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On the Rates of Convergence of the q-Lupaş-Stancu Operators

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Abstract. We introduce a Stancu type generalization of the Lupaş operators based on the q-integers, rate of convergence of this modification are obtained by means of the modulus of continuity, Lipschitz class functions and Peetre's K-functional. We will also introduce r-th order generalization of these operators and obtain its statistical approximation properties.

1. Introduction

Firstly, we give some definitions about q-integers. For any non-negative integer r, the q-integer of the number r is defined by

$$[r]_q := [r] = \begin{cases} \frac{1-q^r}{1-q} & if\ q \neq 1 \\ r & if\ q = 1. \end{cases}$$

The q-factorial is defined as

$$[r]! = \begin{cases} [1]\,[2]\ldots[r] & \text{if } k = 1, 2, \ldots \\ 1 & \text{if } k = 0 \end{cases}$$

and the q-binomial coefficient is defined as

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{[n]!}{[r]!\,[n-r]!}$$

$(r, n \in \mathbb{N})$ for $q \in (0,1]$. It is obvious that q-binomial coefficient reduce to the ordinary case when $q = 1$. Details on q-integers can be found in [2], [10], [12], [18], [19], [16] and [14].

The q-analogue of the classical Bernstein operators [3] is defined by Lupaş [15] as follows:

$$R_{n,q}(f;x) = \sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right) b_{n,k}(q;x) \tag{1}$$

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($f \in C\,[0,1]$ and $x \in [0,1]$) where

$$b_{n,k}(q;x) = \frac{1}{\prod\limits_{s=0}^{n-1}(1-x+q^s x)} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} x^k (1-x)^{n-k}. \tag{2}$$

In [15], Lupaş proved the following Lemma.

Lemma 1.1. *Let $e_i(x) = x^i$, $(i = 0,1,2)$. Then we have*

$$\begin{aligned} R_{n,q}(e_0;x) &= 1, & (3)\\ R_{n,q}(e_1;x) &= x, & (4)\\ R_{n,q}(e_2;x) &= x^2 + \frac{x(1-x)}{n}\left(\frac{1-x+q^n x}{1-x+qx}\right). & (5) \end{aligned}$$

Stancu type generalization of linear positive operators has been studied in several years (for instance see [11]). Now, we introduce the Stancu type generalization of the Lupaş operators based on q-integers as

$$R_{n,q}^{\alpha,\beta}(f;x) = \sum_{k=0}^{n} f\left(\frac{[k]+[\alpha]}{[n]+[\beta]}\right) b_{n,k}(q;x) \tag{6}$$

where $0 < \alpha < \beta$ and $b_{n,k}(q;x)$ is given by (2).

We give some equalities for operators (6) in the following lemma.

Lemma 1.2. *Let $e_i(x) = x^i$, $(i = 0,1,2)$. The following equalities are true:*

$$\begin{aligned} R_{n,q}^{\alpha,\beta}(e_0;x) &= 1 & (7)\\ R_{n,q}^{\alpha,\beta}(e_1;x) &= \tfrac{[n]x+[\alpha]}{[n]+[\beta]} & (8)\\ R_{n,q}^{\alpha,\beta}(e_2;x) &= \left(\tfrac{[n]}{[n]+[\beta]}\right)^2 \left\{x^2 + \frac{x(1-x)}{n}\left(\frac{1-x+q^n x}{1-x+qx}\right)\right\} \\ &\quad + \tfrac{2[\alpha][n]}{([n]+[\beta])^2}x + \left(\tfrac{[\alpha]}{[n]+[\beta]}\right)^2. & (9) \end{aligned}$$

Proof. From (6), for the case $f(s) = e_0(s)$, we can easily get the equality (7).

If we take $f(s) = e_1(s)$ in operators (6), then we have

$$\begin{aligned} R_{n,q}^{\alpha,\beta}(e_1(s);x) &= \sum_{k=0}^{n} \frac{[k]+[\alpha]}{[n]+[\beta]} b_{n,k}(q;x) \\ &= \tfrac{[n]}{[n]+[\beta]} R_{n,q}(e_1;x) + \tfrac{[\alpha]}{[n]+[\beta]} R_{n,q}(e_0;x). \end{aligned}$$

So, from the equalities (3) and (4), we obtain (8).

