



The Minimal Total Irregularity of Some Classes of Graphs

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Abstract. In [1], Abdo and Dimitov defined the total irregularity of a graph $G = (V, E)$ as

$$\text{irr}_t(G) = \frac{1}{2} \sum_{u,v \in V} |d_G(u) - d_G(v)|,$$

where $d_G(u)$ denotes the vertex degree of a vertex $u \in V$. In this paper, we investigate the minimal total irregularity of the connected graphs, determine the minimal, the second minimal, the third minimal total irregularity of trees, unicyclic graphs, bicyclic graphs on n vertices, and propose an open problem for further research.

1. Introduction

Let $G = (V, E)$ be a simple undirected graph with vertex set V and edge set E . For any vertices $v \in V$, the degree of a vertex v in G , denoted by $d_G(v)$, is the number of edges of G incident with v . If $V = \{v_1, v_2, \dots, v_n\}$, then the sequence $(d_G(v_1), d_G(v_2), \dots, d_G(v_n))$ is called a degree sequence of G ([5]). Without loss of generality, we assume $d_G(v_1) \geq d_G(v_2) \geq \dots \geq d_G(v_n)$.

A graph is regular if all its vertices have the same degree, otherwise it is irregular. Several approaches that characterize how irregular a graph is have been proposed. In [4], Albersson defined the imbalance of an edge $e = uv \in E$ as $|d_G(u) - d_G(v)|$ and the irregularity of G as

$$\text{irr}(G) = \sum_{uv \in E} |d_G(u) - d_G(v)|. \quad (1)$$

More results on the imbalance, the irregularity of a graph G can be found in [2, 4, 7, 8].

Inspired by the structure and meaning of the equation (1), Abdo and Dimitov [1] introduced a new irregularity measure, called the total irregularity. For a graph G , it is defined as

$$\text{irr}_t(G) = \frac{1}{2} \sum_{u,v \in V} |d_G(u) - d_G(v)|. \quad (2)$$

Although the two irregularity measures capture the irregularity only by a single parameter, namely the degree of a vertex, the new measure is more superior than the old one in some aspects. For example, (2) has an expected property of an irregularity measure that graphs with the same degree sequences have the same total irregularity, while (1) does not have. Both measures also have common properties, including that they are zero if and only if G is regular.

2010 *Mathematics Subject Classification.* Primary O5C35; Secondary O5C50, O5C07

Keywords. Total irregularity; minimal; tree; unicyclic graph; bicyclic graph.

Received: 22 March 2014; Accepted: 03 May 2015

Communicated by Francesco Belardo

Research supported by Foundation of China (Grant No. 11571123) and the Guangdong Provincial Natural Science Foundation (Grant No. 2015A030313377).

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Obviously, $\text{irr}_t(G)$ is an upper bound of $\text{irr}(G)$. In [6], the authors derived relation between $\text{irr}_t(G)$ and $\text{irr}(G)$ for a connected graph G with n vertices, that is, $\text{irr}_t(G) \leq n^2 \text{irr}(G)/4$. Furthermore, they showed that $\text{irr}_t(T) \leq (n - 2)\text{irr}(T)$ for any tree T .

Let P_n, C_n and S_n be the path, cycle and star on n vertices, respectively. In [1], the authors obtained the upper bound of the total irregularity among all graphs on n vertices, and they showed the star graph S_n is the tree with the maximal total irregularity among all trees on n vertices.

Theorem 1.1. ([1]) *Let G be a simple, undirected graph on n vertices. Then*

- (1) $\text{irr}_t(G) \leq \frac{1}{12}(2n^3 - 3n^2 - 2n + 3)$.
- (2) *If G is a tree, then $\text{irr}_t(G) \leq (n - 1)(n - 2)$, with equality holds if and only if $G \cong S_n$.*

In [9], the authors investigated the total irregularity of unicyclic graphs, and determined the graph with the maximal total irregularity $n^2 - n - 6$ among all unicyclic graphs on n vertices. In [10], the authors investigated the total irregularity of bicyclic graphs, and determined the graph with the maximal total irregularity $n^2 + n - 16$ among all bicyclic graphs on n vertices.

Recently, Abdo and Dimitrov ([3]) also obtained the upper bounds on the total irregularity of graphs under several graph operations including join, lexicographic product, Cartesian product, strong product, direct product, corona product, disjunction and symmetric difference and so on.

In this paper, we introduce an important transformation to investigate the minimal total irregularity of graphs in Section 2, determine the minimal, the second minimal, the third minimal total irregularity of trees, unicyclic graphs, bicyclic graphs on n vertices in Sections 3-5, and propose an open problem for further research.

