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On the Structure of the Pointwise Density Sets on the Real Line

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Abstract. The paper concerns some local properties of the sets with pointwise density points in terms of measure and category on the real line. We also construct nonmeasurable and not having the Baire property sets with pointwise density point.

1. Preliminaries

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{N}$ be the set of positive integers, $\mathbb{Q}$ denote the set of rational numbers and λ be the Lebesgue measure on $\mathbb{R}$. By λ^* and λ_* we shall denote the inner and outer Lebesgue measure on $\mathbb{R}$, respectively. Let $\mathcal{L}$ be the σ-algebra of Lebesgue measurable sets and $\mathcal{B}_a$ the σ-algebra of sets having the Baire property on $\mathbb{R}$. We say that a set has the Baire property if it is a symmetric difference between an open set and a set of the first category. Let $\mathbb{L}$ denote the σ-ideal of Lebesgue null sets, $\mathbb{K}$ denote the σ-ideal of the first category sets on the real line. Let $\mathcal{T}_{nat}$ be the natural topology on $\mathbb{R}$. If $A \subset \mathbb{R}$ and $\alpha, x \in \mathbb{R}$, then $\alpha A = \{\alpha a : a \in A\}$, $A - x = \{a - x : a \in A\}$ and A' denote the complement of A in $\mathbb{R}$. We shall denote the characteristic function of a set $A \subset \mathbb{R}$ by χ_A.

2. Introduction

The following equivalences are well known (cf. [5, p. 681])

$$\lim_{h\to 0^+} \frac{\lambda(A \cap [x_0 - h, x_0 + h])}{2h} = 1$$

$\Leftrightarrow$

$$\lim_{n\to\infty} \frac{\lambda(A \cap [x_0 - \frac{1}{n}, x_0 + \frac{1}{n}])}{\frac{2}{n}} = 1$$

$\Leftrightarrow$

a sequence of characteristic functions $\left\{\chi_{n(A-x_0)\cap[-1,1]}\right\}_{n\in\mathbb{N}}$
is convergent in measure λ to the function $\chi_{[-1,1]}$,

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for $A \in \mathcal{L}$ and $x_0 \in \mathbb{R}$ and they mean that x_0 is a density point of a set A.
The last characterisation of the Lebesgue density point become the motivation to introduce the concept of pointwise density point on the real line (denote briefly by p-density point).

Definition 2.1. *Let $A \subset \mathbb{R}$ and $x_0 \in \mathbb{R}$. We shall say that*

a) x_0 is a pointwise density point of a set A if the sequence $\left\{\chi_{n(A-x_0)\cap[-1,1]}\right\}_{n\in\mathbb{N}}$ is convergent everywhere to the function $\chi_{[-1,1]}$,

b) x_0 is a left-side pointwise density point of a set A if the sequence $\left\{\chi_{n(A-x_0)\cap[-1,0]}\right\}_{n\in\mathbb{N}}$ is convergent everywhere to the function $\chi_{[-1,0]}$,

c) x_0 is a right-side pointwise density point of a set A if the sequence $\left\{\chi_{n(A-x_0)\cap[0,1]}\right\}_{n\in\mathbb{N}}$ is convergent everywhere to the function $\chi_{[0,1]}$,

d) x_0 is a pointwise dispersion point (left-side pointwise dispersion point, right-side pointwise dispersion point) of a set A if x_0 is a pointwise density point (left-side pointwise density point, right-side pointwise density point) of a set A'.

Observation 2.2. *A point $x_0 \in \mathbb{R}$ is a pointwise density point of a set $A \subset \mathbb{R}$ if and only if $[-1, 1] \subset \liminf_{n\to\infty} n(A-x_0)$.*

Similarly

Observation 2.3. *A point $x_0 \in \mathbb{R}$ is a left-side (right-side) pointwise density point of a set $A \subset \mathbb{R}$ if and only if $[-1, 0] \subset \liminf_{n\to\infty} n(A - x_0)$ ($[0, 1] \subset \liminf_{n\to\infty} n(A - x_0)$).*

Let us define the operator of p-density point of a set $A \subset \mathbb{R}$ by

$$\Phi_p(A) = \{x \in \mathbb{R} : x \text{ is a pointwise density point of } A\}.$$

As the consequence of definition we have the following proposition.

