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Univalent Functions in Dirichlet-Type Spaces

Ruishen Qian[a], Yecheng Shi[b]

[a]*School of Mathematics and Computation Science Lingnan Normal University Guangdong Zhanjiang 524048, P. R. China*
[b]*Department of Mathematics and application Shanwei Vocational and Technical College Shanwei 516600, China*

Abstract. In this note, under certain conditions on weighted function K, we give an equivalent characterizations on univalent functions in Dirichlet type spaces D_K.

1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the class of functions analytic in $\mathbb{D}$.

For $0 < p < \infty$, the Hardy space H^p is the sets of $f \in H(\mathbb{D})$ with

$$\|f\|_{H^p}^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

As usual, H^∞ is the sets of $f \in H(\mathbb{D})$ with $\sup_{z\in\mathbb{D}} |f(z)| < \infty$.

Suppose that $K : [0, \infty) \to [0, \infty)$ is a right-continuous and strictly increasing function with $K(0) = 0$. We say that a function $f \in H(\mathbb{D})$ belongs to the space D_K if

$$\|f\|_{D_K}^2 = |f(0)| + \int_{\mathbb{D}} |f'(z)|^2 K\left(1 - |z|^2\right) dA(z) < \infty.$$

In the case $K(t) = t^\alpha$, $0 \le \alpha < 1$, the space D_K gives the space D_α. Note that D_0 is the Dirichlet space $\mathcal{D}$. D_K spaces was introduced by Kerman and Sawyer in [13] to study Multipliers of D_K. And then, studied by Aleman in [1]. For more informations about D_K, we refer to [1], [5], [13], [25], [26], [27] and [31].

Throughout this paper, we assume that weighted function K satisfies:

$$\int_0^1 \frac{\varphi_K(s)}{s} ds < \infty \tag{1.1}$$

and

$$\int_1^\infty \frac{\varphi_K(s)}{s^2} ds < \infty, \tag{1.2}$$

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Email addresses: qianruishen@sina.cn (Ruishen Qian), 09ycshi@sina.cn (Yecheng Shi)

where

$$\varphi_K(s) = \sup_{0\le t\le 1} K(st)/K(t), \quad 0 < s < \infty.$$

If K satisfies (1.2), we get $K(2t) \approx K(t)$ for $t > 0$ and we can assume that K is differentiable up to any desired order. For more informations about weighted function K, we refer to [4], [5], [9], [10], [14] and [31].

For $f \in H(\mathbb{D})$ is said to be univalent if it is an one-to-one there. Let the class of all univalent functions in $H(\mathbb{D})$ denoted by $\mathcal{U}$.

Hardy and Littlewood [12], Pommerenke [21] and Prawitz [23] give the characterization of univalent functions in Hardy spaces. Pommerenke [22] also give the characterization of univalent functions in Bloch spaces, which coincident with the univalent functions in *BMOA* spaces, the space of those analytic functions f in the Hardy space H^1 whose boundary functions have bounded mean oscillation on $\partial\mathbb{D}$. Later, the result improved by Aulaskari, Lappan, Xiao and Zhao in [2]. Univalent functions in the weighted Bergman spaces was gave by Baernstein, Girela and Pelaez in [3] and Pelaez in [16]. Pérez-González and Rättyä charaterized univalent functions in Dirichlet type spaces in [17]. To learn more about the basis of theorem of univalent functions, we refer to [8] and [19].

Motivated by [18], we deduce the following result.

Main Theorem. *Let (1.1) and (1.2) hold for K. Suppose that $f \in \mathcal{U}$, then $f \in D_K$ if and only if*

$$\int_0^1 M_\infty^2(r, f) K_3'(1-r)dr < \infty,$$

where

$$K_3(t) = t\int_t^\infty \frac{K_2(s)}{s^2}ds, \quad 0 < t < \infty$$

and

$$K_2(t) = t\int_t^\infty \frac{K(s)}{s^2}ds, \quad 0 < t < \infty.$$

In this paper, the symbol $f \approx g$ means that $f \lesssim g \lesssim f$. We say that $f \lesssim g$ if there exists a constant C such that $f \le Cg$.

2. Auxiliary Results

The following result can be found in Theorem 1.1 of [15].