2. Branch-Transformation

In this section, we introduce an important transformation to investigate the minimal total irregularity of graphs on n vertices.

Let G be a graph on n vertices, T be an induced subtree of G . We call T is a hanging tree of G if G can be formed by connecting a vertex of T and a vertex of $G - T$.

Branch-transformation: Let G be a simple graph with at least two pendent vertices. Without loss of generality, let u be a vertex of G with $d_G(u) \geq 3$, T be a hanging tree of G connecting to u with $|V(T)| \geq 1$, and v be a pendent vertex of G with $v \notin T$. Let G' be the graph obtained from G by deleting T from vertex u and attaching it to vertex v . We call the transformation from G to G' is a branch-transformation on G from vertex u to vertex v (see Figure 1).

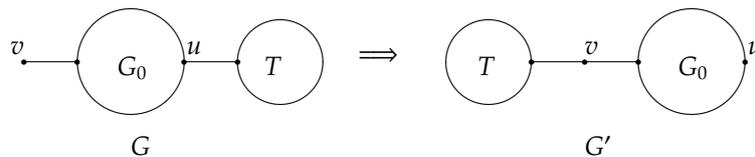


Figure 1. Branch-transformation on G from u to v

Lemma 2.1. *Let G' be the graph obtained from G by branch-transformation from u to v . Then $\text{irr}_t(G) > \text{irr}_t(G')$.*

Proof. Let $G = (V, E)$, $V_1 = \{w | d_G(w) \geq d_G(u), w \in V\}$, $V_2 = \{w | d_G(w) = 1, w \in V\}$, $V_3 = \{w | 2 \leq d_G(w) < d_G(u), w \in V\}$. Clearly, $u \in V_1$, $v \in V_2$, and $V_1 \cup V_2 \cup V_3 = V$. Let $|V_1| = s$, $|V_2| = h$, $|V_3| = r$, then $s \geq 1$, $h \geq 2$ and $s + h + r = n$.

Note that after banch-transformation, only the degrees of u and v have been changed, namely, $d_{G'}(u) = d_G(u) - 1$, $d_{G'}(v) = d_G(v) + 1 = 2$ and $d_{G'}(x) = d_G(x)$ for any $x \in V \setminus \{u, v\}$. Let $U = V \setminus \{u, v\}$. Then

$$\begin{aligned}
 & |d_{G'}(u) - d_{G'}(v)| - |d_G(u) - d_G(v)| = -2, \\
 & \sum_{x \in U} (|d_{G'}(u) - d_{G'}(x)| - |d_G(u) - d_G(x)|) = (s - 1) - (r + h - 1) = s - r - h, \\
 & \sum_{x \in U} (|d_{G'}(v) - d_{G'}(x)| - |d_G(v) - d_G(x)|) = -(s - 1) - r + (h - 1) = -s - r + h.
 \end{aligned}$$

Thus, we have

$$\begin{aligned}
 & \text{irr}_t(G') - \text{irr}_t(G) \\
 &= |d_{G'}(u) - d_{G'}(v)| + \sum_{x \in U} |d_{G'}(u) - d_{G'}(x)| + \sum_{x \in U} |d_{G'}(v) - d_{G'}(x)| \\
 & \quad - (|d_G(u) - d_G(v)| + \sum_{x \in U} |d_G(u) - d_G(x)| + \sum_{x \in U} |d_G(v) - d_G(x)|) \\
 &= (|d_{G'}(u) - d_{G'}(v)| - |d_G(u) - d_G(v)|) + \sum_{x \in U} (|d_{G'}(u) - d_{G'}(x)| - |d_G(u) - d_G(x)|) \\
 & \quad + \sum_{x \in U} (|d_{G'}(v) - d_{G'}(x)| - |d_G(v) - d_G(x)|) \\
 &= -2 + (s - r - h) + (-s - r + h) \\
 &= -2r - 2 \\
 &< 0. \quad \square
 \end{aligned}$$

Remark 2.2. Let G' be the graph obtained from G by branch-transformation from u to v . Then by branch-transformation and Lemma 2.1, we have $d_{G'}(u) = d_G(u) - 1 \geq 2$ and $d_{G'}(v) = d_G(v) + 1 = 2$, namely, $|\{w | d_{G'}(w) = 1, w \in V\}| = |\{w | d_G(w) = 1, w \in V\}| - 1$. If $d_{G'}(u) \geq 3$, G' has at least two pendent vertices, and there exists a hanging tree of G' connecting to the vertex u , we can repeat branch-transformation on G' from the vertex u , till the degree of u is equal to 2, or there is only one pendent vertex in the resulting graph, or there is not any hanging tree connecting to the vertex u .