Now, we take $f(s) = e_2(s)$ in operators (6), we get

$$\begin{aligned} R_{n,q}^{\alpha,\beta}(e_2(s);x) &= \sum_{k=0}^{n} \left(\frac{[k]+[\alpha]}{[n]+[\beta]}\right)^2 b_{n,k}(q;x) \\ &= \left(\tfrac{[n]}{[n]+[\beta]}\right)^2 R_{n,q}(e_2;x) + \tfrac{2[\alpha]}{([n]+[\beta])^2} R_{n,q}(e_1;x) + \left(\tfrac{[\alpha]}{[n]+[\beta]}\right)^2 R_{n,q}(e_0;x). \end{aligned}$$

So, from the equalities (3), (4) and (5), we have (9). □

In the light of the Lemma 2, we can give the following theorem for the convergence of $R_{n,q}^{\alpha,\beta}$ operators.

Theorem 1.3. *Let $f \in C[0,1]$ and (q_n) be a sequence, $0 < q_n \leq 1$, satisfying the following expressions:*

$$\lim_n q_n = 1 \text{ and } \lim_n q_n^n = c \text{ (c is a constant)}.$$

Then we have

$$\lim_n \left| R_{n,q}^{\alpha,\beta}(f;x) - f(x) \right| = 0.$$

Proof. From Lemma 2 and Korovkin's theorem, the proof is obvious. □

2. The Rates of Convergence

In this section, we compute the rates of convergence of the operators $R_{n,q}^{\alpha,\beta}$ to the function f by means of modulus continuity, elements of Lipschitz class and Peetre's K-functional.

Let $f \in C[0,1]$. The modulus of continuity of f denotes by $\omega(f,\delta)$, is defined to be

$$\omega(f,\delta) = \sup_{\substack{y,x \in [0,b] \\ |y-x|<\delta}} \left| f(y) - f(x) \right|.$$

It is well known that a necessary and sufficient condition for a function $f \in C[0,1]$ is

$$\lim_{\delta \to 0} \omega(f,\delta) = 0.$$

It is also well known that for any $\delta > 0$ and each $y \in [0,1]$

$$\left| f(y) - f(x) \right| \leq \omega(f,\delta) \left(1 + \frac{|y-x|}{\delta} \right). \tag{10}$$

Recall that, in [15], for every $f \in C[0,1]$ and $\delta > 0$ Lupaş obtained the following rate of convergence for the operators (1).

$$\left| R_{n,q}(f;x) - f(x) \right| \leq \omega(f,\delta) \left\{ 1 + \frac{1}{\delta} \sqrt{\frac{x(1-x)}{[n]}} \right\}. \tag{11}$$

Theorem 2.1. *Let (q_n) be a sequence, $0 < q_n \leq 1$, satisfying the following conditions:*

$$\lim_n q_n = 1 \text{ and } \lim_n q_n^n = c \text{ (c is a constant)}. \tag{12}$$

For $f \in C[0,1]$ and $\delta_n > 0$, we have

$$\left\| R_{n,q}^{\alpha,\beta}(f;x) - f(x) \right\|_{C[0,1]} \leq 2\omega(f,\delta_n)$$

where

$$\delta_n = \left(\left(\frac{[\alpha]+[\beta]}{[n]+[\beta]}\right)^2 + \frac{[n]}{([n]+[\beta])^2} \right)^{1/2}.$$

Proof. From (7), (8) and (9), we have

$$\begin{aligned} R_{n,q}^{\alpha,\beta}((t-x)^2;x) &= \left(\frac{[\beta]}{[n]+[\beta]}\right)^2 x^2 - \frac{2[\alpha][\beta]}{([n]+[\beta])^2} x \\ &\quad + \frac{[n]x(1-x)}{([n]+[\beta])^2} \left(\frac{1-x+q^n x}{1-x+qx}\right) + \left(\frac{[\alpha]}{[n]+[\beta]}\right)^2. \end{aligned} \tag{13}$$

