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New Inequalities of Hermite-Hadamard Type for n-Times Differentiable Convex and Concave Functions with Applications

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Abstract. In this paper, a new identity for n-times differntiable functions is established and by using the obtained identity, some new inequalities Hermite-Hadamard type are obtained for functions whose nth derivatives in absolute value are convex and concave functions. From our results, several inequalities of Hermite-Hadamard type can be derived in terms of functions whose first and second derivatives in absolute value are convex and concave functions as special cases. Our results may provide refinements of some results already exist in literature. Applications to trapezoidal formula and special means of established results are given.

1. Introduction

A function $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

The double inequality (1.1) is known as the Hermite-Hadamard inequality (see [8]). The inequalities (1.1) hold in reversed direction if f is concave.

A number of papers have been written to extend, to generalize and to improve the inequality (1.1). Many papers can be found in literature to answer an important question of estimating the difference between the middle and the rightmost terms in (1.1) see for example [1, 2], [4, 5], [7], [9–11], [15], [17] and [18] and the references therein.

Dragomir and Agarwal [4], proved the following results to estimate the difference between the middle and the rightmost terms in (1.1).

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Theorem 1. *[4, Theorem 2.2] Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function and $a, b \in I^\circ$ with $a < b$. If $f' \in L([a,b])$ and $\left|f'\right|$ is convex on $[a,b]$. Then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)}{8}\left[\left|f'(a)\right| + \left|f'(b)\right|\right]. \tag{1.2}$$

Theorem 2. *[4, Theorem 2.3] Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function and $a, b \in I^\circ$ with $a < b$. If $f' \in L([a,b])$ and $\left|f'\right|^{p/(p-1)}$ is convex on $[a,b]$ for $p > 1$. Then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)}{2(p+1)^{1/p}}\left[\frac{\left|f'(a)\right|^{p/(p-1)} + \left|f'(b)\right|^{p/(p-1)}}{2}\right]^{(p-1)/p}. \tag{1.3}$$

Pearce and Pečarić [15], improved the inequality (1.3) as the following result.

Theorem 3. *[15, Theorem 1] Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function and $a, b \in I^\circ$ with $a < b$. If $f' \in L([a,b])$ and $\left|f'\right|^q$ is convex on $[a,b]$ for $q \ge 1$. Then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)}{4}\left[\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2}\right]^{1/q}. \tag{1.4}$$

Most recently, Hwang [9] established the following results for functions whose nth derivatives in absolute value are convex and concave functions.

Theorem 4. *[9] Suppose $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$, $a, b \in I^\circ$ with $a < b$. If $f^{(n)}$ exists on I°, $f^{(n)} \in L([a,b])$ and $\left|f^{(n)}\right|^q$ is convex on $[a,b]$ for $q \ge 1$, $n \in \mathbb{N}$, $n \ge 2$. Then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx - \sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right| \le \frac{(n-1)^{1-\frac{1}{q}}(b-a)^n}{2(n+1)!}\left[\frac{\left(n^2-2\right)\left|f^{(n)}(a)\right|^q + n\left|f^{(n)}(b)\right|^q}{n+2}\right]^{\frac{1}{q}}. \tag{1.5}$$

Theorem 5. *[9] Suppose $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$, $a, b \in I^\circ$ with $a < b$. If $f^{(n)}$ exists on I°, $f^{(n)} \in L([a,b])$ and $\left|f^{(n)}\right|^q$ is concave on $[a,b]$ for $q \ge 1$, $n \in \mathbb{N}$, $n \ge 2$. Then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx - \sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right| \le \frac{(n-1)(b-a)^n}{2(n+1)!}\left|f^{(n)}\left(\frac{\left(n^2-2\right)a + nb}{(n-1)(n+2)}\right)\right|. \tag{1.6}$$

Following results can be derived directly from Theorem 4 and Theorem 5 for $n = 2$.

Corollary 1. *[9] Under the assumptions of Theorem 4, for $n = 2$, we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)^2}{12}\left[\frac{\left|f''(a)\right|^q + \left|f''(b)\right|^q}{2}\right]^{\frac{1}{q}}. \tag{1.7}$$

Corollary 2. *[9] Under the assumptions of Theorem 4, for $n = 2$, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)^2}{12}\left|f''\left(\frac{a+b}{2}\right)\right|. \tag{1.8}$$

It can be noticed that the inequalities (1.7) and (1.8) may be better than those given in Theorem 1, Theorem 2 and Theorem 3.

