Then \( \lim_{n \to \infty} x_i = 0 \) and for each \( i \in \mathbb{N} \)

\[
\|x_{\psi(\alpha)}\| \leq \limsup_{\mu \to \infty} \|x_i^{(o)}\|,
\]

where \( \{\|x_{\psi(\alpha)}\|\} \) and \( \{\|x_i^{(o)}\|\} \) are the non-increasing rearrangements of \( \{\|x_i\|\} \) and \( \{\|x_i^{(o)}\|\} \), respectively.

**Lemma 1.3.** Let \( f \) be any modulus function and \( 0 < \delta < 1 \). Then

\[
f(x) \leq \frac{2f(1)}{\delta}x
\]

for all \( x \geq 0 \) [9].

**Lemma 1.4.** For any modulus \( f \) there exists \( \lim_{t \to \infty} \frac{f(t)}{t} \) [9].

**Lemma 1.5.** Let \( f \) be any modulus with \( \lim_{t \to \infty} \frac{f(t)}{t} = \alpha > 0 \). Then there is a constant \( \beta > 0 \) such that

\[
f(t) \geq \beta t
\]

for all \( t \geq 0 \) [9].

2. **Main Results**

**Theorem 2.1.** The space \( L(f, v, p) \) for \( 0 < p < \infty \) is a linear space over the field \( K = \mathbb{R} \) or \( \mathbb{C} \).

**Proof.** Let \( x, y \in L(f, v, p) \) and let \( \{\|x_{\psi(\alpha)}\|\}, \{\|y_{\psi(\alpha)}\|\} \) and \( \{\|x_{\psi(\alpha)} + y_{\psi(\alpha)}\|\} \) be the non-increasing rearrangements of the sequences \( \{\|x_n\|\}, \{\|y_n\|\} \) and \( \{\|x_n + y_n\|\} \), respectively. Since \( v \) is non-increasing and \( f \) is increasing, by the Lemma 1 we have

\[
\sum_{n=1}^{\infty} \left[ f(\|x_{\psi(\alpha)} + y_{\psi(\alpha)}\|) \right]^p v(n) \leq \sum_{n=1}^{\infty} \left[ f(\|x_{\psi(\alpha)}\|) + f(\|y_{\psi(\alpha)}\|) \right]^p v(n)
\]

\[
\leq \sum_{n=1}^{\infty} \left[ f(\|x_{\psi(\alpha)}\|) \right]^p v(n) + \sum_{n=1}^{\infty} \left[ f(\|y_{\psi(\alpha)}\|) \right]^p v(n)
\]

\[
\leq D \sum_{n=1}^{\infty} \left( \left[ f(\|x_{\psi(\alpha)}\|) \right]^p v(n) + \left[ f(\|y_{\psi(\alpha)}\|) \right]^p v(n) \right)
\]

\[
\leq D \left( \sum_{n=1}^{\infty} \left[ f(\|x_{\psi(\alpha)}\|) \right]^p v(n) + \sum_{n=1}^{\infty} \left[ f(\|y_{\psi(\alpha)}\|) \right]^p v(n) \right)
\]

\[
< \infty,
\]
where $D = \max\{1, 2^{\alpha-1}\}$. Let $\alpha \in K$, then there exists $M_\alpha \in \mathbb{N}$ such that $|\alpha| \leq M_\alpha$. Hence we get

$$\begin{align*}
\sum_{n=1}^{\infty} \left[ f \left( \|x_{\phi(n)}\| \right) \right]^p v(n) & \leq \sum_{n=1}^{\infty} \left[ M_\alpha \|x_{\phi(n)}\| \right]^p v(n) \\
& \leq M_\alpha^{p}\sum_{n=1}^{\infty} \left[ \|x_{\phi(n)}\| \right]^p v(n) \\
& < \infty.
\end{align*}$$

This shows that $x + y \in L(f, v, p)$, $\alpha x \in L(f, v, p)$ and so $L(f, v, p)$ is a linear space. $\square$

**Theorem 2.2.** The space $L(f, v, p)$ for $1 \leq p < \infty$ is paranormed space with the paranorm

$$g(x) = \left( \sum_{n=1}^{\infty} \left[ f \left( \|x_{\phi(n)}\| \right) \right]^p v(n) \right)^{\frac{1}{p}},$$

where $\{\|x_{\phi(n)}\|\}$ denotes the non-increasing rearrangements of $\{\|x_n\|\}$.