Here one can observe that

$$\max_{x\in[0,1]} \tfrac{1-x+q^n x}{1-x+qx} = 1 \tag{14}$$

and

$$\max_{x\in[0,1]} x(1-x) = \frac{1}{4}. \tag{15}$$

By using (13), (14) and (15), we get

$$\max_{x\in[0,1]} R_{n,q}^{\alpha,\beta}((t-x)^2;x) \leq (\tfrac{[\alpha]+[\beta]}{[n]+[\beta]})^2 + \tfrac{[n]}{([n]+[\beta])^2}. \tag{16}$$

For $x \in [0,1]$, If we take the maximum of both side of the following inequality

$$\left|R_{n,q}^{\alpha,\beta}(f;x) - f(x)\right| \leq \omega(f,\delta)\left\{1 + \frac{1}{\delta}\left(R_{n,q}^{\alpha,\beta}((t-x)^2;x)\right)^{1/2}\right\},$$

then we get

$$\begin{aligned} &\left\|R_{n,q}^{\alpha,\beta}(f;x) - f(x)\right\|_{C[0,1]} \\ \leq\ & \omega(f,\delta)\left\{1 + \frac{1}{\delta}\left(\max_{x\in[0,1]} R_{n,q}^{\alpha,\beta}((t-x)^2;x)\right)^{1/2}\right\} \\ \leq\ & \omega(f,\delta)\left\{1 + \frac{1}{\delta}\left((\tfrac{[\alpha]+[\beta]}{[n]+[\beta]})^2 + \tfrac{[n]}{([n]+[\beta])^2}\right)^{1/2}\right\}. \end{aligned}$$

If we choose

$$\delta_n = \left((\tfrac{[\alpha]+[\beta]}{[n]+[\beta]})^2 + \tfrac{[n]}{([n]+[\beta])^2}\right)^{1/2} \tag{17}$$

then we have

$$\left\|R_{n,q}^{\alpha,\beta}(f;x) - f(x)\right\|_{C[0,1]} \leq 2\omega(f,\delta_n).$$

So we have the desired result. □

Now, we compute the approximation order of operator $R_{n,q}^{\alpha,\beta}$ in term of the elements of the usual Lipschitz class.

Let $f \in C[0,1]$ and $0 < \alpha \leq 1$. We recall that f belongs to $Lip_M(\rho)$ if the inequality

$$\left|f(x) - f(y)\right| \leq M\left|x-y\right|^{\rho};\ \forall x,y \in [0,1] \tag{18}$$

holds.

Theorem 2.2. *For all $f \in Lip_M(\rho)$, we have*

$$\left\|R_{n,q}^{\alpha,\beta}(f;x) - f(x)\right\|_{C[0,1]} \leq M\delta_n^{\rho}$$

where

$$\delta_n = \left((\tfrac{[\alpha]+[\beta]}{[n]+[\beta]})^2 + \tfrac{[n]}{([n]+[\beta])^2}\right)^{1/2}$$

and M is a positive constant.

Proof. Let $f \in Lip_M(\rho)$ and $0 < \rho \leq 1$. By (18) and linearity and monotonicity of $R_{n,q}^{\alpha,\beta}$ then we have

$$\begin{aligned} \left| R_{n,q}^{\alpha,\beta}(f;x) - f(x) \right| &\leq R_{n,q}^{\alpha,\beta}\left(\left| f(t) - f(x) \right| ; x \right) \\ &\leq M R_{n,q}^{\alpha,\beta}\left(|t-x|^{\rho} ; x \right). \end{aligned}$$

Applying the Hölder inequality with $m = \frac{2}{\rho}$ and $n = \frac{2}{2-\rho}$, we get

$$\left| R_{n,q}^{\alpha,\beta}(f;x) - f(x) \right| \leq M \left(R_{n,q}^{\alpha,\beta}((t-x)^2;x) \right)^{\rho/2}. \tag{19}$$

For $x \in [0,1]$, if we take the maximum of both side of (19) then we have

$$\left\| R_{n,q}^{\alpha,\beta}(f;x) - f(x) \right\|_{C[0,1]} \leq M \left(\max_x R_{n,q}^{\alpha,\beta}((t-x)^2;x) \right)^{\rho/2}.$$

If we use (13) and choose $\delta = \delta_n$ as in (17), then proof is completed. □

Finally, we will study the rate of convergence of the positive linear operators $R_{n,q}^{\alpha,\beta}$ by means of the Peetre's K-functionals.