For more recent results on Hermite-Hadamard type inequalities for functions whose nth derivatives in absolute value are s-convex functions and m-convex, we refer the interested reader to the works of Jiang *et. al* [11] and Hong *et al.* [17] and the references therein.

The main purpose of the present paper is to establish new Hermite-Hadamard type inequalities for functions whose nth derivatives in absolute value are convex and concave. We believe that the results presented in this paper are better than those established in [9] and hence better than those given in [4] and [15]. Applications of our results to trapezoidal formula and to special means are given in Section 3 and Section 4.

2. Main Results

The following Lemma is essential in establishing our main results in this section:

Lemma 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a function such that $f^{(n)}$ exists on I° for $n \in \mathbb{N}$ and $f^{(n)} \in L([a,b])$, where $a, b \in I^\circ$ with $a < b$, we have the identity*

$$\begin{aligned}
&\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\\
&=\frac{(b-a)^n}{2^{n+1}n!}\int_0^1(1-t)^{n-1}(n-1+t)\,f^{(n)}\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt\\
&\qquad+\frac{(-1)^n(b-a)^n}{2^{n+1}n!}\int_0^1(1-t)^{n-1}(n-1+t)\,f^{(n)}\left(\frac{1-t}{2}b+\frac{1+t}{2}a\right)dt,
\end{aligned} \tag{2.1}$$

where an empty sum is understood to be nil.

Proof. Suppose

$$I_n=\frac{(b-a)^n}{2^{n+1}n!}\int_0^1(1-t)^{n-1}(n-1+t)\,f^{(n)}\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt$$

and

$$J_n=\frac{(-1)^n(b-a)^n}{2^{n+1}n!}\int_0^1(1-t)^{n-1}(n-1+t)\,f^{(n)}\left(\frac{1-t}{2}b+\frac{1+t}{2}a\right)dt.$$

For $n = 1$, we have

$$I_1=\frac{b-a}{4}\int_0^1 tf'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt$$

and

$$J_1=\frac{(-1)(b-a)}{4}\int_0^1 tf'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt.$$

By integration by parts and using the substitution $x = \frac{1-t}{2}a + \frac{1+t}{2}b$ for I_1 and $x = \frac{1+t}{2}a + \frac{1-t}{2}b$ for J_1, we obtain

$$I_1 = \frac{1}{2}f(b) - \frac{1}{b-a}\int_{\frac{a+b}{2}}^{b} f(x)$$

and

$$J_1 = \frac{1}{2}f(a) - \frac{1}{b-a}\int_{a}^{\frac{a+b}{2}} f(x).$$

Hence

$$I_1 + J_1 = \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx,$$

which coincides with the L.H.S of (2.1) for $n = 1$.
Similarly for $n = 2$, and using similar arguments as above, we have

$$I_2 + J_2 = \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx$$

which coincides with the L.H.S of (2.1) for $n = 2$.
Suppose (2.1) holds for $n = m - 1 \geq 3$.
Now for $n = m$, we have

$$\begin{aligned}
&\frac{(b-a)^m}{2^{m+1}m!}\int_0^1 (1-t)^{m-1}(m-1+t)\,f^{(m)}\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt \\
&+\frac{(-1)^m(b-a)^m}{2^{m+1}m!}\int_0^1 (1-t)^{m-1}(m-1+t)\,f^{(m)}\left(\frac{1-t}{2}b+\frac{1+t}{2}a\right)dt \\
&= -\frac{(b-a)^{m-1}(m-1)\left[1+(-1)^{m-1}\right]}{2^m m!}f^{(m-1)}\left(\frac{a+b}{2}\right) \\
&+\frac{(b-a)^{m-1}}{2^m(m-1)!}\int_0^1 (1-t)^{m-2}(m-2+t)\,f^{(m)}\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt \\
&+\frac{(-1)^{m-1}(b-a)^{m-1}}{2^m(m-1)!}\int_0^1 (1-t)^{m-2}(m-2+t)\,f^{(m)}\left(\frac{1-t}{2}b+\frac{1+t}{2}a\right)dt \\
&= \frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{m-2}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right) \\
&-\frac{(b-a)^{m-1}(m-1)\left[1+(-1)^{m-1}\right]}{2^m m!}f^{(m-1)}\left(\frac{a+b}{2}\right) \\
&= \frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{m-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right).
\end{aligned}$$

This completes the proof of the lemma. □

Now we state and prove some new Hermite-Hadamard type inequalities for functions whose nth derivatives in absolute value are convex and concave.