Proposition 2.4. *Let A,$B \subset \mathbb{R}$ and $y \in \mathbb{R}$. Then*

1. $\Phi_p(\emptyset) = \emptyset$, $\Phi_p(\mathbb{R}) = \mathbb{R}$,
2. $\Phi_p(A \cap B) = \Phi_p(A) \cap \Phi_p(B)$,
3. $\Phi_p(A) \subset A$,
4. $\Phi_p(A) + y = \Phi_p(A + y)$,
5. $\Phi_p(A) = \emptyset$, *whenever* $A \in \mathbb{L}$ *or* $A \in \mathbb{K}$.

Recall that the set of density points of a measurable set $A \in \mathcal{L}$ is always a measurable set i.e. $\Phi_d(A) \in \mathcal{L}$ (cf. [5, p. 682]). The next theorem shows that the operator Φ_p is significantly different from the corresponding operator of Lebesgue density Φ_d.

Theorem 2.5 (cf. [2]). *There exists a set $A \in \mathcal{L}$ such that $\Phi_p(A) \notin \mathcal{L}$.*

The dual theorem for the sets with the Baire property is also true.

Theorem 2.6 (cf. [2]). *There exists a set $A \in \mathcal{B}_a$ such that $\Phi_p(A) \notin \mathcal{B}_a$.*

Although it is possible to find the sets from theorem 2.5, 2.6 it turns out that the family $\mathcal{T}_p = \{A \in \mathcal{L} : A \subset \Phi_p(A)\}$ forms topology containing $\mathcal{T}_{nat}$. The crucial properties of this topology are investigated in [2].

3. The Main Results

3.1. Local properties of sets with pointwise density points

Definition 3.1 (cf. [1]). *The sets of the form*

$$\bigcup_{i\in\mathbb{N}}(a_i, b_i) \text{ and } \bigcup_{i\in\mathbb{N}}[a_i, b_i]$$

are called a right-side open interval set and right-side closed interval set at a point $x_0 \in \mathbb{R}$*, respectively, if* $b_{i+1} < a_i < b_i$ *for* $i \in \mathbb{N}$ *and* $\lim_{i\to\infty} b_i = x_0$.

A left-side open interval set and left-side closed interval set at a point $x_0 \in \mathbb{R}$ is defined in the same way.

Definition 3.2 (cf. [1]). *The sets of the form*

$$\bigcup_{i\in\mathbb{N}}(a_i, b_i) \text{ and } \bigcup_{i\in\mathbb{N}}[a_i, b_i]$$

are called a left-side open interval set and left-side closed interval set at a point $x_0 \in \mathbb{R}$*, respectively, if* $a_i < b_i < a_{i+1}$ *for* $i \in \mathbb{N}$ *and* $\lim_{i\to\infty} a_i = x_0$.

The union of left-side or right-side open interval (closed interval) sets at the same point x_0 is called an open (closed) interval set at a point x_0. Notice that if $A = \bigcup_{i\in\mathbb{N}}[a_i, b_i]$ is a closed interval set at the point x_0, then $\operatorname{int}(A) = \bigcup_{i\in\mathbb{N}}(a_i, b_i)$ and $A \triangle \operatorname{int}(A) = A \setminus \operatorname{int}(A) = \{a_1, a_2, ...\} \cup \{b_1, b_2, ...\}$. Thus the results proved below for closed interval sets, $\bigcup_{i\in\mathbb{N}}[a_i, b_i]$, are also true for open interval sets, $\bigcup_{i\in\mathbb{N}}(a_i, b_i)$. In the sequel we call open and closed interval sets simply interval sets.