First of all, we recall the definition of $R_{n,q}^{\alpha,\beta}$.

$C^2[0,1]$: The space of those functions f for which $f, f', f'' \in C[0,1]$. We recall the following norm in the space $C^2[0,1]$:

$$\|f\|_{C^2[0,1]} = \|f\|_{C[0,1]} + \|f'\|_{C[0,1]} + \|f''\|_{C[0,1]}.$$

We consider the following Peetre's K-functional

$$K(f,\delta) := \inf_{g \in C^2[0,1]} \left\{ \|f-g\|_{C[0,1]} + \delta \|g\|_{C^2[0,1]} \right\}.$$

Theorem 2.3. *Let $f \in C[0,1]$. Then we have*

$$\left\| R_{n,q}^{\alpha,\beta}(f;x) - f(x) \right\|_{C[0,1]} \leq 2K(f,\delta_n)$$

where $K(f,\delta_n)$ is Peetre's K-functional and

$$\delta_n = \frac{1}{2} \frac{[\alpha]+[\beta]}{[n]+[\beta]} + \frac{1}{4} \left(\frac{[\alpha]+[\beta]}{[n]+[\beta]} \right)^2 + \frac{[n]}{4([n]+[\beta])^2}.$$

Proof. Let $g \in C^2[0,1]$. If we use the Taylor expansion of the function g at $s = x$, we have

$$g(s) = g(x) + (s-x)g'(x) + \frac{(s-x)^2}{2!} g''(x).$$

Hence, we get

$$\begin{aligned} \left\| R_{n,q}^{\alpha,\beta}(g;x) - g(x) \right\|_{C[0,1]} &\leq \left\| R_{n,q}^{\alpha,\beta}((s-x);x) \right\|_{C[0,1]} \|g(x)\|_{C^2[0,1]} \\ &\quad + \frac{1}{2} \left\| R_{n,q}^{\alpha,\beta}((s-x)^2;x) \right\|_{C[0,1]} \|g(x)\|_{C^2[0,1]}. \end{aligned} \tag{20}$$

From the equality (8), we have

$$\left\| R_{n,q}^{\alpha,\beta}((s-x);x) \right\|_{C[0,1]} \leq \frac{[\alpha]+[\beta]}{[n]+[\beta]}. \tag{21}$$

So if we use (16) and (21) in (20), then we get

$$\left\|R_{n,q}^{\alpha,\beta}(g;x)-g\left(x\right)\right\|_{C[0,1]}\leq\left[\tfrac{1}{2}(\tfrac{[\alpha]+[\beta]}{[n]+[\beta]})^{2}+\tfrac{1}{2}\tfrac{[n]}{([n]+[\beta])^{2}}\ +\tfrac{[\alpha]+[\beta]}{[n]+[\beta]}\right]\left\|g(x)\right\|_{C^{2}[0,1]}. \tag{22}$$

On the other hand, we can write

$$\begin{aligned}\left|R_{n,q}^{\alpha,\beta}(f;x)-f\left(x\right)\right| &\leq \left|R_{n,q}^{\alpha,\beta}(f-g;x)\right|+\left|R_{n,q}^{\alpha,\beta}(g;x)-g\left(x\right)\right|\\ &\quad+\left|f(x)-g\left(x\right)\right|.\end{aligned}$$

If we take the maximum on $[0,1]$, we have

$$\left\|R_{n,q}^{\alpha,\beta}(f;x)-f\left(x\right)\right\|_{C[0,1]}\leq 2\left\|f-g\right\|_{C[0,1]}+\left\|R_{n,q}^{\alpha,\beta}(g;x)-g\left(x\right)\right\|_{C[0,1]}. \tag{23}$$

