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Hybrid Functions Approach for the Fractional Riccati Differential Equation

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Abstract. In this paper, we state an efficient method for solving the fractional Riccati differential equation. This equation plays an important role in modeling the various phenomena in physics and engineering. Our approach is based on operational matrices of fractional differential equations with hybrid of block-pulse functions and Chebyshev polynomials. Convergence of hybrid functions and error bound of approximation by this basis are discussed. Implementation of this method is without ambiguity with better accuracy than its counterpart other approaches. The reliability and efficiency of the proposed scheme are demonstrated by some numerical experiments.

1. Introduction

Study on fractional differential equation has been increasing over the past few decades due to its various applications in some branches of science such as image processing and signal identification, optical systems, mechanical systems and etc. [8]. Furthermore, in the real world the fractional order equations are more appropriate than classical integer equations for modeling various phenomena; for instance see [19]. Recent contributions in the fractional calculus field have been reported by some researchers [2, 12].
In this paper, a novel framework has been presented for solving the fractional Riccati differential equation of arbitrary order. The Riccati differential equation has significant importance in classical, as well as, modern science and engineering applications, such as stochastic realization theory [20] and financial mathematics [6]. Also, some review of fundamental theories of the Riccati equation with applications are given by Ried [24].
The general form of the Riccati differential equations can be written as

$$D^{\alpha} f(t) = A(t) + B(t) f(t) + C(t) f^2(t), \quad n-1 < \alpha \leq n, \tag{1}$$

with the initial conditions $f^{(i)}(0) = d_i, i = 0, 1, \ldots, n-1$. Where A(t), B(t) and C(t) are the given real functions, d_i for $i = 0, 1, \ldots, n-1$, are constants and α is a parameter describing the order of the fractional derivative. In the case of $\alpha = 1$, the fractional equation becomes to the classical Riccati differential equation. The value of order $\alpha = 0.5$ has a special importance, because some of practical problems have been developed by

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applying this particular order of the derivative.
Therefore, some approaches for numerical and analytical solutions of the Riccati differential equation are investigated by some scientists, such as Adomian's decomposition method [16], collocation method [26], the variational iteration method [5] and etc [4, 27, 28]. In the present work, we introduce a new numerical method to solve the Riccati differential equation of the fractional order. The method consists of reducing the differential equation to a set of algebraic equations by expanding the fractional derivative term as hybrid functions with unknown coefficients. The operational matrices of hybrid functions are utilized to evaluate the unknown coefficients and then, to find the approximate solution.
It is necessary to introduce some definitions and relations which are used in this article [7, 21]. In case of fractional calculus, the Riemann-Liouville fractional integral operator of order α for $f \in L_1[0, b]$, can be written as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \quad 0 < t \leq b,$$

and Riemann-Liouville fractional derivative of order $\alpha \geq 0$ is

$$D^{\alpha} f = D^n I^{n-\alpha} f,$$

where $n-1 < \alpha \leq n$ and $n \in N$.
The other type of fractional derivative is the Caputo fractional derivative. The Caputo derivative is suitable for the real world physical problems in contrast the Riemann-Liouville differential operator. Nowadays, this type of derivative is frequently used in applications, because by using this one can specify the initial conditions of fractional differential equations in classical form. The Caputo fractional derivative of $f \in L_1[0, b]$, is defined as

$$D_*^{\alpha} f(t) = \begin{cases} I^{n-\alpha} D^n f(t), & n-1 < \alpha < n,\ n \in \mathbb{N}, \\ \frac{d^n}{dt^n} f(t), & \alpha = n. \end{cases}$$

Note that $I^{\alpha} D_*^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{t^k}{k!}$, $n-1 < \alpha \leq n$, $n \in \mathbb{N}$. In this study we point that the approximate solutions will be found by using the Caputo fractional derivative and its properties. The rest of the paper is organized as follows:
In the next section, we briefly introduce some properties of hybrid functions and use these functions to approximate an arbitrary function $f(t) \in L^2[0, T)$. Also, the hybrid operational matrix of the fractional integration is proposed. In Section 3, convergency and error estimate of this approach are given, and adaptability of presented method for solving the Riccati differential equation will be described. Numerical study by using two example has been discussed in Section 4. Finally, ends this paper with a brief conclusion and some remarks.