In the similar line of thought to the proof of Lemma 2 in [1] we can prove the below lemma.

Lemma 3.3. *If* $A \subset \mathbb{R}$ *is a right-side (left-side) interval set at* 0 *then* $\limsup_{n\to\infty} nA$ *is the residual set on* $(0, \infty)$ $((-\infty, 0))$.

By Lemma 3.3 and Observations 2.2, 2.3 we obtain the following proposition.

Proposition 3.4. *If* $A \subset \mathbb{R}$ *is an interval set (right-side interval, left-side interval) at* 0 *then* 0 *is neither pointwise dispersion point (right-side pointwise dispersion point, left-side pointwise dispersion point) of the set A nor pointwise density point (right-side pointwise density point, left-side pointwise density point) of the set* $A \cup \{0\}$.

Proof. To obtain a contradiction, suppose that there exists a right-side interval set $A \subset \mathbb{R}$ such that 0 is a p-dispersion point of this set. Hence $0 \in \Phi_p(A')$ and this means that

$$[0, 1] \subset \liminf_{n\to\infty} nA' = \mathbb{R} \setminus \limsup_{n\to\infty} nA.$$

Clearly, $\limsup_{n\to\infty} nA$ is not a residual set on $(0, \infty)$. This contradicts the Lemma 3.3. In the cases of the left-side interval and the interval set the proofs runs as before. □

Theorem 3.5. *If* 0 *is a pointwise density point of a set* $A \in \mathcal{B}_a$*, then there exists* $\varepsilon > 0$ *such that* $A \cap (-\varepsilon, \varepsilon)$ *is a residual set on the interval* $(-\varepsilon, \varepsilon)$.

Proof. Let $A \in \mathcal{B}_a$ be a set such that $0 \in \Phi_p(A)$. By contradiction, suppose the assertion of the Theorem 3.5 is false. There is no loss of generality if we assume that $A \cap (0, \varepsilon)$ is not a residual set on $(0, \varepsilon)$ for every $\varepsilon > 0$. Let $B = \bigcup_{i=1}^{\infty}[a_i, b_i]$ be right-side interval set at 0 such that $[a_i, b_i] \subset (0, \varepsilon)$ and $A \cap [a_i, b_i] \in \mathbb{K}$ for every $i \in \mathbb{N}$ (the existence of set B follows from the definition of the sets having the Baire property, residual and meager sets). Let $E = A \cap B$. Then $E \in \mathbb{K}$ and by Lemma 3.3 $\limsup_{k\to\infty} kB$ is a residual set on $(0, \infty)$ and

$$\limsup_{k\to\infty} k\,(A' \cap B) = \limsup_{k\to\infty} k\,(B \setminus E) =$$

$$= \limsup_{k\to\infty} (kB \setminus kE) \supset \left(\limsup_{k\to\infty} kB\right) \setminus \left(\bigcup_{k\in\mathbb{N}} kE\right).$$

It implies that $\limsup_{k\to\infty} k(A' \cap B)$, $\limsup_{k\to\infty} kA'$ are the residual sets on $(0,\infty)$. Hence

$$\liminf_{k\to\infty}(kA \cap (0,1)) = (0,1) \setminus \limsup_{k\to\infty}(kA' \cap (0,1)) \in \mathbb{K}.$$

This contradicts the fact that 0 is a pointwise density point of A. Similarly, there exists $\varepsilon > 0$ such that $A \cap (-\varepsilon, 0)$ is a residual set on $(-\varepsilon, 0)$. Finally, there exists $\varepsilon > 0$ such that $A \cap (-\varepsilon, \varepsilon)$ is a residual set on $(-\varepsilon, \varepsilon)$. □

The next theorem show us that analogue of Theorem 3.5 in terms of measure is not valid.