If we consider (22) in (23), we obtain

$$\begin{aligned}\left\|R_{n,q}^{\alpha,\beta}(f;x)-f\left(x\right)\right\|_{C[0,1]} &\leq 2\left\{\left\|f-g\right\|_{C[0,1]}+\left[\tfrac{1}{4}(\tfrac{[\alpha]+[\beta]}{[n]+[\beta]})^{2}+\tfrac{1}{4}\tfrac{[n]}{([n]+[\beta])^{2}}\right.\right.\\ &\quad\left.\left.+\tfrac{1}{2}\tfrac{[\alpha]+[\beta]}{[n]+[\beta]}\right]\left\|g(x)\right\|_{C^{2}[0,1]}\right\}.\end{aligned}$$

If we choose

$$\delta_{n}=\tfrac{1}{2}\tfrac{[\alpha]+[\beta]}{[n]+[\beta]}+\tfrac{1}{4}(\tfrac{[\alpha]+[\beta]}{[n]+[\beta]})^{2}+\tfrac{1}{4}\tfrac{[n]}{([n]+[\beta])^{2}},$$

then we get

$$\left\|R_{n,q}^{\alpha,\beta}(f;x)-f\left(x\right)\right\|_{C[0,1]}\leq 2\left\{\left\|f-g\right\|_{C[0,1]}+\delta_{n}\left\|g(x)\right\|_{C^{2}[0,1]}\right\}.$$

Finally, one can observe that if we take the infimum of both side above inequality for the function $g\in C^{2}\left[0,1\right]$, we can find

$$\left\|R_{n,q}^{\alpha,\beta}(f;x)-f\left(x\right)\right\|_{C[0,1]}\leq 2K(f,\delta_{n}).$$

□

3. The $r-th$ Order Generalization of the Operators $R_{n,q}^{\alpha,\beta}$

By $C^{r}\left[0,1\right]$ $(r=0,1,2,...)$ we denote the set of functions f having continuous r-th derivatives f^{r} $(f^{0}(x)=f(x))$ on the segment $[0,1]$ (see [4] and [13]).

We consider the following generalization of the positive linear operators $R_{n,q}^{\alpha,\beta}$ defined by (6).

$$R_{n,q,r}^{\alpha,\beta}(f;x)=\sum_{k=0}^{n}\left[\sum_{i=0}^{r}f^{(i)}\left(\tfrac{[k]+[\alpha]}{[n]+[\beta]}\right)\frac{(x-\frac{[k]+[\alpha]}{[n]+[\beta]})^{i}}{i!}\right]b_{n,k}(q;x) \tag{24}$$

where $b_{n,k}(q;x)$ is given by (2), $f\in C^{r}\left[0,1\right]$ $(r=0,1,2,...)$ and $n\in\mathbb{N}$. We call the operators (24) the r-th order of the operators $R_{n,q}^{\alpha,\beta}$. Taking $r=0$, we get the sequence $R_{n,q}^{\alpha,\beta}$ defined by (6).

Theorem 3.1. *Let $f^{(r)}\in Lip_{M}(\alpha)$ and $f\in C^{r}\left[0,1\right]$. We have*

$$\left\|R_{n,q,r}^{\alpha,\beta}(f;x)-f\left(x\right)\right\|_{C[0,1]}\leq\frac{M}{(r-1)!}\frac{\alpha}{\alpha+r}B(\alpha,r)\left\|R_{n,q}^{\alpha,\beta}(\left|s-x\right|^{\alpha+r};x)\right\|_{C[0,1]}$$

here $B(\alpha,r)$ is Beta function $r,n\in\mathbb{N}$.

Proof. By (24), we get

$$\begin{aligned} & f(x) - R_{n,q}^{\alpha,\beta}(f;x) \\ = & \sum_{k=0}^{n} \left[f(x) - \sum_{i=0}^{r} f^{(i)}\left(\tfrac{[k]+[\alpha]}{[n]+[\beta]}\right) \frac{(x - \frac{[k]+[\alpha]}{[n]+[\beta]})^i}{i!} \right] b_{n,k}(q;x). \end{aligned} \tag{25}$$