2. Hybrid Functions and Operational Matrix

In this section, we review some properties of the hybrid functions of block-pulse and Chebyshev polynomials and we expand a function with this hybrid basis. Also, operational matrix of the fractional integration is obtained.

2.1. Hybrid functions of block-pulse and Chebyshev polynomials

Hybrid functions $h_{nm}, n = 1, 2, \ldots, N,\ m = 0, 1, \ldots, M-1$, are defined on the interval $[0, t_f)$ as [14],

$$h_{nm}(t) = \begin{cases} T_m\left(\frac{2N}{t_f} t - 2n + 1\right), & t \in \left[\frac{n-1}{N} t_f, \frac{n}{N} t_f\right), \\ 0, & \text{o.w.}, \end{cases}$$

where n and m are the order of block-pulse functions and Chebyshev polynomials, respectively. Also, $T_m(t)$s are the well-known Chebyshev polynomials of order m, which are orthogonal with respect to the weight function $w(t) = 1/\sqrt{1-t^2}$ on the interval $[-1, 1]$, and satisfy the following recursive formulas

$$T_0(t) = 1, \qquad T_1(t) = t,$$

$$T_{m+1}(t) = 2tT_m(t) - T_{m-1}(t), \qquad m = 1, 2, \dots.$$

The orthogonality trait is

$$\int_{-1}^{1} \frac{T_i(t)T_j(t)}{\sqrt{1-t^2}} dt = \begin{cases} \frac{\pi}{2}, & i = j \neq 0, \\ \pi, & i = j = 0, \\ 0, & i \neq j. \end{cases}$$

Any function $f(t) \in L^2_\omega[0, t_f)$, may be expanded as [13],

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}h_{nm}(t) \approx \sum_{n=1}^{N}\sum_{m=0}^{M-1} c_{nm}h_{nm}(t) = C^T H(t). \tag{2}$$

Where

$$C = \left[c_{10}, \dots, c_{1(M-1)}, c_{20}, \dots, c_{2(M-1)}, \dots, c_{N0}, \dots, c_{N(M-1)}\right]^T,$$

and

$$H(t) = \left[h_{10}(t), \dots, h_{1(M-1)}(t), h_{20}(t), \dots, h_{2(M-1)}(t), \dots, h_{N0}(t), \dots, h_{N(M-1)}(t)\right]^T.$$

Furthermore in the following, without less of generality, we assume that $t_f = 1$.

2.2. Operational matrix of the fractional integration

By taking $\mu = NM$, integration of the vector $H(t)$ defined in Eq. (2), can be approximated by [3],

$$\int_0^t H(s)ds \approx P_{\mu\times\mu}H(t),$$

$P_{\mu\times\mu}$ is called the operational matrix for integration. Consider $B(t) = [b_1(t), b_2(t), ..., b_\mu(t)]$, which $b_i(t)$s are block-pulse functions on the interval $[0, 1)$. Hybrid functions may be expanded into an μ-term block-pulse functions, as

$$H(t) \approx \Phi_{\mu\times\mu}B(t), \tag{3}$$

where matrix $\Phi_{\mu\times\mu}$, by using the collocation points $t_i = \frac{2i-1}{2\mu}, i = 1, 2, \dots, \mu$ and vector $H(t)$, is defined as follows

$$\Phi_{\mu\times\mu} = \left[H\left(\frac{1}{2\mu}\right) \ H\left(\frac{3}{2\mu}\right) \dots H\left(\frac{2\mu-1}{2\mu}\right)\right].$$

On the other hand, we denote the block-pulse operational matrix of the fractional integration by F^α, then

$$(I^\alpha B)(t) \approx F^\alpha B(t), \tag{4}$$

where

$$F^\alpha = \frac{1}{\mu^\alpha}\frac{1}{\Gamma(\alpha+2)}\begin{bmatrix} 1 & \varepsilon_1 & \varepsilon_2 & \dots & \varepsilon_{\mu-1} \\ 0 & 1 & \varepsilon_1 & \dots & \varepsilon_{\mu-2} \\ 0 & 0 & 1 & \dots & \varepsilon_{\mu-3} \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix},$$

with $\varepsilon_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$, for $k = 1, 2, \ldots, \mu - 1$ [10]. Next, we derive the hybrid function operational matrix of the fractional integration. Let