From the above arguments, we see that we can do branch-transformation on G if and only if the following three conditions hold:

- (1) there exists a vertex u with $d_G(u) \geq 3$;
- (2) there exists a hanging tree of G connecting to the vertex u ;
- (3) G has at least two pendent vertices.

3. The Minimal Total Irregularity of Trees

In this section, we determine the minimal, the second minimal, the third minimal total irregularity of trees on n vertices and characterize the extremal graphs.

Lemma 3.1. ([5]) Let $G = (V, E)$ be a graph and $|E| = m$. Then $\sum_{v \in V} d_G(v) = 2m$.

Let $G = (V, E)$ be a tree. Then for any vertex $u \in V$, $d_G(u) \geq 2$ implies there must exist a hanging tree of G connecting to the vertex u , thus we can obtain the following results by branch-transformation.

Theorem 3.2. Let $G = (V, E)$ be a tree on n vertices. Then $\text{irr}_t(G) \geq 2n - 4$, and the equality holds if and only if $G \cong P_n$.

Proof. Clearly, $2(n-1) = \sum_{v \in V} d_G(v)$ by Lemma 3.1. Let $s = |\{w | d_G(w) \geq 3, w \in V\}|$, and $h = |\{w | d_G(w) = 1, w \in V\}|$. Then $s \geq 0$ and $h \geq 2$. Let $\Delta(G)$ be the maximum degree of the vertices of G . Now we complete the proof by the following two cases.

Case 1: $s = 0$.

Then $h = 2$ by $2(n-1) = \sum_{v \in V} d_G(v) = 2(n-h) + h$, and the degree sequence of G is $(2, \dots, 2, 1, 1)$. Thus $G \cong P_n$ and $\text{irr}_t(G) = 2n - 4$.

Case 2: $s \geq 1$.

Then $\Delta(G) \geq 3$ by $s \geq 1$, and $h \geq \Delta(G) + s - 1 \geq 3$ by $2(n-1) = \sum_{v \in V} d_G(v) \geq \Delta(G) + 3(s-1) + 2(n-s-h) + h$. So we can do branch-transformation $h - 2$ times on G till the degree sequence of the resulting graph is $(2, \dots, 2, 1, 1)$, denoted by H_1 , and thus $\text{irr}_t(G) > \text{irr}_t(H_1) = 2n - 4$ by Lemma 2.1. \square

Theorem 3.3. Let $n \geq 5$, $G = (V, E)$ be a tree on n vertices and $G \not\cong P_n$. Then $\text{irr}_t(G) \geq 4n - 10$, and the equality holds if and only if the degree sequence of G is $(3, 2, \dots, 2, 1, 1, 1)$.

Proof. It is obvious that $2(n-1) = \sum_{v \in V} d_G(v)$ by Lemma 3.1. Let $s = |\{w | d_G(w) \geq 3, w \in V\}|$, and $h = |\{w | d_G(w) = 1, w \in V\}|$. Then $\Delta(G) \geq 3$ and $s \geq 1$ since $G \not\cong P_n$. Now we complete the proof by the following two cases.

Case 1: $s + \Delta(G) = 4$.

Clearly, $s = 1, \Delta(G) = 3$. Then $h = 3$ by $2(n-1) = \sum_{v \in V} d_G(v) = 3 + 2(n-1-h) + h$, and the degree sequence of G is $(3, 2, \dots, 2, 1, 1, 1)$. Thus $\text{irr}_t(G) = 4n - 10$.

Case 2: $s + \Delta(G) \geq 5$.

Then $h \geq \Delta(G) + s - 1 \geq 4$ by $2(n-1) = \sum_{v \in V} d_G(v) \geq \Delta(G) + 3(s-1) + 2(n-s-h) + h$. Now we can do branch-transformation $h - 3$ times on G till the degree sequence of the resulting graph is $(3, 2, \dots, 2, 1, 1, 1)$, denoted by H_2 , and thus $\text{irr}_t(G) > \text{irr}_t(H_2) = 4n - 10$ by Lemma 2.1. \square

Theorem 3.4. Let $n \geq 6, G = (V, E)$ be a tree on n vertices, $G \not\cong P_n$. If the sequence $(3, 2, \dots, 2, 1, 1, 1)$ is not the degree sequence of G , then $\text{irr}_t(G) \geq 6n - 20$, and the equality holds if and only if the degree sequence of G is $(3, 3, 2, \dots, 2, 1, 1, 1)$.