It is known from Taylor's formula that

$$\begin{aligned} & f(x) - \left[\sum_{i=0}^{r} f^{(i)}\left(\tfrac{[k]+[\alpha]}{[n]+[\beta]}\right) \frac{(x - \frac{[k]+[\alpha]}{[n]+[\beta]})^i}{i!} \right] \\ = & \frac{(x - \frac{[k]+[\alpha]}{[n]+[\beta]})^r}{(r-1)!} \int_0^1 (1-z)^{r-1} \\ & \times \left[f^{(r)}\left(\tfrac{[k]+[\alpha]}{[n]+[\beta]} + z(x - \tfrac{[k]+[\alpha]}{[n]+[\beta]})\right) - f^{(r)}\left(\tfrac{[k]+[\alpha]}{[n]+[\beta]}\right) \right] dz. \end{aligned} \tag{26}$$

Because of $f^{(r)} \in Lip_M(\alpha)$, one can get

$$\begin{aligned} & \left| f^{(r)}\left(\tfrac{[k]+[\alpha]}{[n]+[\beta]} + z(x - \tfrac{[k]+[\alpha]}{[n]+[\beta]})\right) - f^{(r)}\left(\tfrac{[k]+[\alpha]}{[n]+[\beta]}\right) \right| \\ \leq & Mz^{\alpha} \left| x - \tfrac{[k]+[\alpha]}{[n]+[\beta]} \right|^{\alpha}. \end{aligned} \tag{27}$$

From the well known expansion of the Beta function, we can write

$$\int_0^1 (1-z)^{r-1} z^{\alpha} dz = B(\alpha+1, r) = \frac{\alpha}{\alpha+r} B(\alpha, r). \tag{28}$$

Now, by using (28) and (27) in (26), we conclude that

$$\begin{aligned} & \left| f(x) - \left[\sum_{i=0}^{r} f^{(i)}\left(\tfrac{[k]+[\alpha]}{[n]+[\beta]}\right) \frac{(x - \frac{[k]+[\alpha]}{[n]+[\beta]})^i}{i!} \right] \right| \\ \leq & \frac{M}{(r-1)!} \frac{\alpha}{\alpha+r} B(\alpha, r) \left| x - \tfrac{[k]+[\alpha]}{[n]+[\beta]} \right|^{\alpha+r}. \end{aligned} \tag{29}$$

Taking into consideration (29) and (25), we have the desired result. □

Now consider the function $g \in C[0,1]$ defined by

$$g(s) = |s - x|^{\alpha+r}. \tag{30}$$

Since $g(x) = 0$, Theorem 1 yields

$$\lim_n \left\| R_{n,q}^{\alpha,\beta}(g;x) \right\|_{C[0,1]} = 0.$$

So, it follows from above Theorem that, for all $f \in C^r[0,1]$ such that $f^{(r)} \in Lip_M(\alpha)$, we have

$$\lim_n \left\| R_{n,q,r}^{\alpha,\beta}(f;x) - f(x) \right\|_{C[0,1]} = 0.$$

4. The Rates of Statistical Convergence

At this point, let us recall the concept of statistical convergence.

The statistical convergence which was introduced by Fast [8] in 1951, is an important research area in approximation theory. In [9], Gadjiev and Orhan used the concept of statistical convergence in approximation theory. They proved a Bohman-Korovkin type theorem for statistical convergence.

Recently, statistical approximation properties of many operators are investigated (see for instance, [1, 6, 7, 15]).

A sequence $x = (x_k)$ is said to be statistically convergent to a number L if for every $\varepsilon > 0$,

$$\delta\left\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\right\} = 0,$$

where $\delta(K)$ is the natural density of the set $K \subset \mathbb{N}$.

The density of subset $K \subset \mathbb{N}$ is defined by

$$\delta(K) := \lim_n \frac{1}{n}\left\{\text{the number } k \leq n : k \in K\right\}$$

whenever the limit is exists (see [17]).

For instance, $\delta(\mathbb{N}) = 1$, $\delta\left\{2k : k \in \mathbb{N}\right\} = \frac{1}{2}$ and $\delta\left\{k^2 : k \in \mathbb{N}\right\} = 0$. To emphasize the importance of the statistical convergence, one can give the following example: The sequence

$$x_k = \begin{cases} L_1; & \text{if } k = m^2 \\ L_2; & \text{if } k \neq m^2 \end{cases}, \ (m = 1, 2, 3, ...)$$

is statistically convergent to L_2 but not convergent in ordinary sense when $L_1 \neq L_2$. We note that any convergent sequence is statistically convergent but not conversely. Details can be found in [5] and [6].