Theorem 6. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a function such that $f^{(n)}$ exisits on I° and $f^{(n)} \in L([a,b])$ for $n \in \mathbb{N}$, where $a, b \in I^{\circ}$ with $a < b$. If $\left|f^{(n)}\right|^q$ is convex on $[a,b]$ for $q \in [1,\infty)$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx - \sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{(b-a)^n n^{1-1/q}}{2^{n+\frac{1}{q}+1}(n+1)!}\left\{\left(\frac{\left(n^2+n-1\right)\left|f^{(n)}(a)\right|^q + \left(n^2+3n+1\right)\left|f^{(n)}(b)\right|^q}{n+2}\right)^{1/q}\right.$$
$$\left. + \left(\frac{\left(n^2+n-1\right)\left|f^{(n)}(b)\right|^q + \left(n^2+3n+1\right)\left|f^{(n)}(a)\right|^q}{n+2}\right)^{1/q}\right\}. \quad (2.2)$$

Proof. From Lemma 1, the Hölder inequality and the convexity of $\left|f^{(n)}\right|^q$ on $[a,b]$, we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right.$$
$$\left. - \sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{(b-a)^n}{2^{n+1}n!}\left(\int_0^1 (1-t)^{n-1}(n-1+t)\,dt\right)^{1-1/q}$$
$$\times\left\{\left(\int_0^1 (1-t)^{n-1}(n-1+t)\left[\left(\frac{1-t}{2}\right)\left|f^{(n)}(a)\right|^q + \left(\frac{1+t}{2}\right)\left|f^{(n)}(b)\right|^q\right]dt\right)^{1/q}\right.$$
$$\left. + \left(\int_0^1 (1-t)^{n-1}(n-1+t)\left[\left(\frac{1-t}{2}\right)\left|f^{(n)}(b)\right|^q + \left(\frac{1+t}{2}\right)\left|f^{(n)}(a)\right|^q\right]\right)^{1/q}\right\}$$
$$= \frac{(b-a)^n n^{1-1/q}}{2^{n+\frac{1}{q}+1}(n+1)!}\left\{\left(\frac{n^2+n-1}{n+2}\left|f^{(n)}(a)\right|^q + \frac{n^2+3n+1}{n+2}\left|f^{(n)}(b)\right|^q\right)^{1/q}\right.$$
$$\left. + \left(\frac{n^2+n-1}{n+2}\left|f^{(n)}(b)\right|^q + \frac{n^2+3n+1}{n+2}\left|f^{(n)}(a)\right|^q\right)^{1/q}\right\}.$$

Which is the desired result. This completes the proof of the Theorem. □

Corollary 3. *Under the assumptions of Theorem 6, if $q = 1$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx - \sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{n(b-a)^n}{2^{n+1}(n+1)!}\left[\left|f^{(n)}(a)\right| + \left|f^{(n)}(b)\right|\right]. \quad (2.3)$$

Corollary 4. *Under the assumptions of Theorem 6, if $n = 1$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)}{8}\left\{\left(\frac{\left|f'(a)\right|^q + 5\left|f'(b)\right|^q}{6}\right)^{1/q} + \left(\frac{\left|f'(b)\right|^q + 5\left|f'(a)\right|^q}{6}\right)^{1/q}\right\}. \quad (2.4)$$

Corollary 5. *If we take $q = 1$ in Corollary 4, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)}{8}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]. \tag{2.5}$$

Corollary 6. *Suppose the assumptions of Theorem 6 are fulfilled and if $n = 2$, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)^2}{96}\left\{\left(\frac{5\left|f''(a)\right|^q+11\left|f''(b)\right|^q}{16}\right)^{1/q}+\left(\frac{5\left|f''(b)\right|^q+11\left|f''(a)\right|^q}{16}\right)^{1/q}\right\}. \tag{2.6}$$

Corollary 7. *If $q = 1$ in Corollary 6, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)^2}{96}\left[\left|f''(a)\right|+\left|f''(b)\right|\right]. \tag{2.7}$$

Remark 1. *It is obvious that the inequality (2.7) gives better estimate than that of (1.7) for $q = 1$.*

