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Additive Property of Drazin Invertibility of Elements in a Ring

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Abstract. In this article, we investigate additive properties of the Drazin inverse of elements in rings and algebras over an arbitrary field. The necessary and sufficient condition for the Drazin invertibility of $a - b$ is considered under the condition of $ab = \lambda ba$ in algebras over an arbitrary field. Moreover, we give explicit representations of $(a + b)^D$, as a function of a, b, a^D and b^D, whenever $a^3 b = ba$ and $b^3 a = ab$.

1. Introduction

Throughout this article, $\mathcal{A}$ denotes an algebra over an arbitrary field $\mathbb{F}$ and R denotes an associative ring with unity. Recall that the Drazin inverse of $a \in R$ is the element $b \in R$ (denoted by a^D) which satisfies the following equations [12]:

$$bab = b, \quad ab = ba, \quad a^k = a^{k+1} b.$$

for some nonnegative integer k. The smallest integer k is called the Drazin index of a, denoted by ind(a). If ind(a) = 1, then a is group invertible and the group inverse of a is denoted by $a^\sharp$. It is well known that the Drazin inverse is unique, if it exists. The conditions in the definition of Drazin inverse are equivalent to:

$$bab = b, \quad ab = ba, \quad a - a^2 b \text{ is nilpotent.}$$

The study of the Drazin inverse of the sum of two Drazin invertible elements was first developed by Drazin [12]. It was proved that $(a + b)^D = a^D + b^D$ provided that $ab = ba = 0$. In recent years, many papers focused on the problem under some weaker conditions. For two complex matrices A, B, Hartwig et al.[15] expressed $(A + B)^D$ under one-sided condition $AB = 0$. This result was extended to bounded linear operators on an arbitrary complex Banach space by Djordjević and Wei [10], and was extended for morphisms on arbitrary additive categories by Chen et al. [4]. In the article of Wei and Deng [22] and Zhuang et al. [24], the commutativity $ab = ba$ was assumed. In [22], they characterized the relationships of the Drazin inverse

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between $A + B$ and $I + A^D B$ by Jordan canonical decomposition for complex matrices A and B. In [24], Zhuang et al. extended the result in [22] to a ring R , and it was shown that if $a, b \in R$ are Drazin invertible and $ab = ba$, then $a + b$ is Drazin invertible if and only if $1 + a^D b$ is Drazin invertible. More results on the Drazin inverse can also be found in [1-3, 6, 7, 9, 11, 13, 14, 16, 17, 19-24]. The motivation for this article was the results in Deng [8], Cvetković-Ilić [5] and Liu et al. [18]. In [5, 8] the commutativity $ab = \lambda ba$ was assumed. In [8], the author characterized the relationships of the Drazin inverse between $a \pm b$ and $aa^D(a \pm b)bb^D$ by the space decomposition for operator matrices a and b. In [18], the author gave explicit representations of $(a + b)^D$ of two matrices a and b, as a function of a, b, a^D and b^D, under the conditions $a^3 b = ba$ and $b^3 a = ab$. In this article, we extend the results in [8, 18] to more general settings.

As usual, the set of all Drazin invertible elements in an algebra $\mathcal{A}$ is denoted by $\mathcal{A}^D$. Similarly, R^D indicates the set of all Drazin invertible elements in a ring R. Given $a \in \mathcal{A}^D$ (or $a \in R^D$), it is easy to see that $1 - aa^D$ is an idempotent, which is denoted by a^π.

2. Under the Condition $ab = \lambda ba$

In this section, we will extend the result in [8] to an algebra $\mathcal{A}$ over an arbitrary field $\mathbb{F}$.

Lemma 2.1. *Let $a, b \in \mathcal{A}$ be such that $ab = \lambda ba$ and $\lambda \in \mathbb{F}\backslash\{0\}$. Then*
(1) $ab^i = \lambda^i b^i a$ *and* $a^i b = \lambda^i ba^i$.
(2) $(ab)^i = \lambda^{-\frac{i(i-1)}{2}} a^i b^i$ *and* $(ba)^i = \lambda^{\frac{i(i-1)}{2}} b^i a^i$.