Theorem 3.6. *There exists a set $B \in \mathcal{L}$ such that $0 \in \Phi_p(B)$ and $B \cap (-\varepsilon, \varepsilon)$ is not a full measure set on an interval $(-\varepsilon, \varepsilon)$ for every $\varepsilon > 0$.*

Proof. Let $A = [-1,1] \setminus \bigcup_{n=2}^{\infty} \left[\frac{1}{n} - \frac{1}{n^4}, \frac{1}{n}\right]$ and A' be complement of set A with respect to the interval $[-1,1]$. Hence $A' = \bigcup_{n=2}^{\infty} \left[\frac{1}{n} - \frac{1}{n^4}, \frac{1}{n}\right]$. Let us fix $k \in \mathbb{N}$. Hence

$$\lambda((kA') \cap [-1,1]) = \lambda\left(k \bigcup_{n=k}^{\infty} \left[\frac{1}{n} - \frac{1}{n^4}, \frac{1}{n}\right]\right) =$$
$$= \sum_{n=k}^{\infty} \lambda\left(k\left[\frac{1}{n} - \frac{1}{n^4}, \frac{1}{n}\right]\right) = \sum_{n=k}^{\infty} \frac{k}{n^4}.$$

Thus

$$\sum_{k=1}^{\infty} \lambda((kA') \cap [-1,1]) = \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \frac{k}{n^4} =$$
$$= \sum_{n=1}^{\infty} \frac{n(n+1)}{2n^4} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^2} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^3} < \infty.$$

By the Borel-Cantelli lemma

$$\lambda\left(\limsup_{k\to\infty}((kA') \cap [-1,1])\right) = 0.$$

Let $E = \limsup_{k\to\infty}((kA') \cap [-1,1])$ and $D = \bigcup_{k=1}^{\infty}\left(\frac{1}{k}E\right)$. Then $\lambda(D) = 0$. Let $\varepsilon > 0$. It is easy to observe that $\lambda(A \cap (-\varepsilon, \varepsilon)) < 2\varepsilon$. Set $B = A \cup D$, then we get that $B \in \mathcal{L}$ and $0 \in \Phi_p(B)$. Certainly,

$$[-1,1] \setminus E = [-1,1] \setminus \limsup_{k\to\infty}((kA') \cap [-1,1]) =$$
$$= \liminf_{k\to\infty}((kA) \cap [-1,1]) \subset \liminf_{k\to\infty}((kB) \cap [-1,1]).$$

Moreover, if $x \in E$, then $\frac{x}{k} \in D \subset B$ for every $k \in \mathbb{N}$.
Hence $E \subset \liminf_{k\to\infty}((kB) \cap [-1,1])$. Finally, $[-1,1] \subset \liminf_{k\to\infty}((kB) \cap [-1,1])$. Thus $0 \in \Phi_p(B)$. □

Before we formulate a remark on the proof of Theorem 3.6 we recall the following definitions.

Definition 3.7 (cf. [6]). *We shall say that the sequence $\{\chi_{A_n}\}_{n\in\mathbb{N}}$ of characteristic functions of subsets of the interval $[-1,1]$ completely converges to $\chi_{[-1,1]}$ if an only if*

$$\sum_{n=1}^{\infty} \lambda([-1,1] \setminus A_n) < \infty.$$

Definition 3.8 (cf. [7]). *We shall say that a point $x \in \mathbb{R}$ is a complete density point of a set $A \in \mathcal{L}$ if and only if the sequence $\left\{\chi_{n(A-x_0)\cap[-1,1]}\right\}_{n\in\mathbb{N}}$ completely converges to the function $\chi_{[-1,1]}$.*

Remark 3.9. *Observe that in the proof of Theorem 3.6 we can use any interval set A, for which 0 is a complete density point.*

3.2. Nonmeasurable sets with pointwise density points

Definition 3.10 (cf. [4]). *We shall say that a set $H \subset \mathbb{R}$ is a Burstin set if and only if H is both a Bernstein set and a Hamel base.*

It is well known that in ZFC there always exists a Burstin set (cf. [3], [4]).