Theorem 7. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a function such that $f^{(n)}$ exisits on I° and $f^{(n)} \in L([a,b])$ for $n \in \mathbb{N}$, where a, $b \in I^\circ$ with $a < b$. If $\left|f^{(n)}\right|^q$ is convex on $[a,b]$ for $q \in (1,\infty)$, we have the inequality*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right|\\&\le\frac{(b-a)^n\left[n^{(2q-1)/(q-1)}-(n-1)^{(2q-1)/(q-1)}\right]^{1-1/q}}{2^{n+\frac{1}{q}+1}n!}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\\&\quad\times\left\{\left[P\left|f^{(n)}(a)\right|^q+Q\left|f^{(n)}(b)\right|^q\right]^{1/q}+\left[P\left|f^{(n)}(b)\right|^q+Q\left|f^{(n)}(a)\right|^q\right]^{1/q}\right\},\end{aligned} \tag{2.8}$$

where

$$P=\frac{1}{nq-q+2} \text{ and } Q=\frac{nq-q+3}{(nq-q+1)(nq-q+2)}.$$

Proof. Using Lemma 1, the Hölder inequality and the convexity of $\left|f^{(n)}\right|^q$ on $[a,b]$, we have

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right|\\&\le\frac{(b-a)^n}{2^{n+1}n!}\left(\int_0^1(n-1+t)^{q/(q-1)}\,dt\right)^{1-1/q}\\&\times\left\{\left(\int_0^1(1-t)^{q(n-1)}\left[\left(\frac{1-t}{2}\right)\left|f^{(n)}(a)\right|^q+\left(\frac{1+t}{2}\right)\left|f^{(n)}(b)\right|^q\right]dt\right)^{1/q}\right.\\&\left.\quad+\left(\int_0^1(1-t)^{q(n-1)}\left[\left(\frac{1-t}{2}\right)\left|f^{(n)}(b)\right|^q+\left(\frac{1+t}{2}\right)\left|f^{(n)}(a)\right|^q\right]\right)^{1/q}\right\}.\end{aligned} \tag{2.9}$$

From (2.9) we get the inequality (2.8), since

$$\int_0^1 (n-1+t)^{q/(q-1)}\,dt = \left(\frac{q-1}{2q-1}\right)\left[n^{(2q-1)/(q-1)} - (n-1)^{(2q-1)/(q-1)}\right],$$

$$\int_0^1 (1-t)^{q(n-1)+1} = \frac{1}{nq-q+2} = P$$

and

$$\int_0^1 (1-t)^{q(n-1)}(1+t)\,dt = \frac{nq-q+3}{(nq-q+1)(nq-q+2)} = Q.$$

□

Corollary 8. *Suppose the conditions of Theorem 7 are satisfied and if $n=1$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q} \times \left\{\left(\frac{\left|f'(a)\right|^q + 3\left|f'(b)\right|^q}{4}\right)^{1/q} + \left(\frac{\left|f'(b)\right|^q + 3\left|f'(a)\right|^q}{4}\right)^{1/q}\right\}, \quad (2.10)$$

Corollary 9. *Under the assumptions of Theorem 7, if $n=2$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)^2\left[2^{(2q-1)/(q-1)}-1\right]^{1-1/q}}{2^{1/q+4}}\left(\frac{q-1}{2q-1}\right)^{1-1/q} \times \left\{\left[P\left|f''(a)\right|^q + Q\left|f''(b)\right|^q\right]^{1/q} + \left[P\left|f''(b)\right|^q + Q\left|f''(a)\right|^q\right]^{1/q}\right\}, \quad (2.11)$$

where

$$P = \frac{1}{q+2} \text{ and } Q = \frac{q+3}{(q+1)(q+2)}.$$

Theorem 8. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a function such that $f^{(n)}$ exisits on I° and $f^{(n)}\in L([a,b])$ for $n\in\mathbb{N}$, where a, $b\in I^\circ$ with $a<b$. If $\left|f^{(n)}\right|^q$ is convex on $[a,b]$ for $q\in(1,\infty)$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx - \sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)^n}{2^{n+\frac{1}{q}+1}n!}\left(\frac{q-1}{nq-1}\right)^{1-1/q}\cdot\left\{\left[R\left|f^{(n)}(a)\right|^q + S\left|f^{(n)}(b)\right|^q\right]^{1/q} + \left[R\left|f^{(n)}(b)\right|^q + S\left|f^{(n)}(a)\right|^q\right]^{1/q}\right\}, \quad (2.12)$$