Proof. (1) By hypothesis, we have

$$ab^i = abb^{i-1} = \lambda bab^{i-1} = \lambda babb^{i-2} = \lambda^2 b^2 ab^{i-2} = \cdots = \lambda^i b^i a.$$

Similarly, we can obtain that $a^i b = \lambda^i ba^i$.
(2) By hypothesis, it follows that

$$\begin{aligned}(ab)^i &= abab(ab)^{i-2} = \lambda^{-1} a^2 b^2 (ab)^{i-2} = \lambda^{-(1+2)} a^3 b^3 (ab)^{i-3} \\ &= \cdots = \lambda^{-\sum_{k=0}^{k=i-1} k} a^i b^i = \lambda^{-\frac{i(i-1)}{2}} a^i b^i.\end{aligned}$$

Similarly, it is easy to get $(ba)^i = \lambda^{\frac{i(i-1)}{2}} a^i b^i$. □

Lemma 2.2. *Let $a, b \in \mathcal{A}$ be Drazin invertible and $\lambda \in \mathbb{F}\backslash\{0\}$. If $ab = \lambda ba$, then*
(1) $a^D b = \lambda^{-1} ba^D$.
(2) $ab^D = \lambda^{-1} b^D a$.
(3) $(ab)^D = b^D a^D = \lambda^{-1} a^D b^D$.

Proof. Assume $k = \max\{\text{ind}(a), \text{ ind}(b)\}$.
(1) By hypothesis, we have

$$\begin{aligned}a^D(a^k b) &= a^D(\lambda^k ba^k) = \lambda^k a^D(ba^{k+1}a^D) = \lambda^k a^D(\lambda^{-(k+1)} a^{k+1} ba^D) \\ &= \lambda^{-1} a^D a^{k+1} ba^D = \lambda^{-1} a^k ba^D.\end{aligned}$$

It follows that

$$\begin{aligned}a^D b &= (a^D)^{k+1} a^k b = (a^D)^k a^D a^k b = \lambda^{-1}(a^D)^k a^k ba^D = \cdots \\ &= \lambda^{-(k+1)} a^k b(a^D)^{k+1} = \lambda^{-1} ba^k (a^D)^{k+1} = \lambda^{-1} ba^D.\end{aligned}$$

Moreover,

$$(ba^D)^i = \lambda^{-\frac{i(i-1)}{2}} b^i (a^D)^i \text{ and } (a^D b)^i = \lambda^{\frac{i(i-1)}{2}} (a^D)^i b^i.$$

(2) The proof is similar to (1).
(3) By (1), we have $a^Db = \lambda^{-1}ba^D$, then $(aa^D)b = \lambda^{-1}aba^D = b(aa^D)$. By [12], we get $aa^Db^D = b^Daa^D$. Similarly, we can obtain that $ab^Db = \lambda^{-1}b^Dab = b^Dba$ and $a^Dbb^D = bb^Da^D$. This implies that

$$abb^Da^D = bb^Daa^D = b^Da^Dab.$$

$$b^Da^Dabb^Da^D = b^Dbb^Da^Daa^D = b^Da^D.$$

and

$$\begin{aligned}(ab)^{k+1}b^Da^D &= \lambda^{-\frac{k(k+1)}{2}}a^{k+1}b^{k+1}b^Da^D = \lambda^{-\frac{k(k+1)}{2}}a^{k+1}b^ka^D \\ &= \lambda^{-\frac{k(k+1)}{2}}a^{k+1}(\lambda^ka^Db^k) = \lambda^{-\frac{k(k-1)}{2}}a^{k+1}a^Db^k \\ &= \lambda^{-\frac{k(k-1)}{2}}a^kb^k = (ab)^k.\end{aligned}$$

Then we get $(ab)^D = b^Da^D$. Similarly, we can check that $(ab)^D = \lambda^{-1}a^Db^D$. □

Theorem 2.3. *Let a, b be Drazin invertible in $\mathcal{A}$. If $ab = \lambda ba$ and $\lambda \neq 0$, then $a - b$ is Drazin invertible if and only if $w = aa^D(a-b)bb^D$ is Drazin invertible. In this case,*