Definition 3.11 (cf. [3]). *A set $A \subset \mathbb{R}$ is a saturated nonmeasurable set if*

$$\lambda_*(A) = \lambda_*(A') = 0.$$

Clearly, every Burstin set is saturated nonmeasurable set. Recall the following characterisation.

Theorem 3.12 (cf. [3]). *A set $A \subset \mathbb{R}$ is saturated nonmeasurable set if and only if $\lambda^*(A \cap (a,b)) = \lambda^*(A' \cap (a,b)) = b - a$ for every interval $(a,b) \subset \mathbb{R}$.*

As a simple exercise one can find the following proposition.

Proposition 3.13. *If the sets $A_k \subset \mathbb{R}$, $k \in \mathbb{N}$ are pairwise disjoint and are saturated nonmeasurable, then for every $n \in \mathbb{N}$ the set $\bigcup_{k=1}^{n} A_k$ is a saturated nonmeasurable.*

Theorem 3.14. *There exists a set $E \subset \mathbb{R}$ such that $E \notin \mathcal{L}$, $\lambda_*(E) = 0$ and $0 \in \Phi_p(E)$.*

Proof. Let $H \subset \mathbb{R}$ be a Burstin set. For every $n \in \mathbb{N}$ we define

$$A_n = \{x \in \mathbb{R} : x = \sum_{i=1}^{n} q_i h_i, q_i \in \mathbb{Q}\backslash\{0\}, h_i \in H, 1 \le i \le n\}.$$

The sets of the sequence $\{A_n\}_{n\in\mathbb{N}}$ are pairwise disjoint, $\bigcup_{n=1}^{\infty} A_n = \mathbb{R}\backslash\{0\}$ and $H \subset A_1$. We prove that A_n is a saturated nonmeasurable set for every $n \in \mathbb{N}$. Let $h_1, h_2, ..., h_{n-1}$ be mutually different elements of H. If $H_n = H\backslash\{h_1, h_2, ..., h_{n-1}\}$, then $\lambda_*(H_n) = \lambda_*(H_n') = 0$. Moreover, $H_n + (h_1 + h_2 + ... + h_{n-1}) \subset A_n$ is a saturated nonmeasurable set. Hence, by Theorem 3.12 we have that for every interval $(a,b) \subset \mathbb{R}$

$$\lambda^*(A_n \cap (a,b)) \ge \lambda^*(H_n + (h_1 + h_2 + ... + h_{n-1}) \cap (a,b)) = b - a.$$

Since $A_n' \supset A_{n+1}$, Theorem 3.12 now leads to

$$\lambda^*(A_n' \cap (a,b)) \ge \lambda^*(A_{n+1} \cap (a,b)) = b - a.$$

In that way we conclude that A_n is a saturated nonmeasurable set and by Proposition 3.13 the set $\bigcup_{k=1}^{n} A_k$ is a saturated nonmeasurable for every $n \in \mathbb{N}$. Let

$$E = \bigcup_{n=1}^{\infty} \left(A_n \cap \left(-\frac{1}{n}, \frac{1}{n}\right)\right) \cup \{0\}.$$

Clearly,

$$E \cap \left(\left(-\frac{1}{n}, -\frac{1}{n+1}\right] \cup \left[\frac{1}{n+1}, \frac{1}{n}\right)\right) = \left(\left(-\frac{1}{n}, -\frac{1}{n+1}\right] \cup \left[\frac{1}{n+1}, \frac{1}{n}\right)\right) \cap \bigcup_{k=1}^{n} A_k.$$

Then,

$$\begin{aligned}\lambda_*(E) = \lambda_*(E \cap [-1,1]) &= \lambda_*\left(E \cap \bigcup_{n=1}^{\infty}\left(\left(-\frac{1}{n}, -\frac{1}{n+1}\right] \cup \left[\frac{1}{n+1}, \frac{1}{n}\right)\right)\right) \\ &= \sum_{n=1}^{\infty} \lambda_*\left(E \cap \left(\left(-\frac{1}{n}, -\frac{1}{n+1}\right] \cup \left[\frac{1}{n+1}, \frac{1}{n}\right)\right)\right) \\ &= \sum_{n=1}^{\infty} \lambda_*\left(\left(\left(-\frac{1}{n}, -\frac{1}{n+1}\right] \cup \left[\frac{1}{n+1}, \frac{1}{n}\right)\right) \cap \bigcup_{k=1}^{n} A_k\right) = 0.\end{aligned}$$