$$R = \frac{n^{q+2} - (n+q+1)(n-1)^{q+1}}{(q+1)(q+2)}$$

and

$$S = \frac{(2q-n+4)\,n^{q+1} - (q-n+3)(n-1)^{q+1}}{(q+1)(q+2)}.$$

Proof. Using Lemma 1, the Hölder inequality and the convexity of $\left|f^{(n)}\right|^q$ on $[a,b]$, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right|\\
&\leq \frac{(b-a)^n}{2^{n+1}n!}\left(\int_0^1 (1-t)^{q(n-1)/(q-1)}\,dt\right)^{1-1/q}\\
&\times\left\{\left(\int_0^1 (n-1+t)^q\left[\left(\frac{1-t}{2}\right)\left|f^{(n)}(a)\right|^q+\left(\frac{1+t}{2}\right)\left|f^{(n)}(b)\right|^q\right]dt\right)^{1/q}\right.\\
&\left.\qquad+\left(\int_0^1 (n-1+t)^q\left[\left(\frac{1-t}{2}\right)\left|f^{(n)}(b)\right|^q+\left(\frac{1+t}{2}\right)\left|f^{(n)}(a)\right|^q\right]\right)^{1/q}\right\}. \qquad (2.13)
\end{aligned}$$

Since

$$\int_0^1 (1-t)^{q(n-1)/(q-1)}\,dt=\frac{q-1}{nq-1},$$

$$\int_0^1 (n-1+t)^q(1-t)\,dt=\frac{n^{q+2}-(n+q+1)(n-1)^{q+1}}{(q+1)(q+2)}$$

and

$$\int_0^1 (n-1+t)^q(1+t)\,dt=\frac{(2q-n+4)\,n^{q+1}-(q-n+3)(n-1)^{q+1}}{(q+1)(q+2)}$$

and hence (2.12) follows from (2.13). □

Corollary 10. *Suppose the assumptions of Theorem 8 are satisfied and if $n=1$, we have the inequality*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\
&\qquad\leq\frac{(b-a)}{2^{2+1/q}}\left\{\left[R\left|f'(a)\right|^q+S\left|f'(b)\right|^q\right]^{1/q}+\left[R\left|f'(b)\right|^q+S\left|f'(a)\right|^q\right]^{1/q}\right\}, \qquad (2.14)
\end{aligned}$$

$$R=\frac{1}{(q+1)(q+2)}\text{ and }S=\frac{2q+3}{(q+1)(q+2)}.$$

Corollary 11. *Suppose the assumptions of Theorem 8 are satisfied and if $n=2$, we have the inequality*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{(b-a)^2}{2^{4+\frac{1}{q}}}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\\
&\qquad\times\left\{\left[R\left|f''(a)\right|^q+S\left|f''(b)\right|^q\right]^{1/q}+\left[R\left|f''(b)\right|^q+S\left|f''(a)\right|^q\right]^{1/q}\right\}, \qquad (2.15)
\end{aligned}$$

$$R=\frac{2^{q+2}-(q+3)}{(q+1)(q+2)}\text{ and }S=\frac{2^{q+2}-1}{q+2}.$$

A result for concave functions is given in the following theorem.

Theorem 9. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a function such that $f^{(n)}$ exisits on I° and $f^{(n)} \in L([a,b])$ for $n \in \mathbb{N}$, where a, $b \in I^{\circ}$ with $a < b$. If $\left|f^{(n)}\right|^q$ is convex on $[a,b]$ for $q \in [1,\infty)$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{n(b-a)^n}{2^{n+1}(n+1)!}\left\{\left|f^{(n)}\left(\frac{\left(n^2+n-1\right)a+\left(n^2+3n+1\right)b}{2n(n+2)}\right)\right|+\left|f^{(n)}\left(\frac{\left(n^2+n-1\right)b+\left(n^2+3n+1\right)a}{2n(n+2)}\right)\right|\right\}. \tag{2.16}$$