$$(a-b)^D = w^D + a^D(1-bb^\pi a^D)^{-1}b^\pi - a^\pi(1-b^Daa^\pi)^{-1}b^D.$$

Proof. Since $w = aa^D(a-b)bb^D$, we have $w = (1-a^\pi)(a-b)(1-b^\pi)$ and

$$a - b = w + (a-b)b^\pi + a^\pi(a-b) - a^\pi(a-b)b^\pi. \tag{1}$$

By the proof of Lemma 2.2 (3), we have $aa^Db = baa^D$ and $abb^D = b^Dba$. This means that $a^\pi b = ba^\pi$ and $b^\pi a = ab^\pi$.
Let $s = \mathrm{ind}(a)$ and $t = \mathrm{ind}(b)$. By Lemma 2.2 (1) and $b^tb^\pi = 0$, we get

$$(bb^\pi a^D)^t = \lambda^{-\frac{t(t-1)}{2}}b^tb^\pi(a^D)^t = 0$$

and $(1-bb^\pi a^D)^{-1} = 1 + ba^Db^\pi + (ba^D)^2b^\pi + \cdots + (ba^D)^{t-1}b^\pi$.
By a similar method, we get $1 - b^Daa^\pi$ and $1 - aa^\pi b^D$ are both invertible.
Note that $wa^\pi = a^\pi w = a^\pi aa^D(a-b)bb^D = 0$ and $b^\pi w = wb^\pi = aa^D(a-b)bb^Db^\pi = 0$ by $a^\pi b = ba^\pi$ and $b^\pi a = ab^\pi$.

Now let us begin the proof of Theorem 2.3. Assume w is Drazin invertible and let

$$x = w^D + a^D(1-bb^\pi a^D)^{-1}b^\pi - a^\pi(1-b^Daa^\pi)^{-1}b^D.$$

Since $ab^\pi = b^\pi a$ and $ba^\pi = a^\pi b$, it is easy to obtain that $w(a-b) = (a-b)w$ and $w^D(a-b) = (a-b)w^D$.
A direct computation yields

$$\begin{aligned}&(a-b)[a^D(1-bb^\pi a^D)^{-1}b^\pi] \\ =\ & aa^D(1-ba^D)b^\pi(1-bb^\pi a^D)^{-1} \\ =\ & aa^D(1-bb^\pi a^D - bbb^Da^D)b^\pi(1-bb^\pi a^D)^{-1} \\ =\ & aa^D(1-bb^\pi a^D)b^\pi(1-bb^\pi a^D)^{-1} \\ =\ & aa^Db^\pi.\end{aligned}$$

Since $(1-b^Daa^\pi)b^D = b^D(1-aa^\pi b^D)$, we have

$$\begin{aligned}(a-b)a^\pi(1-b^Daa^\pi)^{-1}b^D &= (a-b)b^Da^\pi(1-aa^\pi b^D)^{-1} \\ &= -bb^D(1-ab^D)a^\pi(1-aa^\pi b^D)^{-1} \\ &= -bb^Da^\pi.\end{aligned}$$

So, by the above, we can obtain that

$$\begin{aligned}(a-b)x &= (a-b)(w^D + a^D(1-bb^\pi a^D)^{-1}b^\pi - a^\pi(1-b^Daa^\pi)^{-1}b^D) \\ &= (a-b)w^D + aa^Db^\pi + bb^Da^\pi.\end{aligned} \tag{2}$$

Similar to the above way, we also have $[a^D(1-bb^\pi a^D)^{-1}b^\pi](a-b) = a^D ab^\pi$ and $[a^\pi(1-b^D aa^\pi)^{-1}b^D](a-b) = -b^D ba^\pi$.

So, it follows $x(a-b) = w^D(a-b) + a^D ab^\pi + b^D ba^\pi$ and $x(a-b) = (a-b)x$.

We now prove that $x(a-b)x = x$.

Let $(a-b)x = x_1 + x_2$ where $x_1 = w^D(a-b)$ and $x_2 = a^D ab^\pi + b^D ba^\pi$. Note that $wa^\pi = a^\pi w = 0$, $wb^\pi = b^\pi w = 0$ and $w^D(a-b) = (a-b)w^D$. By Eq.(1), we have

$$w^D(a-b) = w^D(w + (a-b)b^\pi + a^\pi(a-b) - a^\pi(a-b)b^\pi) = w^D w \tag{3}$$

Then we have $w^D x_1 = w^D$ and $w^D x_2 = w^D(aa^D b^\pi + bb^D a^\pi) = w^D b^\pi aa^D + w^D a^\pi bb^D = 0$.