$$(I^{\alpha}H)(t) \approx P^{\alpha}_{\mu\times\mu}H(t), \tag{5}$$

matrix $P^{\alpha}_{\mu\times\mu}$ is the operational matrix of the fractional integration for hybrid function of block-pulse and Chebyshev polynomials. Using Eqs. (3) and (4), we have

$$(I^{\alpha}H)(t) \approx \left(I^{\alpha}\Phi_{\mu\times\mu}B\right)(t) = \Phi_{\mu\times\mu}(I^{\alpha}B)(t) \approx \Phi_{\mu\times\mu}F^{\alpha}B(t).$$

From the last two equations, we get

$$P^{\alpha}_{\mu\times\mu}H(t) \approx P^{\alpha}_{\mu\times\mu}\Phi_{\mu\times\mu}B(t) \approx \Phi_{\mu\times\mu}F^{\alpha}B(t).$$

Then, via Eq. (3) the hybrid operational matrix of the fractional integration is given by

$$P^{\alpha}_{\mu\times\mu} \approx \Phi_{\mu\times\mu}F^{\alpha}\Phi^{-1}_{\mu\times\mu}.$$

3. Main Results

The convergence of hybrid Chebyshev and block-pulse functions has been studied by the following theorems. Also, in this section we present the hybrid function approach for solving the fractional Riccati differential equation in general case.

3.1. Convergence analysis

In [1, 25] convergency of wavelet functions has been demonstrated, indeed, we extend it to hybrid functions, then the error estimate of this approach is desired.

Theorem 3.1. *A function $f(t)$, with bounded second derivative, can be expanded as an infinite sum of hybrid Chebyshev and block-pulse functions with the bounded expansion coefficients. In other words, the coefficients $c_{nm} =< f(t), h_{nm}(t) >$, which $< ., . >$ denotes the inner product in $L^2_{\omega}[0,1)$, in series (2) are bounded.*

Proof. Reminding the definition of inner product,

$$c_{nm} = \frac{4N}{\pi}\int_0^1 f(t)h_{nm}(t)w_n(t)dt = \frac{4N}{\pi}\int_{\frac{n-1}{N}}^{\frac{n}{N}} \frac{f(t)T_m(2Nt-2n+1)}{\sqrt{1-(2Nt-2n+1)^2}}dt.$$

Since $T_m(\cos\theta) = \cos m\theta$, with the change variable $2Nt - 2n + 1 = cosx$, we have

$$c_{nm} = \frac{2}{\pi}\int_0^{\pi} f\left(\frac{\cos x + 2n - 1}{2N}\right)\cos mx dx. \tag{6}$$

Using the integration by parts, we obtain

$$c_{nm} = \frac{1}{2Nm\pi}\int_0^{\pi} f'\left(\frac{\cos x + 2n - 1}{2N}\right) \times (\cos(m-1)x - \cos(m+1)x)\, dx.$$

Once again, an integration by parts of above relation, results that

$$c_{nm} = \frac{1}{2^2N^2m\pi}\int_0^{\pi} f''\left(\frac{\cos x + 2n - 1}{2N}\right) \times \sin x\left(\frac{\sin(m-1)x}{m-1} - \frac{\sin(m+1)x}{m+1}\right)dx.$$

Since n up to N, for $m > 1$

$$|c_{nm}| \leq \frac{M_2}{2^2 n^2 m\pi} \int_0^{\pi} \left| \frac{\sin x \sin(m-1)x}{m-1} \right| + \left| \frac{\sin x \sin(m+1)x}{m+1} \right| dx$$
$$\leq \frac{M_2}{2n^2(m^2-1)}, \tag{7}$$

where

$$M_2 = \max_{0 \leq x < 1} |f''(x)|.$$

For $m = 1$, from (6) we get

$$|c_{n1}| \leq \frac{1}{2n} \max_{0 \leq x \leq 1} |f'(x)|.$$

□

Remark 3.2. *According to Theorem 3.1, the series $\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm}$ is absolutely convergent. Also, we know that $|h_{nm}(t)| \leq 1$, therefore*

$$|f(t)| = |\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} h_{nm}(t)| \leq \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} |c_{nm}| < \infty,$$

and the series (2) *converges to the function $f(x)$ uniformly, when M, N trend to infinity.*