Proof. By the concavity of $\left|f^{(n)}\right|^q$ and the power-mean inequality, we have

$$\left|f^{(n)}(\lambda x+(1-\lambda)y)\right| \geq \lambda\left|f^{(n)}(x)\right|+(1-\lambda)\left|f^{(n)}(y)\right|$$

which shows that $\left|f^{(n)}\right|$ is also concave.
Hence by using Lemma 1 and the Jensen integral inequality, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{(b-a)^n}{2^{n+1}n!}\left(\int_0^1(1-t)^{n-1}(n-1+t)\,dt\right)$$
$$\times\left\{\left|f^{(n)}\left(\frac{(1-t)^{n-1}(n-1+t)\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt}{\int_0^1(1-t)^{n-1}(n-1+t)\,dt}\right)\right|+\left|f^{(n)}\left(\frac{(1-t)^{n-1}(n-1+t)\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt}{\int_0^1(1-t)^{n-1}(n-1+t)\,dt}\right)\right|\right\}.$$

The above inequality gives the desired result. □

Corollary 12. *Suppose the assumptions of Theorem 9 are fulfilled and if $n = 1$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)}{8}\left[\left|f'\left(\frac{a+5b}{6}\right)\right|+\left|f'\left(\frac{b+5a}{6}\right)\right|\right]. \tag{2.17}$$

Corollary 13. *Suppose the assumptions of Theorem 9 are fulfilled and if $n = 2$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2}{24}\left[\left|f''\left(\frac{5a+11b}{16}\right)\right|+\left|f''\left(\frac{5b+11a}{16}\right)\right|\right]. \tag{2.18}$$

3. Applications to the Trapezoidal Formula

Let d be a division of the interval $[a,b]$, i.e. $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$, and consider the quadrature formula

$$\int_a^b f(x)dx = T(f,d)+E(f,d),$$

where

$$T(f,d)=\sum_{i=0}^{n-1}(x_{i+1}-x_i)\frac{f(x_i)+f(x_{i+1})}{2}$$

is the trapezoidal versions and $E(f,d)$ is the associated error. Here, we derive some error estimates for the trapezoidal formula in terms of absolute values of the second derivative of f which may be better than those already exist in the literature.

Theorem 10. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L([a,b])$, where $a, b \in I^\circ$ with $a < b$. If $|f''|^q$ is convex on $[a,b]$ for $q \geq 1$, then for every division d of $[a,b]$, we have*

$$\left|E(f,d)\right| \leq \frac{1}{96}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3\left\{\left(\frac{5\left|f''(x_i)\right|^q+11\left|f''(x_{i+1})\right|^q}{16}\right)^{1/q}+\left(\frac{5\left|f''(x_{i+1})\right|^q+11\left|f''(x_i)\right|^q}{16}\right)^{1/q}\right\}. \tag{3.1}$$

Proof. By applying Corollary 6 on the subinterval $[x_i, x_{i+1}]$ $(i = 0, 1, \ldots, n-1)$ of the division d, we have

$$\begin{aligned}
\left|E(f,d)\right| &= \left|\sum_{i=0}^{n-1}\left\{(x_{i+1}-x_i)\frac{f(x_i)+f(x_{i+1})}{2}-\int_{x_i}^{x_{i+1}}f(x)\,dx\right\}\right| \\
&\leq \sum_{i=0}^{n-1}(x_{i+1}-x_i)\left|\frac{f(x_i)+f(x_{i+1})}{2}-\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}f(x)\,dx\right| \\
&\leq \frac{1}{96}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3\left\{\left(\frac{5}{16}\left|f''(x_i)\right|^q+\frac{11}{16}\left|f''(x_{i+1})\right|^q\right)^{1/q}+\left(\frac{5}{16}\left|f''(x_{i+1})\right|^q+\frac{11}{16}\left|f''(x_i)\right|^q\right)^{1/q}\right\}.
\end{aligned} \tag{3.2}$$

Hence the result is proved. □

Corollary 14. *Under the assumptions of Theorem 10, for $q = 1$, we have*

$$\left|E(f,d)\right| \leq \frac{1}{96}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3\left[\left|f''(x_i)\right|+\left|f''(x_{i+1})\right|\right]. \tag{3.3}$$