Similarly, it is easy to get $(a^D(1-bb^\pi a^D)^{-1}b^\pi - a^\pi(1-b^D aa^\pi)^{-1}b^D)w^D = 0$, this shows that $(a^D(1-bb^\pi a^D)^{-1}b^\pi - a^\pi(1-b^D aa^\pi)^{-1}b^D)x_1 = 0$.

$$\begin{aligned}
& [a^D(1-bb^\pi a^D)^{-1}b^\pi - a^\pi(1-b^D aa^\pi)^{-1}b^D]x_2 \\
= \; & [a^D(1-bb^\pi a^D)^{-1}b^\pi - a^\pi(1-b^D aa^\pi)^{-1}b^D](a^D ab^\pi + b^D ba^\pi) \\
= \; & a^D(1-bb^\pi a^D)^{-1}b^\pi - a^\pi(1-b^D aa^\pi)^{-1}b^D.
\end{aligned}$$

So, we get $x(a-b)x = x$.

By Eq.(3), we have $(a-b)^2 w^D = w^2 w^D = w - ww^\pi$ and

$$\begin{aligned}
& (a-b)(aa^D b^\pi + bb^D a^\pi) \\
= \; & (a-b)((1-a^\pi)b^\pi + (1-b^\pi)a^\pi) \\
= \; & ab^\pi - ba^\pi + aa^\pi - bb^\pi - 2aa^\pi b^\pi + 2bb^\pi a^\pi.
\end{aligned}$$

Then by Eq.(1) and Eq.(2), we have

$$\begin{aligned}
& (a-b) - (a-b)^2 x \\
= \; & (a-b) - (a-b)(w^D(a-b) + a^D ab^\pi + b^D ba^\pi) \\
= \; & (a-b) - (w - ww^\pi + ab^\pi - ba^\pi + aa^\pi - bb^\pi - 2aa^\pi b^\pi + 2bb^\pi a^\pi) \\
= \; & (a-b) - [(a-b) - (a-b)b^\pi - a^\pi(a-b) + a^\pi(a-b)b^\pi - ww^\pi \\
& + ab^\pi - ba^\pi + aa^\pi - bb^\pi - 2aa^\pi b^\pi + 2bb^\pi a^\pi] \\
= \; & (a-b) - ((a-b) + bb^\pi a^\pi - aa^\pi b^\pi - ww^\pi) \\
= \; & -(bb^\pi a^\pi - aa^\pi b^\pi - ww^\pi).
\end{aligned}$$

Note that $(bb^\pi a^\pi - aa^\pi b^\pi)^k = (b-a)^k b^\pi a^\pi$ and $(b-a)^k = \sum_{i+j=k} \lambda_{i,j} b^j a^i$.

Let $k \geq 2\max\{s,t\}$. Then we have $(bb^\pi a^\pi - aa^\pi b^\pi)^k = 0$.

Since $(bb^\pi a^\pi - aa^\pi b^\pi)ww^\pi = ww^\pi(bb^\pi a^\pi - aa^\pi b^\pi) = 0$, we have $bb^\pi a^\pi - aa^\pi b^\pi - ww^\pi$ is nilpotent.

Hence, we get $(a-b)^D = w^D + a^D(1-bb^\pi a^D)^{-1}b^\pi - a^\pi(1-b^D aa^\pi)^{-1}b^D$.

For the " only if " part: Assume $(a-b) \in \mathcal{A}^D$. Since $(bb^D)^2 = bb^D$, $bb^D \in \mathcal{A}^D$. By Lemma 2.2 and $(a-b)bb^D = bb^D(a-b)$, we have $(a-b)bb^D \in \mathcal{A}^D$. Similarly, since $aa^D(a-b)bb^D = (a-b)bb^D aa^D$, we have $aa^D(a-b)bb^D \in \mathcal{A}^D$. □

3. Under the Condition $a^3b = ba$, $b^3a = ab$.

In [18], Liu et al. gave the explicit representations of $(a+b)^D$ of two complex matrices under the condition $a^3b = ba$ and $b^3a = ab$. In this section, we will extend the result to a ring R in which 2=1+1 is Drazin invertible for the unity 1.

Lemma 3.1. *Let $a, b \in R$ be such that $a^3b = ba$ and $b^3a = ab$. then for $i \in \mathbb{N}$*

(1) $ba^i = a^{3i}b$ *and* $b^ia = a^{3^i}b^i$.