Theorem 3.3. *Let f(x) be a continuous function defined on [0, 1), with bounded second derivative, $|f''(x)| \leq M_2$, then we have the following accuracy estimation*

$$\|f(t) - C^T H(t)\| \leq \frac{M_2 \pi^{1/2}}{2^2} C_{NM},$$

with

$$C_{NM}^2 = \left(\sum_{n=N+1}^{\infty} \frac{1}{n^5} \sum_{m=M}^{\infty} \frac{1}{(m^2-1)^2} \right).$$

Proof.

$$\|f(t) - C^T H(t)\| = \left(\int_0^1 \left(f(t) - \sum_{n=1}^{N} \sum_{m=0}^{M-1} c_{nm} h_{nm}(t) \right)^2 w_n(t) dt \right)^{1/2}$$
$$= \left(\sum_{n=N+1}^{\infty} \sum_{m=M}^{\infty} c_{nm}^2 \int_0^1 h_{nm}^2(t) w_n(t) dt \right)^{1/2}$$
$$= \left(\sum_{n=N+1}^{\infty} \sum_{m=M}^{\infty} c_{nm}^2 \int_{\frac{n-1}{N}}^{\frac{n}{N}} \frac{T_m^2(2Nt-2n+1)}{\sqrt{1-(2Nt-2n+1)^2}} dt \right)^{1/2}. \tag{8}$$

By substituting $2Nt - 2n + 1 = x$ and using the property of orthogonality of Chebyshev polynomials with considering the relation (7), we get

$$\|f(t) - C^T H(t)\| = \left(\frac{\pi}{4N} \sum_{n=N+1}^{\infty} \sum_{m=M}^{\infty} c_{nm}^2 \right)^{1/2}$$
$$\leq \frac{M_2 \pi^{1/2}}{2^2} \left(\sum_{n=N+1}^{\infty} \frac{1}{n^5} \sum_{m=M}^{\infty} \frac{1}{(m^2-1)^2} \right)^{1/2}.$$

□

3.2. Implementation the method

Consider the nonlinear fractional Riccati differential equation (1), with the Caputo type derivative. To solve this equation, firstly we approximate $D_*^{\alpha} f(t)$ by the hybrid function as

$$D_*^{\alpha} f(t) = K^T H(t), \tag{9}$$

where $K = \left[k_1, k_2, \ldots, k_\mu\right]^T$, is an unknown vector. Accordingly, by applying the fractional integral operator of order α to both sides of the above equation, we get

$$I^{\alpha} D_*^{\alpha} f(t) = K^T I^{\alpha} H(t),$$

and thereupon

$$f(t) = K^T I^{\alpha} H(t) + \sum_{i=0}^{\lceil\alpha\rceil-1} \frac{d_i}{i!} t^i = K^T P^{\alpha}_{\mu\times\mu} H(t) + \sum_{i=0}^{\lceil\alpha\rceil-1} \frac{d_i}{i!} t^i. \tag{10}$$

Substituting Eqs. (9) and (10) in Eq. (1), we obtain

$$K^T H(t) = A_1(t) + B_1(t)\widetilde{K}H(t) + C(t)(\widetilde{K}H(t)H(t)^T\widetilde{K}^T). \tag{11}$$

Where $\widetilde{K} = K^T P^{\alpha}_{\mu\times\mu}$, $A_1(t)$ represents all sentences without unknown function $f(t)$, and $B_1(t)$ is the coefficient of the linear terms of the unknown function $f(t)$. From the property of the product of two hybrid functions vectors, we conclude that

$$H(t)H(t)^T\widetilde{K}^T = \widehat{K}H(t).$$

Block diagonal matrix $\widehat{K}_{\mu\times\mu}$ is called product operational matrix and is given by [14]. In order to find the solution of the fractional Riccati differential equation (1), we collocate Eq. (11) in $t_j = \frac{2j-1}{2\mu}$, $j = 1, 2, ..., \mu$. So, we have

$$K^T H(t_j) = A_1(t_j) + B_1(t_j)\widetilde{K}H(t_j) + C(t_j)(\widetilde{K}\widehat{K}H(t_j)). \tag{12}$$

That is a nonlinear system of algebraic equations which can be solved by Newton's method for finding the unknown coefficients vector K. Finally, by using this vector the unknown function $f(t)$ as a solution of Eq. (1) can be approximate.