**Proof.** It is clear that $g(x) = g(-x)$ and $g(0) = 0$. Let $x, y \in L(f, v, p)$. Since $f$ is increasing and weight sequence $v$ is decreasing, by Lemma 1 we have

$$g(x + y) = \left( \sum_{n=1}^{\infty} \left[ f \left( \|x_{\phi(n)} + y_{\phi(n)}\| \right) \right]^p v(n) \right)^{\frac{1}{p}} \leq \left( \sum_{n=1}^{\infty} \left[ f \left( \|x_{\phi(n)}\| \right) \right]^p v(n) \right)^{\frac{1}{p}} + \left( \sum_{n=1}^{\infty} \left[ f \left( \|y_{\phi(n)}\| \right) \right]^p v(n) \right)^{\frac{1}{p}} \leq g(x) + g(y),$$

where $\{\|x_{\phi(n)}\|\}$, $\{\|y_{\phi(n)}\|\}$ and $\{\|x_{\phi(n)} + y_{\phi(n)}\|\}$ denote the non-increasing rearrangements of $\{\|x_n\|\}$, $\{\|y_n\|\}$ and $\{\|x_n + y_n\|\}$, respectively.

Now we show the continuity of scalar multiplication. Let $\lambda$ be an element in $K$, $\{\lambda^{(m)}\}$ be a sequence in $K$ such that $|\lambda^{(m)} - \lambda| \to 0$ as $m \to \infty$, $x$ be an element in $L(f, v, p)$ and $\{x^{(m)}\}$ be a sequence in $L(f, v, p)$ such that $g(x^{(m)} - x) \to 0$ as $m \to \infty$. Using triangle inequality we have

$$g(\lambda^{(m)}x^{(m)} - \lambda x) \leq g(\lambda^{(m)}x^{(m)} - \lambda^{(m)}x) + g(\lambda^{(m)}x - \lambda x). \quad (1)$$

By monotony of modulus function

$$g(\lambda^{(m)}x^{(m)} - \lambda^{(m)}x) = \left( \sum_{n=1}^{\infty} \left[ f \left( \|\lambda^{(m)}x_{\phi(n)}^{(m)} - \lambda^{(m)}x_{\phi(n)}\| \right) \right]^p v(n) \right)^{\frac{1}{p}} \leq A \cdot \left( \sum_{n=1}^{\infty} \left[ f \left( \|x_{\phi(n)}^{(m)} - x_{\phi(n)}\| \right) \right]^p v(n) \right)^{\frac{1}{p}} = A \cdot g(x^{(m)} - x)$$
where $A = (\|\sup_m |\lambda^{(m)}|\| + 1)$ and $\left\{\|\lambda^{(m)}x^{(n)} - \lambda^{(m)}x^{(m)}\|\right\}_n$ denotes the non-increasing rearrangement of $\left\{\|\lambda^{(m)}x_n - \lambda^{(m)}x\|\right\}_n$. Thus we get

$$g(\lambda^{(m)}x^{(m)} - \lambda^{(m)}x) \to 0$$

as $m \to \infty$.

Since $|\lambda^{(m)} - \lambda| \to 0$ as $m \to \infty$, there exists $T \in \mathbb{N}$ such that $|\lambda^{(m)} - \lambda| \leq T$ for each $m \in \mathbb{N}$. Let us take any $\varepsilon > 0$. Since $x \in L(f, v, p)$, there exists $n_0 \in \mathbb{N}$ such that

$$\sum_{n=n_0}^{\infty} \left[ f\left(|\lambda^{(m)} - \lambda|\|x^{(\phi(n))}\|\right) \right]^p \overline{v}(n) \leq \sum_{n=n_0}^{\infty} \left[ f\left(T\|x^{(\phi(n))}\|\right) \right]^p \overline{v}(n)$$

$$\leq T^p \sum_{n=n_0}^{\infty} \left[ f\left(\|x^{(\phi(n))}\|\right) \right]^p \overline{v}(n)$$

$$< \frac{\varepsilon}{2}$$

and hence we get

$$\sum_{n=n_0}^{\infty} \left[ f\left(\|\lambda^{(m)}x^{(\phi(n))} - \lambda x^{(\phi(n))}\|\right) \right]^p \overline{v}(n) < \frac{\varepsilon}{2}$$

for all $m \in \mathbb{N}$. Also by the continuity of $f$, we have

$$\sum_{n=1}^{n_0-1} \left[ f\left(\|\lambda^{(m)}x^{(\phi(n))} - \lambda x^{(\phi(n))}\|\right) \right]^p \overline{v}(n) < \frac{\varepsilon}{2}$$

as $m \to \infty$, where $\left[\|\lambda^{(m)}x^{(\phi(n))} - \lambda x^{(\phi(n))}\|\right]_n$ is non-increasing rearrangement of $\left[\|\lambda^{(m)}x_n - \lambda x\|\right]_n$. Consequently, by (3) and (4) we have

$$\sum_{n=n_0}^{\infty} \left[ f\left(\|\lambda^{(m)}x^{(\phi(n))} - \lambda x^{(\phi(n))}\|\right) \right]^p \overline{v}(n) \to 0$$

as $m \to \infty$. By (1), (2) and (5), we get $g(\lambda^{(m)}x^{(m)} - \lambda x) \to 0$ as $m \to \infty$. This completes the proof.