(2) $ab^i = b^{3i}a$ *and* $a^ib = b^{3^i}a^i$.

(3) $ab = a^{26i}(ab)b^{2i}$ *and* $ba = b^{26i}(ba)a^{2i}$.

Proof. (1) By induction, it is easy to obtain (1) and (2).
(3) The proof is similar to [18, lemma 2.1]. □

Lemma 3.2. *Let $a, b \in R^D$ be such that $a^3b = ba$ and $b^3a = ab$. Then*
(1) $a^\pi ba^D = 0$ *and* $a^D ba^\pi = 0$.
(2) $b^\pi ab^D = 0$ *and* $b^D ab^\pi = 0$.

Proof. (1) By Lemma 3.1(1), there exists some $i \in \mathbb{N}$, such that $a^\pi ba^D = a^\pi ba^i(a^D)^{i+1} = a^\pi a^{3i}b(a^D)^{i+1} = 0$. Similarly, $a^D ba^\pi = 0$.
(2) It is analogous to the proof of (1). □

Corollary 3.3. *Let $a, b \in R^D$ be such that $a^3b = ba$ and $b^3a = ab$. Then*
(1) $(a^D)^3b = ba^D$ *and* $(b^D)^3a = ab^D$.
(2) aa^D *commutes with* b *and* b^D.
(3) bb^D *commutes with* a *and* a^D.
(4) $ab^D = b^Da^3$ *and* $ba^D = a^Db^3$.
(5) $a^Db^D = b^D(a^D)^3$ *and* $b^Da^D = a^D(b^D)^3$.
(6) $a^Db^D = b^Da^Db^2$ *and* $b^Da^D = a^Db^Da^2$.

Proof. (1) By hypothesis, $(a^D)^3baa^D = (a^D)^3a^3ba^D$. By Lemma 3.2 (1), $(a^D)^3b = ba^D$. Similarly, we have $(b^D)^3a = ab^D$.
(2) By hypothesis and (1), we get $baa^D = a^3ba^D = a^3(a^D)^3b = aa^Db$. Then $b^Daa^D = aa^Db^D$. (3) is analogous to the proof of (2).
(4) By (3), we get $b^Da^3 = b^Da^3bb^D = b^Dbab^D = ab^D$. Similarly, $a^Db = a^Db^3$.
(5) By (1) and (3), we have $b^D(a^D)^3 = b^D(a^D)^3bb^D = b^Dba^Db^D = a^Db^Dbb^D = a^Db^D$. Similarly, $a^D(b^D)^3 = b^Da^D$.
(6) By (5), we have $b^Da^Db^2 = a^D(b^D)^3b^2 = a^Db^D$. Similarly, $b^Da^D = a^Db^Da^2$. □

In Corollary 3.3 (5), one can see that $a^Db^D = b^D(a^D)^3$ and $b^Da^D = a^D(b^D)^3$. In the following, we will consider the analogous condition of $ab^3 = ba$ and $ba^3 = ab$.

Lemma 3.4. *Let $a, b \in R^D$ be such that $ab^3 = ba$ and $ba^3 = ab$. Then $a^Db^D = b^3a$ and $b^Da^D = a^3b$.*

Proof. Similar to Lemma 3.2 and Corollary 3.3, we have $a^Db = b(a^D)^3$ and $b^Da = a(b^D)^3$.
Then we can obtain that $aa^Db = ab(a^D)^3 = ba^3(a^D)^3 = ba^Da$ and $bb^Da = ab^Db$.
This implies that

$$\begin{aligned} a^3b^D &= a^3b(b^D)^2 = ba^9(b^D)^2 \\ &= b^Db^2a^9(b^D)^2 = b^Dba^3b(b^D)^2 \\ &= b^Dabb(b^D)^2 = b^Da(bb^D)^2 = b^Da. \end{aligned}$$

So, we get $a^Db^D = a^Db^Daa^D = a^Da(b^D)^3a^D = (b^D)^3a^D$ and $b^Da^D = (a^D)^3b^D$.
Similar to the proof of Lemma 3.1, we have $ab = a^{2i}(ab)b^{26i}$ for $i \geqslant \max\{ind(a), ind(b)\}$. Then it is easy to get

$$\begin{aligned} &abb^Da^D = bb^Daa^D = b^Dba^Da = b^Da^Dab, \\ &b^Da^Dabb^Da^D = b^Dbb^Da^Daa^D, \\ &(ab)^2b^Da^D = (ab)abb^Da^D = (ab)aa^Dbb^D = a^{2i}(ab)b^{26i}aa^Dbb^D = a^{2i}(ab)b^{26i} = ab. \end{aligned}$$