Proof. Clearly, $2(n-1) = \sum_{v \in V} d_G(v)$ by Lemma 3.1. Let $s = |\{w | d_G(w) \geq 3, w \in V\}|$, and $h = |\{w | d_G(w) = 1, w \in V\}|$. Then $s \geq 1$ and $\Delta(G) \geq 3$ since $G \not\cong P_n$. Now we complete the proof by the following two cases.

Case 1: $s = 1$.

Then $\Delta(G) \geq 4$ because sequence $(3, 2, \dots, 2, 1, 1, 1)$ is not the degree sequence of G .

Subcase 1.1: $\Delta(G) = 4$.

Then $h = 4$ by $2(n-1) = \sum_{v \in V} d_G(v) = 4 + 2(n-1-h) + h$, and the degree sequence of G is $(4, 2, \dots, 2, 1, 1, 1, 1)$. Thus $\text{irr}_t(G) = 6n - 18 > 6n - 20$.

Subcase 1.2: $\Delta(G) \geq 5$.

Then $h = \Delta(G) \geq 5$ by $2(n-1) = \sum_{v \in V} d_G(v) = \Delta(G) + 2(n-1-h) + h$. Now we can do branch-transformation $h - 4$ times on G till the degree sequence of the resulting graph is $(4, 2, \dots, 2, 1, 1, 1, 1)$, denoted by H_3 , and thus $\text{irr}_t(G) > \text{irr}_t(H_3) = 6n - 18 > 6n - 20$ by Lemma 2.1.

Case 2: $s \geq 2$.

Case 2.1: $s + \Delta(G) = 5$.

Then the degree sequence of G is $(3, 3, 2, \dots, 2, 1, 1, 1, 1)$, and thus $\text{irr}_t(G) = 6n - 20$.

Case 2.2: $s + \Delta(G) \geq 6$.

Then $h \geq \Delta(G) + s - 1 \geq 5$ by $2(n-1) = \sum_{v \in V} d_G(v) \geq \Delta(G) + 3(s-1) + 2(n-s-h) + h$. Now we can do branch-transformation $h - 4$ times on G till the degree sequence of the resulting graph is $(3, 3, 2, \dots, 2, 1, 1, 1, 1)$, denoted by H_4 , and thus $\text{irr}_t(G) > \text{irr}_t(H_4) = 6n - 20$ by Lemma 2.1. \square

Remark 3.5. Let $n \geq 6$, by Theorems 3.2-3.4, we know the minimal, the second minimal, the third minimal total irregularity of trees on n vertices are $2n - 4, 4n - 10, 6n - 20$, respectively, and the degree sequences of the corresponding extremal graphs are $(2, \dots, 2, 1, 1), (3, 2, \dots, 2, 1, 1, 1), (3, 3, 2, \dots, 2, 1, 1, 1, 1)$, respectively.

4. The Minimal Total Irregularity of Unicyclic Graphs

In this section, we determine the minimal, the second minimal, the third minimal total irregularity of unicyclic graphs on n vertices and characterize the extremal graphs.

An unicyclic graph is a simple connected graph in which the number of edges equals the number of vertices. Let $G = (V, E)$ be an unicyclic graph. Then for any vertex $u \in V, d_G(u) \geq 3$ implies there must exist a hanging tree of G connecting to the vertex u , thus we can obtain the following results by branch-transformation.

Theorem 4.1. Let $n \geq 3$ and $G = (V, E)$ be an unicyclic graph on n vertices.

- (1) $\text{irr}_t(G) \geq 0$, and the equality holds if and only if $G \cong C_n$.
- (2) Let $n \geq 4$, and $G \not\cong C_n$. Then $\text{irr}_t(G) \geq 2n - 2$, and the equality holds if and only if the degree sequence of G is $(3, 2, \dots, 2, 1)$.

Proof. (1) is obvious. Now we show (2) holds.

It is obvious that $2n = \sum_{v \in V} d_G(v)$ by Lemma 3.1. Let $s = |\{w | d_G(w) \geq 3, w \in V\}|$, and $h = |\{w | d_G(w) = 1, w \in V\}|$. Then $s \geq 1, h \geq 1, \Delta(G) \geq 3$ by $G \not\cong C_n$ and $2n = \sum_{v \in V} d_G(v)$. Now we complete the proof by the following two cases.

Case 1: $s + \Delta(G) = 4$.