**Theorem 2.3.** The space $L(f, v, p)$ for $1 \leq p < \infty$ is complete with respect to its paranorm.

**Proof.** Let $\{x^{(s)}\}$ be an arbitrary Cauchy sequence in $L(f, v, p)$ with $x^{(s)} = \left\{x_n^{(s)}\right\}_{n=1}^{\infty}$ for all $s \in \mathbb{N}$. For any $\varepsilon > 0$ and a fixed $n \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that

$$g(x^{(s)} - x^{(t)}) = \left(\sum_{m=1}^{\infty} \left[ f\left(\|x_n^{(s)} - x_n^{(t)}\|\right) \right]^p \overline{v}(m) \right)^\frac{1}{p} < f(\varepsilon^2 (\overline{v}(n))^{\frac{1}{p}})$$

whenever $s, t \geq n_0$. Here, $\left[\|x_n^{(s)} - x_n^{(t)}\|\right]_n$ denotes non-increasing rearrangement of $\left[\|x_n^{(s)} - x_n^{(t)}\|\right]_n$ and we indicate that $\pi_{s,t}(m)$ is a permutation for $\mathbb{N}$. Thus we have

$$\left[ f\left(\|x_n^{(s)} - x_n^{(t)}\|\right) \right]^p < f(\varepsilon^2 (\overline{v}(n))^{\frac{1}{p}})^p$$

whenever $s, t \geq n_0$. Therefore we get

$$\left\|x_n^{(s)} - x_n^{(t)}\right\| < \varepsilon$$
whenever \( s, t \geq n_0 \). Then \( \{x_n^{(s)}\} \), for a fixed \( n \in \mathbb{N} \), is a Cauchy sequence in \( X \).

Then, there exists \( x_0 \in X \) such that \( x_n^{(s)} \to x_0 \) as \( s \to \infty \). Let \( x = \{x_n\} \). Since \( \lim_{n \to \infty} x_n^{(s)} = 0 \) for each \( s \in \mathbb{N} \), by Lemma 2 we have \( \lim_{n \to \infty} x_n = 0 \). Therefore we can choose the non-increasing rearrangement \( \{\|x_{n_i(n)} - x_{n_{i+1}(n)}^{(s)}\|\} \) of \( \{\|x_n - x_n^{(s)}\|\} \). Also, for an arbitrary \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that

\[
\sum_{n=1}^{\infty} \left[ f \left( \|x_{n_i(n)} - x_{n_{i+1}(n)}^{(s)}\| \right) \right]^p v(n) < \varepsilon^p
\]

(7)

for \( s, t > N \). Let \( t \) be an arbitrary positive integer with \( t > N \) and fixed. If we put

\[
y_n^{(s)} = x_n^{(s)} - x_n^{(t)} \quad \text{and} \quad y_n = x_n - x_n^{(t)},
\]

then we have

\[
\lim_{n \to \infty} y_n^{(s)} = 0 \quad \text{for each } s \in \mathbb{N} \quad \text{and} \quad \lim_{n \to \infty} y_n^{(s)} = y_n \quad \text{(uniformly in } n).\]

Thus by Lemma 2 we get

\[
\|y(x_0)\| \leq \limsup_{n \to \infty} \|y_n^{(s)}\|
\]

for each \( n \in \mathbb{N} \), that is,

\[
\|x_{n_i(n)} - x_{n_{i+1}(n)}^{(s)}\| \leq \limsup_{n \to \infty} \|x_{n_i(n)} - x_{n_{i+1}(n)}^{(s)}\|
\]

(8)

for each \( n \in \mathbb{N} \). Hence, by (7), (8) and continuity of \( f \) we get

\[
g(x - x^{(t)}) = \left( \sum_{n=1}^{\infty} f \left( \|x_{n_i(n)} - x_{n_{i+1}(n)}^{(s)}\| \right) \right)^{1/p} v(n)^{1/p}
\]

\[
\leq \left( \sum_{n=1}^{\infty} f \left( \limsup_{n \to \infty} \|x_{n_i(n)} - x_{n_{i+1}(n)}^{(s)}\| \right) \right)^{1/p} v(n)^{1/p}
\]

\[
= \limsup_{n \to \infty} \left( \sum_{n=1}^{\infty} f \left( \|x_{n_i(n)} - x_{n_{i+1}(n)}^{(s)}\| \right) \right)^{1/p} v(n)^{1/p}
\]

\[
< \varepsilon.
\]

Also, since \( L(f, v, p) \) is a linear space we have \( \{x_n\} = \{x_n - x_n^{(N)}\} + \{x_n^{(N)}\} \in L(f, v, p) \). Hence the space \( L(f, v, p) \) is complete with respect to its paranorm. \( \square \)

**Theorem 2.4.** Let \( f \) and \( h \) be two modulus functions. Then

(i) \( \limsup_{n \to \infty} \frac{f(n)}{h(n)} < \infty \) implies \( L(h, v, p) \subset L(f, v, p) \),

(ii) \( L(f, v, p) \cap L(h, v, p) \subset L(f + h, v, p) \) for \( 1 \leq p < \infty \).