Then this implies that $(ab)^\# = b^Da^D$ and $(ba)^\# = a^Db^D$.
Hence, there exist $i \in \mathbb{N}$ such that

$$\begin{aligned} b^Da^D &= (ab)^\# = ((ab)^\#)^2ab = b^Da^Db^Da^Dba^3 = b^Da^Db^D(a^Da)b^3a^2 = b^Da^Db^Db^3a^2 \\ &= b^Da^D(b^Db)b^2a^2 = b^Da^Db^2a^2 = b^Da^Dbab^3a = b^Da^Dab^6a = b^Db^6a^2a^D \\ &= b^4(b^Db)(ba)(a^Da) = b^4(b^Db)b^{2i}(ba)a^{26i}(a^Da) = b^4b^{2i}(ba)a^{26i} = b^4ba \\ &= b^5a. \end{aligned}$$

and $a^Db^D = (ba)^\# = a^5b$.

So, we have $a^Db^D = (b^D)^3a^D = (b^D)^2b^Da^D = (b^D)^2b^5a = b^2(bb^D)ba = b^2(bb^D)b^{2i}baa^{26i} = b^3a$ and $b^Da^D = a^3b$. $\square$

Lemma 3.5. *Let $a, b \in R^D$ be such that $a^3b = ba$ and $b^3a = ab$. Then the following statements hold:*

(1) $a^Db^D = (b^D)^3a^D = b^Da^Da^2 = b^2b^Da^D$.

(2) $b^Da^D = (a^D)^3b^D = a^Db^Db^2 = a^2a^Db^D$.

Proof. Let $a^D = x \in R^\#$ and $b^D = y \in R^\#$. By Corollary 3.3, we have $xy^3 = yx$ and $yx^3 = xy$. Then by Lemma 3.4, it follows $x^\#y^\# = y^3x$ and $y^\#x^\# = x^3y$, that is, $a^2a^Db^2b^D = (b^D)^3a^D$.

Note that

$$a^2a^Db^2b^D = a^2a^Db^3(b^D)^2 = a^2ba^D(b^D)^2 = a^2(a^D)^3b(b^D)^2 = a^Db^D,$$

and

$$b^Da^Da^2 = b^Da^3(a^D)^2 = ab^D(a^D)^2 = aa^D(b^D)^3a^D = (b^D)^3a^Daa^D = (b^D)^3a^D.$$

So, we get $a^Db^D = (b^D)^3a^D = b^Da^Da^2$. Similarly, $b^Da^D = (a^D)^3b^D = a^Db^Db^2$.
Hence, by $b^Da^D = a^Db^Db^2$ and Corollary 3.3 (6), we have

$$b^2b^Da^D = b^Db(ba^D) = b^Dba^Db^3 = a^Db^Db^4 = b^Da^Db^2 = a^Db^D.$$

Similarly, $a^2a^Db^D = b^Da^D$. $\square$

Lemma 3.6. *Let $a, b \in R^D$ be such that $a^3b = ba$ and $b^3a = ab$. Then the following statements hold:*

(1) $aa^Da^{4+i}b^jbb^D = aa^Da^ib^jbb^D$.

(2) $aa^Da^{2+i}b^{2+j}bb^D = aa^Da^ib^jbb^D$, *where* $i, j \in \mathbb{N}$.

(3) $aa^Dabb^D = a^D(b^D)^2$.

(4) $aa^Da^3bb^D = a^Dbb^D$.

(5) $aa^Da^2bbb^D = aa^Db^D$.

(6) $aa^Dab^2bb^D = a^Dbb^D$.

(7) $aba^\pi = 0$ *and* $bab^\pi = 0$.

Proof. (1) By Lemma 3.5 (2), we have

$$aa^Da^4bb^D = aa^Dabab^D = aba^2a^Db^D = abb^Da^D = aa^Dbb^D.$$

Then we get $aa^Da^{4+i}b^jbb^D = aa^Da^ib^jbb^D$.