Lemma 1. *Let $0 < p < \infty$ and $0 < q \le \infty$. If $\omega : [0,1) \to (0,\infty)$ is differentiable and satisfies*

$$\frac{\omega'(r)}{\omega(r)^2}\int_r^1 \omega(x)dx \le C, \quad C > 0.$$

Then

$$\int_0^1 M_q^p(r, f)\omega(r)dr \approx |f(0)|^p + \int_0^1 M_q^p(r, f')\left(\frac{\int_r^1 \omega(t)dt}{\omega(r)}\right)^p \omega(r)dr.$$

Lemma 2. *Let (1.1) and (1.2) hold for K. There exists $C > 0$, such that*

$$-\frac{K_3''(t)K_3(t)}{\left(K_3'(t)\right)^2} \le C, \quad 0 < t < 1.$$

Proof. Since

$$K_3(t) = t\int_t^\infty \frac{K_2(s)}{s^2}ds, \quad 0 < t < \infty,$$

an easy computation gives

$$K_3'(t) = \int_t^\infty \frac{K_2(s)}{s^2}ds - \frac{K_2(t)}{t} = \frac{K_3(t) - K_2(t)}{t}$$

and

$$K_3''(t) = -\frac{K_2'(t)}{t}.$$

Then

$$-\frac{K_3''(t)K_3(t)}{\left(K_3'(t)\right)^2} = \frac{(K_2(t) - K(t))K_3(t)}{(K_3(t) - K_2(t))^2}.$$

By Lemma 2.2 of [10], we know that there exists a small $c > 0$, such that $K_2(t)/t^{1-c}$ is nonincreasing and $K_2(t) \geq K(t)$. Thus,

$$K_3(t) = t\int_t^\infty \frac{K_2(s)}{s^2}ds \leq K_2(t)t^c \int_t^\infty \frac{1}{s^{1+c}}ds = 1/cK_2(t).$$

Since K is strictly increasing, then

$$K_2'(t) = \int_t^\infty \frac{K(s)}{s^2}ds - \frac{K(t)}{t} > \frac{K(t)}{t} - \frac{K(t)}{t} = 0.$$

Hence, K_2 is also strictly increasing. Thus, there exists a small c_1 such that

$$\begin{aligned} K_3(t) - K_2(t) =& t\int_t^\infty \frac{K_2(s)}{s^2}ds - K_2(t) \\ =& t\int_t^\infty \frac{K_2(s) - (1 + c_1)K_2(t)}{s^2}ds + c_1K_2(t) \\ \geq& t\int_1^\infty \frac{(K_2(s) - K_2(t)) - c_1K_2(t)}{s^2}ds + c_1K_2(t) \geq c_1K_2(t). \end{aligned}$$

Hence,

$$c_1K_2(t) \leq K_3(t) - K_2(t) \leq (1/c - 1)K_2(t).$$

Look back the proof of Lemma 2.2 of [10], we have

$$K_2(t) \leq K(t)\int_1^\infty \frac{\varphi_K(s)}{s^2}ds \leq \max\{\int_1^\infty \frac{\varphi_K(s)}{s^2}ds, 2\}K(t).$$

Therefore, combine with Lemma 2.3 of [10] and $K(t) \approx K_2(t) \approx K_3(t)$, we obtain

$$\begin{aligned} -\frac{K_3''(t)K_3(t)}{\left(K_3'(t)\right)^2} =& \frac{(K_2(t) - K(t))K_3(t)}{(K_3(t) - K_2(t))^2} \\ \lesssim& \frac{K(t)K_3(t)}{(K_2(t))^2} \approx 1. \end{aligned}$$

The proof is completed. □

3. Proof of Main Theorem

Proof. Suppose that $f \in \mathcal{U} \cap D_K$. Applying Lemma 1 with $p = 2$, $q = \infty$ and $\omega(r) = K_3'(1-r)$. By Lemma 2, note that

$$\frac{\omega'(r)}{\omega(r)^2}\int_r^1 \omega(x)dx = -\frac{K_3''(1-r)K_3(1-r)}{\left(K_3'(1-r)\right)^2} \lesssim 1, \ \ 0 < r < 1,$$

we have

$$\int_0^1 M_\infty^2(t,f)K_3'(1-t)dt \lesssim \int_0^1 M_\infty^2(t,f')\frac{(K_3(1-t))^2}{K_3'(1-t)}dt + |f(0)|^2.$$