4. Numerical Discussion of the Fractional Riccati Differential Equation

To demonstrate the effectiveness of the proposed method in the previous section, we consider some cases of fractional Riccati differential equations and compare the results obtained using this scheme with the analytical solution or the estimated solutions by using other schemes. Therefore, in this section we have reported the obtained results to solve practical problems via two examples for comprehensive overview of the methodology.

Example 4.1. In [5, 11, 15, 17, 18, 22, 23], the fractional Riccati differential equation

$$D_*^{\alpha} f(t) = 2f(t) - f^2(t) + 1, \quad 0 < \alpha \leq 1, \tag{13}$$

subject to the initial condition $f(0) = 0$, has been solved by different numerical methods. The exact solution of Eq. (13) when $\alpha = 1$, is

$$f(t) = 1 + \sqrt{2}\tanh\left(\sqrt{2}\,t + \frac{1}{2}\ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right).$$

By implementing the method described in subsection 3.2, the corresponding algebraic system for representation FDE (13), is

$$K^T H(t) + \widetilde{K}\widehat{K}H(t) - 2\widetilde{K}H(t) - 1 = 0.$$

Fig. 1. shows the numerical results for $f(t)$ with $\mu = 16$, $\alpha = 0.25, 0.5, 0.75, 1$.

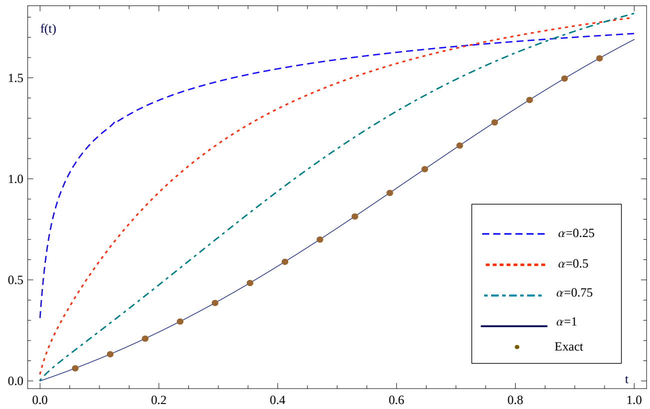

Figure 1: Numerical solutions of the example 4.1, for $\mu = 16$, $\alpha = 0.25, 0.5, 0.75, 1$

According to Table 9 of [22], for $\alpha = 1$, between the methods: variational iteration method (VIM) [5], modified homotopy perturbation method (MHPM) [17], particle swarm optimization (PSO) [22], Chebyshev wavelets (CW) [11], generalized differential transform method (GDTM) [18] and genetic algorithm (GA) [23], the best approximate solution of Eq. (13), from the point of view total absolute errors in $t = 0.1, 0.2, \cdots, 0.9$, is CW method. Therefore, for summarizing the discussion, we give in Table 1, only comparison of our method with CW method (for $\mu = 192$) and fractional variational iteration method (FVIM) presented in [15] for $\alpha = 0.5, 1$.

Table 1: Numerical results of the example 4.1, with comparison to Refs. [11],[15]

	$\alpha = 0.5$			$\alpha = 1$			
t	FVIM	CW	Ours	FVIM	CW	Ours	Exact
0.1	0.577431	0.592756	0.592805	0.110266	0.110311	0.110304	0.110295
0.2	0.912654	0.933179	0.933213	0.241585	0.241995	0.241987	0.241976
0.3	1.166253	1.173983	1.17401	0.393515	0.395123	0.395116	0.395104
0.4	1.353549	1.346654	1.34667	0.564013	0.567829	0.567824	0.567812
0.5	1.482633	1.473887	1.4739	0.749528	0.756029	0.756025	0.756014
0.6	1.559656	1.570571	1.57058	0.945155	0.953576	0.953576	0.953566
0.7	1.589984	1.646199	1.6462	1.144826	1.152955	1.15296	1.152948
0.8	1.578559	1.706880	1.70688	1.341552	1.346365	1.34637	1.346363
0.9	1.530028	1.756644	1.75665	1.527690	1.526909	1.52692	1.526911

Fig. 2. is indicating the absolute error of the numerical solution of example 4.1 by hybrid method. Table 2, shows the sum of the absolute errors in points $t = 0.1, 0.2, \cdots, 0.9$, for all methods listed in this problem with $\alpha = 1$.