Theorem 11. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f'' \in L([a,b])$, where $a, b \in I^\circ$ with $a < b$. If $|f''|^q$ is convex on $[a,b]$ for $q > 1$, then for every division d of $[a,b]$, we have*

$$\begin{aligned}
\left|E(f,d)\right| &\leq \frac{\left[2^{(2q-1)/(q-1)}-1\right]^{1-1/q}}{(2^{1/q})\cdot 8}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3 \\
&\times\left\{\left[\frac{(q+1)\left|f''(x_i)\right|^q+(q+3)\left|f''(x_{i+1})\right|^q}{(q+1)(q+2)}\right]^{1/q}+\left[\frac{(q+1)\left|f''(x_{i+1})\right|^q+(q+3)\left|f''(x_i)\right|^q}{(q+1)(q+2)}\right]^{1/q}\right\}.
\end{aligned} \tag{3.4}$$

Proof. It follows from Corollary 9. □

Theorem 12. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L([a,b])$, where $a, b \in I^\circ$ with $a < b$. If $|f''|^q$ is convex on $[a,b]$ for $q > 1$, then for every division d of $[a,b]$, we have*

$$\begin{aligned}
\left|E(f,d)\right| &\leq \frac{1}{2^{4+\frac{1}{q}}}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3 \\
&\times\left\{\left[\frac{\left(2^{q+2}-q-3\right)\left|f''(x_i)\right|^q+(q+1)\left(2^{q+2}-1\right)\left|f''(x_{i+1})\right|^q}{(q+1)(q+2)}\right]^{1/q}\right. \\
&\left.+\left[\frac{\left(2^{q+2}-q-3\right)\left|f''(x_{i+1})\right|^q+(q+1)\left(2^{q+2}-1\right)\left|f''(x_i)\right|^q}{(q+1)(q+2)}\right]^{1/q}\right\}.
\end{aligned} \tag{3.5}$$

Proof. It is a direct consequence of Corollary 11. □

Error bound for trapezoidal formaula when $\left|f''\right|^q$ for $q \geq 1$ is concave may be given as follows.

Theorem 13. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L([a,b])$, where $a, b \in I^\circ$ with $a < b$. If $\left|f''\right|^q$ is concave on $[a,b]$ for $q \geq 1$, then for every division d of $[a,b]$, we have*

$$\left|E(f,d)\right| \leq \frac{1}{24} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^3 \left[\left|f''\left(\frac{5x_i + 11x_{i+1}}{16}\right)\right| + \left|f''\left(\frac{5x_{i+1} + 11x_i}{16}\right)\right|\right]. \quad (3.6)$$

Proof. It can be proved easily by using Corollary 13. □

4. Applications to the Special Means

Now, we consider applications of our results to special means. We consider the means for positive real numbers $a, b \in \mathbb{R}_+$. We take

1. The arithmetic mean:
$$A(a,b) = \frac{a+b}{2}, a, b > 0.$$
2. The harmonic mean:
$$H(a,b) = \frac{2}{\frac{1}{a} + \frac{1}{b}}, a, b > 0.$$
3. The logarithmic mean:
$$L(a,b) = \frac{b-a}{\ln b - \ln a}, a, b > 0, a \neq b.$$
4. Generalized log-mean:
$$L_p(a,b) = \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, a, b > 0, p \in \mathbb{R}\setminus\{-1, 0\}, a \neq b.$$

Now we apply our results to establish some inequalities for special means.

Theorem 14. *For $a, b \in \mathbb{R}_+$, $a < b$ and $p \in \mathbb{N}$, $p > 1$, we have*

$$\left|A\left(a^{p+1}, b^{p+1}\right) - L_{p+1}^{p+1}\left(a^{p+1}, b^{p+1}\right)\right| \leq \frac{p(p+1)(b-a)^2}{48} A\left(a^{p-1}, b^{p-1}\right).$$

Proof. For the function $f(x) = x^{p+1}$, $x \in \mathbb{R}_+$ and $p \in \mathbb{N}$, then $\left|f''(x)\right| = (p+1)px^{p-1}$ is a convex function on $\mathbb{R}_+$. Applying Corollary 7, we obtain the required result. □

Theorem 15. *For $a, b \in \mathbb{R}_+$, $a < b$ and $p, q \in \mathbb{N}$, $p, q > 1$, we have*

$$\begin{aligned}
&\left|A\left(a^{p+1/q}, b^{p+1/q}\right) - L_{p+1/q}^{p+1/q}\left(a^{p+1/q}, b^{p+1/q}\right)\right| \\
&\leq \frac{(b-a)^2\left[2^{(2q-1)/(q-1)} - 1\right]^{1-1/q}}{16}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left(n - 1 + \frac{1}{q}\right)\left(n + \frac{1}{q}\right) \\
&\times \left\{\left[\frac{A\left(a^{(p-2)q+1}, b^{(p-2)q+1}\right)}{q+2} + \frac{b^{(p-2)q+1}}{(q+1)(q+2)}\right]^{1/q} + \left[\frac{A\left(a^{(p-2)q+1}, b^{(p-2)q+1}\right)}{q+2} + \frac{a^{(p-2)q+1}}{(q+1)(q+2)}\right]^{1/q}\right\}. \quad (4.1)
\end{aligned}$$