**Proof.** (i) By the hypothesis there exists \( K > 0 \) such that \( f(t) \leq Kh(t) \) for all \( t \geq 0 \). Let \( x \in L(h, v, p) \). Then we have

\[
\left( \sum_{n=1}^{\infty} f \left( \|x_{n_i(n)}^{(s)}\| \right) \right)^{1/p} v(n)^{1/p} \leq \left( \sum_{n=1}^{\infty} K \cdot h \left( \|x_{n_i(n)}^{(s)}\| \right) \right)^{1/p} v(n)^{1/p} < \infty.
\]
Hence we get \( x \in L(f, v, p) \).

(ii) Let \( x \in L(h, v, p) \cap L(f, v, p) \). Hence we have

\[
\left( \sum_{n=1}^{\infty} \left( f + h \left( \|x_{\phi(n)}\| \right) \right)^p v(n) \right)^{\frac{1}{p}} = \left( \sum_{n=1}^{\infty} \left( f \left( \|x_{\phi(n)}\| \right) + h \left( \|x_{\phi(n)}\| \right) \right)^p v(n) \right)^{\frac{1}{p}} \\
\leq \left( \sum_{n=1}^{\infty} f \left( \|x_{\phi(n)}\| \right)^p v(n) \right)^{\frac{1}{p}} + \left( \sum_{n=1}^{\infty} h \left( \|x_{\phi(n)}\| \right)^p v(n) \right)^{\frac{1}{p}} < \infty.
\]

Therefore we get \( x \in L(f + h, v, p) \) and this completes the proof. □

**Theorem 2.5.** Let \( f \) be modulus function. Then

(a) If \( \lim_{t \to \infty} \frac{f(t)}{t} > 0 \) then \( L(f, v, p) \subset d(v, p) \),
(b) \( d(v, 1) \subset L(f, v, 1) \).

**Proof.** (a) Let \( x \in L(f, v, p) \). By Lemma 5, there is \( \beta > 0 \) such that \( f(t) \geq \beta t \) for all \( t \geq 0 \). Hence we have

\[
\sum_{n=1}^{\infty} \left( \|x_{\phi(n)}\| \right)^p v(n) \leq \max \left\{ 1, \frac{1}{p^p} \right\} \sum_{n=1}^{\infty} f \left( \|x_{\phi(n)}\| \right)^p v(n) < \infty.
\]

This completes the proof.

(b) Let \( x \in d(v, 1) \). Then there exists \( n_0 \in \mathbb{N} \) such that

\[
\sum_{n=n_0}^{\infty} \|x_{\phi(n)}\| v(n) < \varepsilon
\]

for all \( n \geq n_0 \). Since \( f \) is continuous on \([0, \infty)\), we have for all \( \varepsilon > 0 \) there exists \( 0 < \delta < 1 \) such that \( f(t) < \varepsilon \) for all \( t \in [0, \delta) \). Also, by Lemma 3 we have

\[
f \left( \|x_{\phi(n)}\| \right) < \frac{2f(1)}{\delta} \|x_{\phi(n)}\|
\]

for \( \|x_{\phi(n)}\| > \delta \), where \( \left( \|x_{\phi(n)}\| \right) \) is the non-increasing rearrangement of \( \left( \|x_n\| \right) \). Hence we get

\[
\sum_{n=1}^{\infty} f \left( \|x_{\phi(n)}\| \right) v(n) = \sum_{\|x_{\phi(n)}\| \leq \delta} f \left( \|x_{\phi(n)}\| \right) v(n) + \sum_{\|x_{\phi(n)}\| > \delta} f \left( \|x_{\phi(n)}\| \right) v(n) < \varepsilon + \frac{2f(1)}{\delta} \sum_{\|x_{\phi(n)}\| > \delta} \|x_{\phi(n)}\| v(n) < \infty
\]

and so we get \( x \in L(f, v, 1) \). □

**Corollary 2.6.** If \( \lim_{t \to \infty} \frac{f(t)}{t} > 0 \) then \( L(f, v, 1) \subset d(v, 1) \).
References