Notice the fact that

$$\frac{(K_3(1-t))^2}{K_3'(1-t)} \lesssim \int_t^1 K_3(1-s)ds, \tag{$*$}$$

combine with Fubini's theorem and the inequality (see [3, page 841])

$$\int_0^s M_\infty^2(t,f')dt \le \pi M_2^2(s,f'),$$

it follows that

$$\begin{aligned}
\int_0^1 M_\infty^2(t,f)K_3'(1-t)dt &\lesssim \int_0^1 M_\infty^2(t,f')\int_t^1 K_3(1-s)dsdt + |f(0)|^2 \\
&= \int_0^s M_\infty^2(t,f')dt \int_0^1 K_3(1-s)ds + |f(0)|^2 \\
&\lesssim \int_0^1 M_2^2(s,f')K_3(1-s)ds + |f(0)|^2 \\
&\approx \int_0^1 M_2^2(s,f')K(1-s)ds + |f(0)|^2 < \infty.
\end{aligned}$$

Now, we are going to prove ($*$). Let C as in Lemma 2 and

$$F(t) = (2+C)\int_t^1 K_3(1-s)ds - \frac{(K_3(1-t))^2}{K_3'(1-t)}.$$

From Lemma 2, it is not hard to deduce that

$$F'(t) = K_3(t)\left(-C - \frac{K_3''(1-t)K_3(1-t)}{\left(K_3'(1-t)\right)^2}\right) < 0.$$

Therefore, $F(t)$ is decreasing. Since $F(t) \to 0$ as $t \to 1^-$, hence, ($*$) holds.

On the other hand, by Lemma 2.3 of [10], we have $K \approx K_3$. Using Fubini's theorem, we obtain

$$\begin{aligned}
\int_{\mathbb{D}} |f'(z)|^2 K(1-|z|^2)dA(z) &\approx \int_{\mathbb{D}} |f'(z)|^2 K_3(1-|z|^2)dA(z) \\
&\approx \int_0^1 M_2^2(r,f')K_3(1-r)rdr \\
&= \int_0^1 M_2^2(r,f')\int_r^1 K_3'(1-t)dtrdr \\
&= \int_0^1 \int_{D(0,t)} |f'(z)|^2 dA(z) K_3'(1-t)dt,
\end{aligned}$$

where $D(0,t)=\{z:|z|<t\}$. Since $f\in\mathcal{U}$, we can easily to obtain

$$\begin{aligned}\frac{1}{\pi}\int_{D(0,t)}|f'(z)|^2dA(z)\lesssim&\frac{1}{\pi}\int_{|w|\le M_\infty(t,f)}dA(w)\\ \approx&\int_0^{M_\infty(t,f)}2sds\\ =&M_\infty^2(t,f).\end{aligned}$$

Therefore,

$$\int_{\mathbb{D}}|f'(z)|^2K(1-|z|^2)dA(z)\lesssim\int_0^1 M_\infty^2(t,f)K_3'(1-t)dt<\infty.$$

Thus, $f\in D_K$. The proof is completed. □

We say that a function $f\in H(\mathbb{D})$ belongs to the space Q_K if

$$\|f\|_{Q_K}^2=|f(0)|+\int_{\mathbb{D}}|f'(z)|^2K\left(1-|\varphi_a(z)|^2\right)dA(z)<\infty,$$

where $\varphi_a(z)=\frac{a-z}{1-\bar{a}z}$. For more informations about Q_K spaces, we refer to [4], [9], [10] and [14].

Corollary 1. *Let (1.1) and (1.2) hold for K. Suppose that $f\in\mathcal{U}$, then $f\in Q_K$ if and only if*

$$\sup_{a\in\mathbb{D}}\int_0^1 M_\infty^2(r,f\circ\varphi_a-f(a))K_3'(1-r)dr<\infty.$$

Proof. Using the fact that

$$\begin{aligned}&\|f-f(0)\|_{Q_K}^2\approx\|f-f(0)\|_{Q_{K_3}}^2\\ \approx&\sup_{a\in\mathbb{D}}\|f\circ\varphi_a-f(a)\|_{D_{K_3}}^2\approx\sup_{a\in\mathbb{D}}\|f\circ\varphi_a-f(a)\|_{D_K}^2,\end{aligned}$$