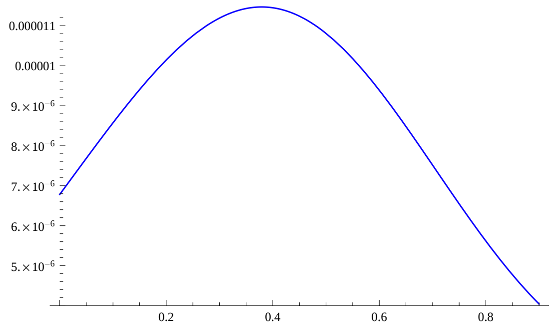

Figure 2: Absolute error of approximate solutions of the example 4.1, with hybrid method for $\mu = 192$, $\alpha = 1$

Table 2: Sum of the absolute errors in points $t = 0.1, 0.2, \cdots, 0.9$ for different methods of the example 4.1

Method	Ours	CW	PSO	GA	VIM	FVIM	GDTM	MHPM
Error	9.3×10^{-5}	1.07×10^{-4}	1.49×10^{-4}	7.71×10^{-4}	8.2×10^{-4}	3.44×10^{-2}	4.64×10^{-2}	6.39×10^{-2}

Clearly, the approximations obtained by the hybrid method are in agreement with other mentioned numerical methods and in total this approach has high accuracy.

Example 4.2. We consider the fractional-order Riccati differential equation [9, 17, 29],

$$D_*^{\alpha} f(t) = 1 - f^2(t), \quad 0 < \alpha \le 2, \tag{14}$$

with the initial conditions $f(0) = 0, f'(0) = 1$. For classical first order case of Eq. (14), the exact solution is $f(t) = \dfrac{\exp(2t)-1}{\exp(2t)+1}$, and similar to the previous example, for fractional order cases the exact solutions are not available. Here we use the hybrid operational matrices of the fractional integration to solve it. System of algebraic equations corresponding to Eq. (14) is

$$\begin{cases} K^T H(t) + \widetilde{K}\widetilde{K}H(t) - 1 = 0, & 0 < \alpha \le 1, \\ K^T H(t) + \widetilde{K}\widetilde{K}H(t) + 2t\widetilde{K}H(t) + t^2 - 1 = 0, & 1 < \alpha \le 2. \end{cases}$$

The numerical results for solution of Eq. (14) by hybrid method with $\mu = 12$, $\alpha = 0.5, 1, 1.5, 2$, are provided graphically in Fig. 3.

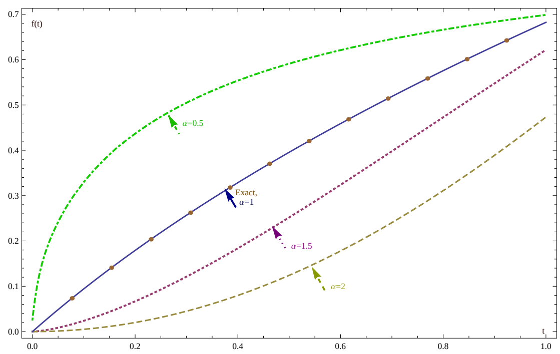

Figure 3: Numerical solutions of the example 4.2, for $\mu = 12$, $\alpha = 0.5, 1, 1.5, 2$

We solve the Eq. (14) for $\alpha = 1$, by using the third Chebyshev polynomial and $N = 5, 10, 20$. Results of absolute errors in some different value of t are reported in Table 3.

Table 3: Absolute error with $N = 5, 10, 20$ in different values of t for the example 4.2 with $\alpha = 1$

t Values	$N = 5$	$N = 10$	$N = 20$
0.1	7.2546×10^{-5}	3.9554×10^{-5}	7.2701×10^{-6}
0.2	2.7085×10^{-4}	5.2483×10^{-5}	1.0922×10^{-5}
0.3	1.8559×10^{-4}	6.0540×10^{-5}	1.3476×10^{-5}
0.4	2.7724×10^{-4}	6.3308×10^{-5}	1.4755×10^{-5}
0.5	2.2507×10^{-4}	6.1141×10^{-5}	1.4778×10^{-5}
0.6	2.1289×10^{-4}	5.4970×10^{-5}	1.3730×10^{-5}
0.7	1.9436×10^{-4}	4.6039×10^{-5}	1.1891×10^{-5}
0.8	1.1771×10^{-4}	3.5636×10^{-5}	9.57512×10^{-6}
0.9	1.2601×10^{-4}	2.4905×10^{-5}	7.07323×10^{-6}