Then $(3, 2, \dots, 2, 1)$ is the degree sequence of G by $2n = \sum_{v \in V} d_G(v) = 3 + 2(n - 1 - h) + h$, and thus $\text{irr}_t(G) = 2n - 2$.

Case 2: $s + \Delta(G) \geq 5$.

Then $h \geq \Delta(G) + s - 3 \geq 2$ by $2n = \sum_{v \in V} d_G(v) \geq \Delta(G) + 3(s - 1) + 2(n - s - h) + h$, and we can do branch-transformation $h - 1$ times on G till the degree sequence of the resulting graph is $(3, 2, \dots, 2, 1)$, denoted by H_5 , and thus $\text{irr}_t(G) > \text{irr}_t(H_5) = 2n - 2$ by Lemma 2.1. \square

Theorem 4.2. *Let $n \geq 5, G = (V, E)$ be an unicyclic graph on n vertices with $G \not\cong C_n$. If the sequence $(3, 2, \dots, 2, 1)$ is not the degree sequence of G , then $\text{irr}_t(G) \geq 4n - 8$, and the equality holds if and only if the degree sequence of G is $(3, 3, 2, \dots, 2, 1, 1)$.*

Proof. Clearly, $2n = \sum_{v \in V} d_G(v)$ by Lemma 3.1. Let $s = |\{w | d_G(w) \geq 3, w \in V\}|$, and $h = |\{w | d_G(w) = 1, w \in V\}|$. Then $s \geq 1, h \geq 1$ by $G \not\cong C_n$ and $2n = \sum_{v \in V} d_G(v)$. Now we complete the proof by the following two cases.

Case 1: $s = 1$.

Then $\Delta(G) \geq 4$ because sequence $(3, 2, \dots, 2, 1)$ is not the degree sequence of G .

Subcase 1.1: $\Delta(G) = 4$.

Then $h = 2$ and the degree sequence is $(4, 2, \dots, 2, 1, 1)$ by $2n = \sum_{v \in V} d_G(v) = 4 + 2(n - 1 - h) + h$, and thus $\text{irr}_t(G) = 4n - 6 > 4n - 8$.

Subcase 1.2: $\Delta(G) \geq 5$.

Then $h = \Delta(G) - 2 \geq 3$ by $2n = \sum_{v \in V} d_G(v) = \Delta(G) + 2(n - 1 - h) + h$, and we can do branch-transformation $h - 2$ times on G till the degree sequence of the resulting graph is $(4, 2, \dots, 2, 1, 1)$, denoted by H_6 , and thus $\text{irr}_t(G) > \text{irr}_t(H_6) = 4n - 6$ by Lemma 2.1.

Case 2: $s \geq 2$.

Subcase 2.1: $s + \Delta(G) = 5$.

Then $h = 2$ and the degree sequence is $(3, 3, 2, \dots, 2, 1, 1)$ by $2n = \sum_{v \in V} d(v) = 6 + 2(n - 2 - h) + h$, and thus $\text{irr}_t(G) = 4n - 8$.

Subcase 2.2: $s + \Delta(G) \geq 6$.

Then $h \geq \Delta(G) + s - 3 \geq 3$ by $2n = \sum_{v \in V} d_G(v) \geq \Delta(G) + 3(s - 1) + 2(n - s - h) + h$, and we can do branch-transformation $h - 2$ times on G till the degree sequence of the resulting graph is $(3, 3, 2, \dots, 2, 1, 1)$, denoted by H_7 , and thus $\text{irr}_t(G) > \text{irr}_t(H_7) = 4n - 8$ by Lemma 2.1. \square

Remark 4.3. *Let $n \geq 5$, by Theorems 4.1-4.2, we know the minimal, the second minimal, the third minimal total irregularity of unicyclic graphs on n vertices are $0, 2n - 2, 4n - 8$, respectively, and the degree sequences of the corresponding extremal graphs are $(2, \dots, 2), (3, 2, \dots, 2, 1), (3, 3, 2, \dots, 2, 1, 1)$, respectively.*

5. The Minimal Total Irregularity of Bicyclic Graphs

In this section, we determine the minimal, the second minimal, the third minimal total irregularity of bicyclic graphs on n vertices and characterize the extremal graphs.

A bicyclic graph is a simple connected graph in which the number of edges equals the number of vertices plus one. There are two basic bicyclic graphs: ∞ -graph and Θ -graph. An ∞ -graph, denoted by $\infty(p, q, l)$ (see Figure 2), is obtained from two vertex-disjoint cycles C_p and C_q by connecting one vertex of C_p and one

of C_q with a path P_l of length $l - 1$ (in the case of $l = 1$, identifying the above two vertices, see Figure 3) where $p, q \geq 3$ and $l \geq 1$; and a Θ -graph, denoted by $\theta(p, q, l)$ (see Figure 4), is a graph on $p + q - l$ vertices with the two cycles C_p and C_q have l common vertices, where $p, q \geq 3$ and $l \geq 2$.