We shall prove that $0 \in \Phi_p(E)$. Let $x \in [-1,1]\backslash\{0\}$, then there exists a unique $n_0 \in \mathbb{N}$ such that $x \in A_{n_0}$. By definition of the set A_{n_0}, we conclude that $qx \in A_{n_0}$ for every $q \in \mathbb{Q}\backslash\{0\}$. Then $[-1,1] \setminus \{0\} \subset \liminf_{n\to\infty} nE$ and $0 \in E$. It implies the $0 \in \Phi_p(E)$ and obviously $E \notin \mathcal{L}$. □

Definition 3.15 (cf. [3]). *A set $A \subset \mathbb{R}$ has $(*)$ property, if for every $B \in \mathcal{B}_a$ such that $B \subset A$ or $B \subset A'$ one has $B \in \mathbb{K}$.*

It is easy to get the following proposition.

Proposition 3.16. *If the sets $A_k \subset \mathbb{R}$, $k \in \mathbb{N}$ are pairwise disjoint and have $(*)$ property, then for every $n \in \mathbb{N}$ the set $\bigcup_{k=1}^{n} A_k$ has $(*)$ property.*

Theorem 3.17 (cf. [3]). *If a set $A \subset \mathbb{R}$ has $(*)$ property and $B \in \mathcal{B}_a \setminus \mathbb{K}$, then $A \cap B \notin \mathcal{B}_a$.*

Theorem 3.18 (cf. [3]). *Every Burstin set has $(*)$ property.*

The analogue of Theorem 3.14 in terms of category is the following theorem.

Theorem 3.19. *There exists a set $E \subset \mathbb{R}$ such that $E \notin \mathcal{B}_a$, $0 \in \Phi_p(E)$ and $E \cap (-\varepsilon, \varepsilon)$ is not a residual set on an interval $(-\varepsilon, \varepsilon)$ for an arbitrary $\varepsilon > 0$.*

Proof. Let $H \subset \mathbb{R}$ be a Burstin set. For every $n \in \mathbb{N}$ let us denote

$$A_n = \{x \in \mathbb{R} : x = \sum_{i=1}^{n} q_i h_i, q_i \in \mathbb{Q}\backslash\{0\}, h_i \in H, 1 \le i \le n\}.$$

The sets of the sequence $\{A_n\}_{n\in\mathbb{N}}$ are pairwise disjoint, $\bigcup_{n=1}^{\infty} A_n = \mathbb{R}\backslash\{0\}$ and $H \subset A_1$. For every $n \in \mathbb{N}$ we show that, if $B \subset A'_n$ and $B \in \mathcal{B}_a$, then $B \in \mathbb{K}$. Let $h_1, h_2, ..., h_{n-1} \in H$ be mutually different elements of H. Let $H_n = H\backslash\{h_1, h_2, ..., h_{n-1}\}$. Then H_n and H'_n have $(*)$ property. The set $H_n + (h_1 + h_2 + ... + h_{n-1})$ has also $(*)$ property and $H_n + (h_1 + h_2 + ... + h_{n-1}) \subset A_n$. Hence $B \in \mathcal{B}_a$ and $B \subset A'_n \subset (H_n + (h_1 + h_2 + ... + h_{n-1}))'$. Thus $B \in \mathbb{K}$. We now show that for every $n \in \mathbb{N}$, if $B \subset A_n$ and $B \in \mathcal{B}_a$, then $B \in \mathbb{K}$. First observe that $A_n \subset A'_{n+1}$. Then, if $B \subset A_n$ and $B \in \mathcal{B}_a$, then by the previous part of proof, we get that $B \in \mathbb{K}$. Therefore, we have that A_n has $(*)$ property for every $n \in \mathbb{N}$. Define

$$E = \bigcup_{n=1}^{\infty}\left(A_n \cap \left(-\frac{1}{n}, \frac{1}{n}\right)\right) \cup \{0\}.$$