we easy to deduce our result. □

References

[1] A. Aleman, Hilbert spaces of analytic functions between the Hardy space and the Dirichlet space, Proc. Amer. Math. Soc. 115 (1992), 97-104.
[2] R. Aulaskari, P. Lappan, J. Xiao and R. Zhao, On α-Bloch spaces and multipliers of Dirichlet spaces, J. Math. Anal. Appl. 209 (1997), 103-121.
[3] A. Baernstein II, D. Girela and J. A. Peláez, Univalent functions, Hardy spaces and spaces of Dirichlet type, Illionis. J. Math., 48 (2004), 837-859.
[4] G. Bao, Z. Lou, R. Qian and H. Wulan, Improving multipliers and zero sets in Q_K spaces, submitted.
[5] G. Bao, Z. Lou, R. Qian and H. Wulan, On multipliers of Dirichlet type spaces, preprint.
[6] J. J. Donaire, D. Girela and D. Vukotić, On univalent functions in some Möbius invariant spaces, J. Reine Angew. Math. 553 (2002), 43-72.
[7] P. Duren, Theory of H^p Spaces, Academic Press, New York, 1970.
[8] P. Duren, Univalent Functions, Grundlehren der Mathematischen Wissenschaften, vol. 259. Springer, New York 1983.
[9] M. Essen and H. Wulan, On analytic and meromorphic function and spaces of Q_K-type, Illionis. J. Math. 46 (2002), 1233-1258.
[10] M. Essen, H. Wulan and J. Xiao, Several function-theoretic characterizations of Möbius invariant Q_K spaces, J. Funct. Anal. 230 (2006), 78-115.
[11] J. Garnett, Bounded Analytic Functions, Academic Press, New York, 1981.
[12] G. H. Hardy and J. E. Littlewood, Some properties of fractional integrals II, Math. Z. 34 (1932), 403-439.
[13] R. Kerman and E. Sawyer, Carleson measures and multipliers of Dirichlet-type spaces, Trans. Amer. Math. Soc. 309 (1988), 87-98.
[14] Z. Lou and W. Chen, Distances from Bloch functions to Q_K-type spaces, Integral Equations Operator Theory, 67 (2010), 171-181.

[15] M. Pavlovic and J. A. Pelaez, An equivalence for weighted integrals of an analytic function and its derivative, Math. Nachr. 281 (2008), 1612-1623.
[16] J. A. Peláez, Contribuciones a la teoría de ciertos espacios de funciones analíticas, PhD thesis, Universidad de Málaga, 2004.
[17] F. Pe´rez-González and J. Rättyä, Univalent functions in Hardy, Bergman, Bloch and related spaces. J. Anal. Math. 105 (2008), 125-148.
[18] F. Pe´rez-González and J. Rättyä, Univalent functions in the Möbius invariant Q_K space. Abstr. Appl. Anal. 2011, Art. ID 259796, 11 pp.
[19] C. Pommerenke, Univalent Functions, Vandenhoeck and Ruprecht, Göttingen 1975.
[20] C. Pommerenke, Boundary behaviour of conformal maps, Springer-Verlag, Berlin 1992.
[21] C. Pommerenke, Uber die Mittelwerte und Koeffizienten multivalenter Funktionen, Math. Ann. 145 (1961/62), 285-296.
[22] C. Pommerenke, Schlichte Funktionen and analytische Funktionen von beschränkten mittlerer Oszillation, Comment. Math. Helv. 52 (1977), 591-602.
[23] H. Prawitz, Über Mittelwerte analytischer Funktionen, Ark. Mat. Astr. Fys. 20 (1927), 1-12.
[24] R. Qian, Composition operators on Dirichlet type spaces, submitted.
[25] R. Rochberg and Z. Wu, A new characterization of Dirichlet type spaces and applications, Illinois J. Math. 37 (1993), 101-122.
[26] D. Stegenga, Multipliers of the Dirichlet space, Illinois J. Math. 24 (1980), 113-139.
[27] G. Taylor, Multipliers on D_α, Trans. Amer. Math. Soc. 123 (1966), 229-240.
[28] D. Walsh, A property of univalent functions in A_p, Glasg. Math. J. 42 (2000), 121-124.
[29] J. Xiao, Holomorphic $\mathcal{Q}$ Classes, Springer, LNM 1767, Berlin, 2001.
[30] J. Xiao, Geometric $\mathcal{Q}_p$ functions, Birkhäuser Verlag, Basel-Boston-Berlin, 2006.
[31] J. Zhou and Y. Wu, Decomposition theorems and conjugate pair in D_K spaces, to appear.
[32] K. Zhu, Operator Theory in Function Spaces, American Mathematical Society, Providence, RI, 2007.