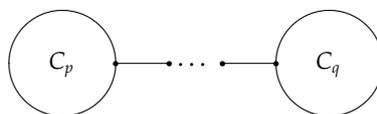


Figure 2. The graph $\infty(p, q, l)$ with $l \geq 2$

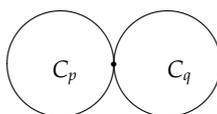


Figure 3. The graph $\infty(p, q, 1)$

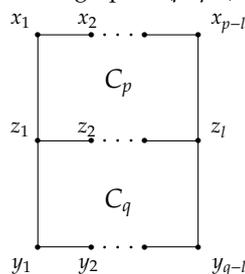


Figure 4. The graph $\theta(p, q, l)$

Denoted by \mathcal{B}_n is the set of all bicyclic graphs on n vertices. Obviously, \mathcal{B}_n consists of three types of graphs: first type denoted by B_n^+ , is the set of those graphs each of which is an ∞ -graph, $\infty(p, q, l)$, with trees attached when $l = 1$; second type denoted by B_n^{++} , is the set of those graphs each of which is an ∞ -graph, $\infty(p, q, l)$, with trees attached when $l \geq 2$; third type denoted by Θ_n , is the set of those graphs each of which is a Θ -graph, $\theta(p, q, l)$, with trees attached. Then $\mathcal{B}_n = B_n^+ \cup B_n^{++} \cup \Theta_n$.

5.1. The graph with minimal total irregularity in B_n^+

In this subsection, the minimal, the second minimal total irregularity of the bicyclic graphs in B_n^+ are determined.

Theorem 5.1. Let $n \geq 6$, $G = (V, E) \in B_n^+$.

- (1) $\text{irr}_t(G) \geq 2n - 2$, and the equality holds if and only if the degree sequence of G is $(4, 2, \dots, 2)$.
- (2) If $(4, 2, \dots, 2)$ is not the degree sequence of G , then $\text{irr}_t(G) \geq 4n - 6$, and the equality holds if and only if the degree sequence of G is $(4, 3, 2, \dots, 2, 1)$.

Proof. Clearly, $\sum_{v \in V} d_G(v) = 2(n + 1)$ by Lemma 3.1. Let $s = |\{w | d_G(w) \geq 3, w \in V\}|$, $h = |\{w | d_G(w) = 1, w \in V\}|$ and $t = |\{w | d_G(w) = \Delta(G), w \in V\}|$. Then $s \geq 1$, $h \geq 0$, $1 \leq t \leq s$ and $\Delta(G) \geq 4$ by $G \in B_n^+$.

Note that $G \in B_n^+$, if $s = 1$, $\Delta(G) \geq 5$ or $s \geq 2$, there must exist a vertex u with $d_G(u) \geq 3$ and there exists a hanging tree of G connecting to u . Then we complete the proof by the following two cases.

Case 1: $s = 1$.

Subcase 1.1: $\Delta(G) = 4$.

Then $h = 0$ and the degree sequence of G is $(4, 2, \dots, 2)$ by the fact $2(n + 1) = \sum_{v \in V} d_G(v) = 4 + 2(n - 1 - h) + h$,

and thus $\text{irr}_t(G) = 2n - 2$.

Subcase 1.2: $\Delta(G) = 5$.

Then $h = 1$ and the degree sequence of G is $(5, 2, \dots, 2, 1)$ by the fact $2(n+1) = \sum_{v \in V} d_G(v) = 5 + 2(n-1-h) + h$, and thus $\text{irr}_t(G) = 4n - 4 > 4n - 6$.

Subcase 1.3: $\Delta(G) \geq 6$.

Then $h = \Delta(G) - 4 \geq 2$ by the fact $2(n+1) = \sum_{v \in V} d_G(v) = \Delta(G) + 2(n-1-h) + h$, and we can do branch-transformation $h - 1$ times on G till the degree sequence of the resulting graph is $(5, 2, \dots, 2, 1)$, denoted by H_8 , and thus $\text{irr}_t(G) > \text{irr}_t(H_8) = 4n - 4$ by Lemma 2.1.

Case 2: $s \geq 2$.

Subcase 2.1: $s + \Delta(G) = 6$.