By proof of Theorem 3.14, $0 \in \Phi_p(E)$. We prove that $E \cap (-\varepsilon, \varepsilon)$ is not a residual set on an interval $(-\varepsilon, \varepsilon)$ for every $\varepsilon > 0$. Suppose contrary to our claim that there exists $\varepsilon > 0$ such that $E \cap (-\varepsilon, \varepsilon)$ is a residual set on interval $(-\varepsilon, \varepsilon)$. Evidently $0 \in \Phi_p(E \cap (-\varepsilon, \varepsilon))$ and $E \cap (-\varepsilon, \varepsilon) \in \mathcal{B}_a$. By Theorem 3.5 there exists $\delta > 0$ such that $E \cap (-\varepsilon, \varepsilon) \cap (-\delta, \delta)$ is a residual set on the open interval $(-\delta, \delta)$. Let $n_0 \in \mathbb{N}$ be a such that

$\frac{1}{n_0} < \min(\delta, \varepsilon)$. Then $E \cap \left(-\frac{1}{n_0}, \frac{1}{n_0}\right)$ is a residual set on $\left(-\frac{1}{n_0}, \frac{1}{n_0}\right)$ and thus $E \cap \left(-\frac{1}{n_0}, \frac{1}{n_0}\right) \in \mathcal{B}_a$. It implies that $E \cap \left(-\frac{1}{n_0+1}, \frac{1}{n_0+1}\right) \in \mathcal{B}_a$. We thus get

$$D = \left(E \cap \left(-\frac{1}{n_0}, \frac{1}{n_0}\right)\right) \setminus \left(E \cap \left(-\frac{1}{n_0+1}, \frac{1}{n_0+1}\right)\right) \in \mathcal{B}_a.$$

At the same time we have

$$D = \left(\left(-\frac{1}{n_0}, -\frac{1}{n_0+1}\right] \cup \left[\frac{1}{n_0+1}, \frac{1}{n_0}\right)\right) \cap \bigcup_{n=1}^{n_0} A_n.$$

By Proposition 3.16 the set $\bigcup_{n=1}^{n_0} A_n$ has $(*)$ property and we obtain by Theorem 3.17 that $D \notin \mathcal{B}_a$. This contradiction means that $E \cap (-\varepsilon, \varepsilon)$ is not a residual set on interval $(-\varepsilon, \varepsilon)$ for every $\varepsilon > 0$. Finally by Theorem 3.5 we have that $E \notin \mathcal{B}_a$. $\square$

References

[1] M. Filipczak, Some remarks on Foran's sequence conditions, *Tatra Mt. Math. Publ.*, 8, (1996), 143–146.
[2] M. Górajska, Pointwise density topology, Open Mathematics, vol. 13, no. 1, (2015), 75–82.
[3] M. Kuczma, *An introduction to the theory of functional equation and inequalities*, 2en ed., Birkhäuser Verlag AG, Basel-Boston-Berlin, 2009.
[4] A. Nowik, Vitali set can be homeomorphic to its complement, *Acta Math. Hungar.*, vol. 115, no. 1-2, (2007), 145–154.
[5] W. Wilczyński, Density topologies, Chapter 15 in *Handbook of measure theory*, vol. I, North Holland, Amsterdam, 2002, 675–702.
[6] W. Wilczyński, Density topologies, *Scientific Bulletyn Of Chełm Section Of Mathematics And Computer Science*, 1, (2007), 223–227.
[7] W. Wilczyński, W. Wojdowski, Complete density topology, *Indag.Math.*, vol.18, issue 2, (2007), 295–303.