Proof. Let $f(x) = x^{p+1/q}$, $x \in \mathbb{R}_+$, $p, q \in \mathbb{N}$, $p, q > 1$, then

$$\left|f''(x)\right|^q = \left(p - 1 + \frac{1}{q}\right)^q \left(p + \frac{1}{q}\right)^q x^{(p-2)q+1}$$

is convex on $\mathbb{R}_+$ and the result follows directly from Corollary 9. □

Theorem 16. *For $a, b \in \mathbb{R}_+$, $a < b$ and $p, q \in \mathbb{N}$, $p, q > 1$, we have*

$$\begin{aligned}&\left|A\left(a^{p+1/q}, b^{p+1/q}\right) - L_{pn+1/q}^{p+1/q}\left(a^{p+1/q}, b^{p+1/q}\right)\right| \\ &\leq \frac{(b-a)^2}{2^{4+1/q}} \left(\frac{q-1}{2q-1}\right)^{1-1/q} \left(p - 1 + \frac{1}{q}\right)\left(p + \frac{1}{q}\right) \\ &\times \left\{\left[\frac{2^{q+2} - (q+3)}{(q+1)(q+2)} a^{(p-2)q+1} + \frac{2^{q+2}-1}{q+2} b^{(p-2)q+1}\right]^{1/q}\right. \\ &\qquad \left. + \left[\frac{2^{q+2} - (q+3)}{(q+1)(q+2)} b^{(p-2)q+1} + \frac{2^{q+2}-1}{q+2} a^{(p-2)q+1}\right]^{1/q}\right\} \quad (4.2)\end{aligned}$$

Proof. Let $f(x) = x^{p+1/q}$, $x \in \mathbb{R}_+$, $p, q \in \mathbb{N}$, $p, q > 1$, then

$$\left|f''(x)\right|^q = \left(p - 1 + \frac{1}{q}\right)^q \left(p + \frac{1}{q}\right)^q x^{(p-2)q+1}$$

is convex on $\mathbb{R}_+$ and the result follows directly from Corollary 11. □

Theorem 17. *For $a, b \in \mathbb{R}_+$, $a < b$ and $p, q \in \mathbb{N}$, $p, q > 1$ with $p < q$, we have*

$$\left|A\left(a^{p/q^2+2}, b^{p/q+2}\right) - L_{p/q^2+2}^{p/q^2+2}\left(a^{p/q^2+2}, b^{p/q^2+2}\right)\right| \leq \frac{(b-a)^2}{12}\left(\frac{p}{q^2} + 2\right)\left(\frac{p}{q^2} + 1\right) A\left(a^{p/q^2}, b^{p/q^2}\right). \quad (4.3)$$

Proof. Let $f(x) = x^{p/q^2+2}$, $x \in \mathbb{R}_+$, then $\left|f''(x)\right| = \left(\frac{p}{q^2} + 2\right)\left(\frac{p}{q^2} + 1\right) x^{p/q^2}$. It is obvious that $\left|f''(x)\right|^q$ is concave on $\mathbb{R}_+$. Applying Corollary 13, we get the desired result. □

Theorem 18. *For $a, b \in \mathbb{R}_+$, $a < b$, we have*

$$\left|H^{-1}(a,b) - L^{-1}(a,b)\right| \leq \frac{(b-a)^2}{24} A\left(a^{-3}, b^{-3}\right) \quad (4.4)$$

Proof. Let $f(x) = \frac{1}{x}$, $x \in \mathbb{R}_+$, then $\left|f''(x)\right| = \frac{2}{x^3}$ is convex on $\mathbb{R}_+$ so applying Corollary 7, we get the inequality (4.4). □

Remark 2. *Many other interesting inequalities for means can be obtained by applying the other results to the function $f(x) = \frac{1}{x}$, $x \in \mathbb{R}_+$, however the details are left to the interested reader.*

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