Then $s = 2$, $\Delta(G) = 4$ and $1 \leq t \leq 2$.

If $t = 1$, then $h = 1$ and the degree sequence of G is $(4, 3, 2, \dots, 2, 1)$ by the fact $2(n+1) = \sum_{v \in V} d_G(v) = 4 + 3 + 2(n-2-h) + h$, and thus $\text{irr}_t(G) = 4n - 6$.

If $t = 2$, then $h = 2$ by the fact $2(n+1) = \sum_{v \in V} d_G(v) = 4 + 4 + 2(n-2-h) + h$, and we can do branch-transformation once on G such that the degree sequence of the resulting graph is $(4, 3, 2, \dots, 2, 1)$, denoted by H_9 , and thus $\text{irr}_t(G) > \text{irr}_t(H_9) = 4n - 6$ by Lemma 2.1.

Subcase 2.2: $s + \Delta(G) \geq 7$.

Then $h \geq \Delta(G) + s - 5 \geq 2$ by the fact $2(n+1) = \sum_{v \in V} d_G(v) \geq \Delta(G) + 3(s-1) + 2(n-s-h) + h$, and we can do branch-transformation $h - 1$ times on G such that the degree sequence of the resulting graph is $(4, 3, 2, \dots, 2, 1)$, denoted by H_9 , and thus $\text{irr}_t(G) > \text{irr}_t(H_9) = 4n - 6$ by Lemma 2.1. \square

5.2. The graph with minimal total irregularity in B_n^{++}

In this subsection, the minimal, the second minimal total irregularity of the bicyclic graphs in B_n^{++} are determined.

Theorem 5.2. Let $n \geq 7$, $G = (V, E) \in B_n^{++}$.

(1) $\text{irr}_t(G) \geq 2n - 4$, and the equality holds if and only if the degree sequence of G is $(3, 3, 2, \dots, 2)$.

(2) If $(3, 3, 2, \dots, 2)$ is not the degree sequence of G , then $\text{irr}_t(G) \geq 4n - 10$, and the equality holds if and only if the degree sequence of G is $(3, 3, 3, 2, \dots, 2, 1)$.

Proof. Clearly, $\sum_{v \in V} d_G(v) = 2(n+1)$ by Lemma 3.1. Let $s = |\{w | d_G(w) \geq 3, w \in V\}|$, $h = |\{w | d_G(w) = 1, w \in V\}|$ and $t = |\{w | d_G(w) = \Delta(G), w \in V\}|$. Then $s \geq 2$, $h \geq 0$, $1 \leq t \leq s$ and $\Delta(G) \geq 3$ by $G \in B_n^{++}$.

Note that $G \in B_n^{++}$, if $s = 2$, $\Delta(G) \geq 4$ or $s \geq 3$, there must exist a vertex u with $d_G(u) \geq 3$ and there exists a hanging tree of G connecting to u . Then we complete the proof by the following two cases.

Case 1: $s = 2$.

Subcase 1.1: $\Delta(G) = 3$.

Then $h = 0$ and the degree sequence of G is $(3, 3, 2, \dots, 2)$ by the fact $2(n+1) = \sum_{v \in V} d_G(v) = 3 + 3 + 2(n-2-h) + h$, and thus $\text{irr}_t(G) = 2n - 4$.

Subcase 1.2: $\Delta(G) = 4$.

Then $t = 1$ or $t = 2$ by $1 \leq t \leq s$.

If $t = 1$, then $h = 1$ and the degree sequence of G is $(4, 3, 2, \dots, 2, 1)$ by the fact $2(n+1) = \sum_{v \in V} d_G(v) = 4 + 3 + 2(n-2-h) + h$, and thus $\text{irr}_t(G) = 4n - 6 > 4n - 10$.

If $t = 2$, then $h = 2$ by the fact $2(n+1) = \sum_{v \in V} d_G(v) = 4 + 4 + 2(n-2-h) + h$, and we can do branch-transformation once on G such that the degree sequence of the resulting graph is $(4, 3, 2, \dots, 2, 1)$, denoted by H_{10} , and thus $\text{irr}_t(G) > \text{irr}_t(H_{10}) = 4n - 6$ by Lemma 2.1.

Subcase 1.3: $\Delta(G) \geq 5$.

Then $h \geq \Delta(G) - 3 \geq 2$ by the fact $2(n+1) = \sum_{v \in V} d_G(v) \geq \Delta(G) + 3 + 2(n-2-h) + h$, and we can do branch-transformation $h - 1$ times on G such that the degree sequence of the resulting graph is $(4, 3, 2, \dots, 2, 1)$, denoted by H_{10} , and thus $\text{irr}_t(G) > \text{irr}_t(H_{10}) = 4n - 6$ by Lemma 2.1.