(2) Note that $a^2a^Db^2b^D = a^Db^D$, Then we have $aa^Da^2b^2bb^D = a(a^2a^Db^2b^D)b = aa^Dbb^D$. This implies that $aa^Da^{2+i}b^{2+j}bb^D = aa^Da^ib^jbb^D$.

(3) By Lemma 3.5 (2), we have $aa^Dabb^D = a^2a^Db^Db = b^Da^Db = a^D(b^D)^3b = a^D(b^D)^2$.

(4) $aa^Da^3bb^D = aa^Dbab^D = ba^2a^Db^D = bb^Da^D = a^Dbb^D$.

(5) In the proof of Lemma 3.5 (1), we get $a^2a^Db^2b^D = a^Db^D$. Then we have

$$aa^Da^2bbb^D = a(aa^Dabbb^D) = aa^Db^D.$$

(6) Similar to (5), we have $aa^Dab^2bb^D = (aa^Dabbb^D)b = a^Dbb^D$.

(7) For $k \geqslant \max\{ind(a), ind(b)\}$, we have $aba^\pi = a^\pi ab = a^\pi a^{26k}(ab)b^{2k} = 0$ and $bab^\pi = b^\pi ba = b^\pi b^{26k}(ba)a^{2k} = 0$. $\square$

Theorem 3.7. *Let $a, b \in R^D$ be such that $a^3b = ba$ and $b^3a = ab$. Suppose 2 is Drazin invertible. Then $a+b$ is Drazin invertible and*

$(a+b)^D = (2^D)^3bb^D(3a^3 + 3b^3 - a - b)aa^D + a^D(1 - bb^D) + (1 - aa^D)b^D$.

Proof. Firstly, let $M = M_1 + M_2 + M_3$, where $M_1 = (2^D)^3 bb^D(3a^3 + 3b^3 - a - b)aa^D$, $M_2 = a^D(1 - bb^D)$, $M_3 = (1 - aa^D)b^D$. In what follows, we show that M is the Drazin inverse of $a + b$, i.e. the following conditions hold: (a). $M(a + b) = (a + b)M$, (b). $M(a + b)M = M$ and (c). $(a + b) - (a + b)^2M$ is nilpotent.
For the condition (a), we will show that $(a + b)$ is communicate with M_1, M_2 and M_3. By Corollary 3.3 (2) and (3), we have

$$(a + b)M_1 = (2^D)^3 bb^D(a + b)(3a^3 + 3b^3 - a - b)aa^D$$

and

$$M_1(a + b) = (2^D)^3 bb^D(3a^3 + 3b^3 - a - b)(a + b)aa^D.$$

After a calculation we can obtain

$$\begin{aligned}(a + b)M_1 - M_1(a + b) &= (2^D)^3 bb^D(3ab^3 + 3ba^3 - 3ab - 3ba)aa^D \\ &= (2^D)^3 3(aa^Da^9bbb^D + aa^Db^9abb^D - aa^Dbabb^D - aa^Dabbb^D)\end{aligned}$$

From Lemma 3.6 (1), one can get $aa^Da^9bbb^D = aa^Dabbb^D$. Similar to Lemma 3.6(1), it is easy to check that $aa^Db^{4+i}a^jbb^D = aa^Db^ia^jbb^D$ for $j \in \mathbb{N}$. Then one can see that $aa^Db^9abb^D = aa^Dbabb^D$. This implies $M_1(a + b) = (a + b)M_1$.
Note that $abb^\pi = 0$, we get

$$\begin{aligned}(a + b)M_2 - M_2(a + b) &= (ba^D - a^Db)(1 - bb^D) \\ &= ((a^D)^3b - a^Db)(1 - bb^D) \\ &= ((a^D)^4 - (a^D)^2)ab(1 - bb^D) \\ &= 0.\end{aligned}$$