As it can be seen of Table 3, we achieve a good approximation of the exact solution by using a few terms of approximate function by hybrid basis of block-pulse function and Chebyshev polynomials. Also, the error is being rapidly reduced when the time of simulation or numbers of block-pulse functions are increased. In Table 4, numerical solutions of our method are compared with the solutions of the modified homotopy perturbation method (MHPM) [17], the enhanced homotopy perturbation method (EHPM) [9], the improved Adams-Bashforth-Moulton method (IABMM) [9] and Bernstein polynomials (BP) [29]. Table 5, shows a comparison of our method with the methods; MHPM, PSO, FVIM, BP. Table 4, denotes the values of the solutions for $\alpha = 0.75$ and Table 5, gives the values of the solutions for $\alpha = 1$. We observe from Table 5, that for large value of $\mu = M \times N$, similar to BP [29], our presented method coincide to the exact solution.

Table 4: Comparison of the numerical solutions with the other methods for $\alpha = 0.75$ of Eq.(14)

	Hybrid method, M=6					
t Values	N=8	N=16	MHPM	EHPM	IABMM	BP
0.2	0.309927	0.309963	0.3138	0.3214	0.3117	0.309975
0.4	0.481611	0.481627	0.4929	0.5077	0.4855	0.481631
0.6	0.597775	0.597781	0.5974	0.6259	0.6045	0.597782
0.8	0.678849	0.67885	0.6604	0.7028	0.6880	0.678849

Table 5: Comparison of the numerical solutions with the other methods for $\alpha = 1$ of Eq.(14)

	Hybrid method, N=8					
t Values	M=8	M=24	MHPM	PSO	FVIM	$BP = Exact$
0.2	0.197368	0.197375	0.197375	0.197400	0.197375	0.197375
0.4	0.379937	0.379948	0.379944	0.379954	0.380005	0.379948
0.6	0.537038	0.537049	0.536857	0.537021	0.537923	0.537049
0.8	0.664028	0.664036	0.661706	0.664095	0.669695	0.664036

Finally, in Table 6, we tabulate the sum of estimated absolute errors by different methods for $\alpha = 1, t = 0.2, 0.4, 0.6, 0.8$ of the example 4.2.

Table 6: Sum of the absolute errors in points $t = 0.2, 0.4, 0.6, 0.8$ for different methods of the example 4.2

Method	Ours(μ=64)	MHPM	PSO	FVIM	BP,Ours(μ=192)
Error	0.000037	0.002526	0.000118	0.00659	0

Conclusion

In this paper, the hybrid method of block-pulse function and Chebyshev polynomials has been successfully applied to find the approximate solution of the fractional Riccati differential equations. The presented method in this study is unproblematic to implement and yields very accurate results. To demonstrate the convergence and applicability of the presented technique, some numerical experiments are reported. Comparisons with the exact solution and other methods show that this technique is a powerful and efficient tool for solving the fractional Riccati differential equations.