Case 2: $s \geq 3$.

Subcase 2.1: $s + \Delta(G) = 6$.

Then $h = 1$ and the degree sequence of G is $(3, 3, 3, 2, \dots, 2, 1)$ by the fact $2(n + 1) = \sum_{v \in V} d_G(v) = 3 + 3 + 3 + 2(n - 3 - h) + h$, and thus $\text{irr}_t(G) = 4n - 10$.

Subcase 2.2: $s + \Delta(G) \geq 7$.

Then $h \geq \Delta(G) + s - 5 \geq 2$ by the fact $2(n + 1) = \sum_{v \in V} d_G(v) \geq \Delta(G) + 3(s - 1) + 2(n - s - h) + h$, and we can do branch-transformation $h - 1$ times on G such that the degree sequence of the resulting graph is $(3, 3, 3, 2, \dots, 2, 1)$, denoted by H_{11} , and thus $\text{irr}_t(G) > \text{irr}_t(H_{11}) = 4n - 10$ by Lemma 2.1. \square

5.3. The graph with minimal total irregularity in Θ_n

By the same proof of Theorem 5.2, we can determine the minimal, the second minimal total irregularity of the bicyclic graphs in Θ_n immediately.

Theorem 5.3. Let $n \geq 5$, $G = (V, E) \in \Theta_n$.

- (1) $\text{irr}_t(G) \geq 2n - 4$, and the equality holds if and only if the degree sequence of G is $(3, 3, 2, \dots, 2)$.
- (2) If $(3, 3, 2, \dots, 2)$ is not the degree sequence of G , then $\text{irr}_t(G) \geq 4n - 10$, and the equality holds if and only if the degree sequence of G is $(3, 3, 3, 2, \dots, 2, 1)$.

5.4. The graph with minimal total irregularity in \mathcal{B}_n

By Theorems 5.1-5.3, we can determine the minimal, the second minimal, the third minimal total irregularity of the bicyclic graphs on n vertices immediately.

Theorem 5.4. Let $n \geq 7$, $G \in \mathcal{B}_n$.

- (1) $\text{irr}_t(G) \geq 2n - 4$, and the equality holds if and only if the degree sequence of G is $(3, 3, 2, \dots, 2)$.
- (2) If $(3, 3, 2, \dots, 2)$ is not the degree sequence of G , then $\text{irr}_t(G) \geq 2n - 2$, and the equality holds if and only if the degree sequence of G is $(4, 2, \dots, 2)$.
- (3) If $(3, 3, 2, \dots, 2)$ and $(4, 2, \dots, 2)$ are not the degree sequence of G , then $\text{irr}_t(G) \geq 4n - 10$, and the equality holds if and only if the degree sequence of G is $(3, 3, 3, 2, \dots, 2, 1)$.

6. Open Problem for Further Research

By the results of Sections 3-5, we know the minimal total irregularity of simple, undirected graphs on n vertices is zero, and the corresponding extremal graphs are regular graphs. A nature question is to look for the second minimal total irregularity of simple, undirected graphs on n vertices.

Let G be a simple, undirected graph on n vertices, if n, r are odd positive integers with $3 \leq r < n$, and the degree sequence of G is $(r, \dots, r, r - 1)$ or $(r + 1, r, \dots, r)$, then $\text{irr}_t(G) = n - 1 < 2n - 4$ when $n \geq 5$.

On the other hand, it is well known that the number of vertices of odd degree in a graph G is always even, if the order of G is even, then the number of vertices of even degree of G is also even. Let G be not a regular graph on n vertices, the number of vertices of odd degree of G be n_1 , and the number of vertices of even degree of G be n_2 , where n, n_1, n_2 are even nonnegative integers and $n_1 + n_2 = n$. Then we have $\text{irr}_t(G) \geq n_1 n_2 \geq 2(n - 2) > n - 1$ when $n \geq 4$ and $n_1, n_2 > 0$. Based on the above arguments, we propose the following question.

Conjecture 6.1. Let G be a simple, undirected graph on n vertices. If G is not a regular graph, then

$$\text{irr}_t(G) \geq \begin{cases} n - 1, & \text{if } n \text{ is odd;} \\ 2n - 4, & \text{if } n \text{ is even.} \end{cases}$$

Acknowledgments

The authors would like to thank the referees for their valuable comments, corrections and suggestions, which lead to an improvement of the original manuscript.

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