Similarly, $(a + b)M_3 - M_3(a + b) = ((b^D)^4 - (b^D)^2)ba(1 - aa^D) = 0$. This means that $(a + b)M = M(a + b)$.
(b) By Corollary 3.3 (2)(3), we get

$$M_1(a + b)M_2 = M_1(a + b)M_3 = 0,$$

$$M_2(a + b)M_1 = M_2(a + b)M_3 = 0,$$

$$M_3(a + b)M_1 = M_3(a + b)M_2 = 0.$$

By hypothesis and Lemma 3.6, we can simplify

$$\begin{aligned}M_1(a + b)M_1 &= (2^D)^3bb^D(3a^3 + 3b^3 - a - b)aa^D(a + b)(2^D)^3bb^D(3a^3 + 3b^3 - a - b)aa^D \\ &= (2^D)^6bb^D(3a^3 + 3b^3 - a - b)(a + b)(3a^3 + 3b^3 - a - b)aa^D \\ &= (2^D)^6bb^D(25a^3 + 25b^3 - ab^2 - a^2b - 8a - 8b)aa^D \\ &= (2^D)^6(25a^Dbb^D + 25b^Daa^D - a^Dbb^D - b^Daa^D - 8a^Db^Db^D - 8b^Da^Da^D) \\ &= (2^D)^6(24a^Dbb^D + 24b^Daa^D - 8a^Db^Db^D - 8b^Da^Da^D) \\ &= (2^D)^3bb^D(3a^3 + 3b^3 - a - b)aa^D.\end{aligned}$$

Note that $a^Dba^D(1 - bb^D) = a^D(1 - bb^D)baa^Da^D = 0$ and $b^Dab^D(1 - aa^D) = 0$. After a calculation, we obtain

$$\begin{aligned}M(a + b)M &= M_1(a + b)M_1 + M_2(a + b)M_2 + M_3(a + b)M_3 \\ &= M_1 + a^D(a + b)a^D(1 - bb^D) + b^D(a + b)b^D(1 - aa^D) \\ &= M_1 + (a^D + a^Dba^D)(1 - bb^D) + (b^Dab^D + b^D)(1 - aa^D) \\ &= M.\end{aligned}$$

(c) Note that $(aba^D + baa^D + bba^D)(1 - bb^D) = 0$ and $(aab^D + abb^D + bab^D)(1 - aa^D) = 0$.

Similar to the proof of (b), by Lemma 3.6, we have

$$\begin{aligned}
(a+b)^2M &= (a+b)[(2^D)^3bb^D(3a^4+3ab^3+3ba^3+3b^4-a^2-ab-ba-b^2)aa^D \\
&\quad +(aa^D+ba^D)(1-bb^D)+(1-aa^D)(ab^D+bb^D)] \\
&= (2^D)^3bb^D(3a^5+3a^2b^3+3aba^3+3ab^4+3ba^4+3bab^3+3a^2b^3+3b^5 \\
&\quad -a^3-a^2b-aba-ab^2-ba^2-bab-b^2a-b^3)aa^D+(a^2a^D+aba^D \\
&\quad +baa^D+bba^D)(1-bb^D)+(aab^D+abb^D+bab^D+bbb^D)(1-aa^D) \\
&= (2^D)^3(8a^Db^Db^D+8b^Da^Da^D)+a^2a^D(1-bb^D)+b^2b^D(1-aa^D) \\
&= (2^D)^3(8a^Db^Db^D+8b^Da^Da^D)+a^2a^D-a^Db^Db^D+b^2b^D-b^Da^Da^D \\
&= a^2a^D+b^2b^D-(1-22^D)(a^Db^Db^D+b^Da^Da^D).
\end{aligned}$$

Note that $a^Db^Db^D+b^Da^Da^D=aa^D(a+b)bb^D$ and

$$[(1-22^D)aa^D(a+b)bb^D]^4=2(1-22^D)aa^D(3+2a^3b+2ab+a^2)bb^D.$$

Since $aa^\pi bb^\pi=bb^\pi aa^\pi=0$ and $aa^\pi aa^D(a+b)bb^D=bb^\pi aa^D(a+b)bb^D=0$, it follows that $a+b-(a+b)^2M=aa^\pi+bb^\pi-(1-22^D)aa^D(a+b)bb^D$ is nilpotent. □

Example 3.8. *Suppose $S=\mathbb{Z}_8$ and $R=S_{2\times2}$. Set $a=\begin{pmatrix}4&0\\0&0\end{pmatrix}$ and $b=\begin{pmatrix}2&0\\0&3\end{pmatrix}$. By direct computation, we have $a^2=0$ and $b^3=b^D=\begin{pmatrix}0&0\\0&3\end{pmatrix}$. It is easy to check $a^3b=ba$ and $b^3a=ab$. Then by theorem 3.7, one can obtain that $(a+b)^D=\begin{pmatrix}0&0\\0&3\end{pmatrix}$.*

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