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References

[1] H. Adibi, P. Assari, Chebyshev wavelet method for numerical solution of Fredholm integral equations of the first kind, Math. Probl. Eng. (2010) Article ID 138408.
[2] B. Ahmad, S. K. Ntouyas, Existence of solutions for fractional differential inclusions with four-point nonlocal Riemann-Liouville type integral boundary conditions, Filomat 27 (2013) 1027–1036.
[3] E. Babolian, K. Maleknejad, M. Mordad, B. Rahimi, A numerical method for solving Fredholm-Volterra integral equations in two dimensional spaces using block-pulse functions and an operational matrix, J. Comput. Appl. Math. 235 (2011) 3965–3971.
[4] D. Baleanu, M. Alipour, H. Jafari, The Bernstein operational matrices for solving the fractional quadratic Riccati differential equations with the Riemann-Liouville derivative, Abstr. Appl. Anal. (2013) Article ID 461970.
[5] B. Batiha, M. S. M. Noorani, I. Hashim, Application of variational iteration method to general Riccati equation, Int. Math. Forum 2 (2007) 2759–2770.
[6] P. P. Boyel, W. Tian, F. Guan, The Riccati equation in mathematical finance, J. Symbolic Comput. 33 (2002) 343–355.
[7] K. Diethelm, The Analysis of Fractional Differential Equations, Springer-Verlag, Berlin 2010.
[8] R. Hilfer, Application of Fractional Calculus in Physics, World Scientific, Singapor 2010.
[9] S. H. Hosseinnia, A. Ranjbar, S. Momani, Using an enhanced homotopy perturbation method in fractional differential equations via deforming the linear part, Comput. Math. Appl. 56 (2008) 3138–3149.
[10] A. Kiliçman, Z. A. A. Al Zhour, Kronecker operational matrices for fractional calculus and some applications, Appl. Math. Comput. 187 (2007) 250–265.
[11] Y. Li, Solving a nonlinear fractional differential equation using Chebyshev wavelets, Commun. Nonlinear Sci. Numer. Simul. 15 (2010) 2284–2292.
[12] J. Liu, L. Yan, D. Tang, p-variation of an integral functional associated with bi-fractional Brownian motion, Filomat 27 (2013) 995–1009.
[13] K. Maleknejad, B. Basirat, E. Hashemizadeh, Hybrid Legendre polynomials and block-pulse functions approach for nonlinear Volterra-Fredholm integro-differential equations, Comput. Math. Appl. 61 (2011) 2821–2828.
[14] H. R. Marzban, M. Shahsiah, Solution of piecewise constant delay systems using hybrid of block-pulse and Chebyshev polynomials, Optim. Contr. Appl. Met. 32 (2011) 647–659.
[15] M. Merdan, On the solutions fractional Riccati differential equation with modified Riemann-Liouville derivative, Int. J. Differ. Equ. Appl. (2012) Article ID 346089.
[16] S. Momani, N. T. Shawagfeh, Decomposition method for solving fractional Riccati differential equations, Appl. Math. Comput. 182 (2006) 1083–1092.
[17] Z. Odibat, S. Momani, Modified homotopy perturbation method: application to quadratic Riccati differential equation of fractional order, Chaos Solitons Fractals 36 (2008) 167–174.
[18] Z. Odibat, S. Momani, V. S. Erturk, Generalized differential transform method: application to differential equations of fractional order, Appl. Math. Comput. 197 (2008) 467–477.
[19] K. B. Oldham, J. Spanier, The Fractional Calculus, Academic Press, New York 1974.
[20] M. Pavon, Stochastic realization and invariant directions of the matrix Riccati equation, SIAM J. Control Optim. 18 (1980) 155-180.
[21] I. Podlubny, Fractional Differential Equations, Academic Press, New York 1999.
[22] M.A.Z. Raja, J.A. Khan, I.M. Qureshi, A new stochastic approach for solution of Riccati differential equation of fractional order, Ann. Math. Artif. Intel. 60 (2011) 229–250.

[23] M.A.Z. Raja, J.A. Khan, I.M. Qureshi, Evolutionary computation technique for solving Riccati differential equation of arbitrary order, Proceeding WASET 58 (2009) 531–536.

[24] W. T. Reid, Riccati Differential Equations, Academic Press, New York 1972.

[25] S. Sohrabi, Comparison Chebyshev wavelets method with BPFs method for solving Abel's integral equation, Ain Shams Engineering Journal 2 (2011) 249–254.

[26] Ş. Yüzbaşi, A collocation approach to solve the Riccati-type differential equation systems, Int. J. Comput. Math. 89 (2012) 2180–2197.

[27] Ş. Yüzbaşi, Bessel collocation approach for solving continuous population models for single and interacting species, Appl. Math. Model. 36 (2012) 3787–3802.

[28] Ş. Yüzbaşi, A numerical approximation based on the Bessel functions of first kind for solutions of Riccati type differential-difference equations, Comput. Math. Appl. 64 (2012) 1691–1705.

[29] Ş. Yüzbaşi, Numerical solutions of fractional Riccati type differential equations by means of the Bernstein polynomials, Appl. Math. Comput. 219 (2013) 6328–6343.