Now, we consider a sequence $q := (q_n)$ satisfying the following expressions:

$$st - \lim_n q_n = 1 \text{ and } st - \lim_n q_n^n = a. \tag{31}$$

Gadjiev and Orhan [9] gave the below theorem for linear positive operators which is about statistically Korovkin type theorem. Now, we recall this theorem.

Theorem 4.1. *If the sequence of linear positive operators $A_n : C_B[a, b] \to C_B[a, b]$ satisfies the conditions*

$$st - \lim_n \|A_n(e_\nu; .) - e_\nu\|_{C[a,b]} = 0,$$

for $e_\nu(t) = t^\nu$, $\nu = 0, 1, 2$, then for any $f \in C[a, b]$, we get

$$st - \lim_n \left\|A_n(f; .) - f\right\|_{C[a,b]} = 0.$$

Finally, we investigate the rates of statistical convergence of $R_{n,q}^{\alpha,\beta}$ operators. So we give the below theorem.

Theorem 4.2. *Let $q := (q_n)$, $0 < q_n < 1$ be a sequence satisfying (31) conditions. For any monotone increasing continuous function f defined on* $[0, 1]$, *we have*

$$st - \lim_n \left\|R_{n,q}^{\alpha,\beta}(f, q_n; .) - f\right\|_{C[0,1]} = 0. \tag{32}$$

Proof. We know that $R_{n,q_n}^{\alpha,\beta}$ is a positive linear operator. Here, we need to show that

$$st - \lim_n \left\|R_{n,q}^{\alpha,\beta}(e_\nu, q_n; .) - e_\nu\right\|_{C[0,1]} = 0, \text{ for } \nu = 0, 1, 2. \tag{33}$$

For $\nu = 0$, we get

$$st - \lim_n \left\| R_{n,q}^{\alpha,\beta}(e_0, q_n; .) - e_0 \right\|_{C[0,1]} = 0.$$

For $\nu = 1$, we have

$$R_{n,q}^{\alpha,\beta}(e_1, q_n; x) - e_1(x) = \frac{-[\beta]_{q_n} x}{[n]_{q_n} + [\beta]_{q_n}} + \frac{[\alpha]_{q_n}}{[n]_{q_n} + [\beta]_{q_n}}.$$

If we take the maximum of both side for $x \in [0, 1]$, we obtain

$$\left\| R_{n,q}^{\alpha,\beta}(e_1, q_n; .) - e_1(x) \right\|_{C[0,1]} \leq \frac{[\alpha]_{q_n} + [\beta]_{q_n}}{[n]_{q_n} + [\beta]_{q_n}}. \tag{34}$$

Now, we define the sets

$$T := \left\{ k : \left\| R_{k,q}^{\alpha,\beta}(e_1, q_k; .) - e_1 \right\|_{C[0,1]} \geq \varepsilon \right\},$$

$$T_1 := \left\{ k : \frac{[\alpha]_{q_k} + [\beta]_{q_k}}{[n]_{q_k} + [\beta]_{q_k}} \geq \varepsilon \right\}$$

for $\varepsilon > 0$. From the inequality (34), we have $T \subset T_1$. So, we write

$$\begin{aligned} & \delta \left\{ k \leq n : \left\| R_{n,q}^{\alpha,\beta}(e_1, q_k; .) - e_1 \right\|_{C[0,1]} \geq \varepsilon \right\} \\ \leq \quad & \delta \left\{ k \leq n : \frac{[\alpha]_{q_k} + [\beta]_{q_k}}{[n]_{q_k} + [\beta]_{q_k}} \geq \varepsilon \right\}. \end{aligned} \tag{35}$$

From the conditions (31), we get

$$st - \lim_n \left(\frac{[\alpha]_{q_n} + [\beta]_{q_n}}{[n]_{q_n} + [\beta]_{q_n}} \right) = 0.$$

From the definition of density, we see that

$$\delta \left\{ k \leq n : \frac{[\alpha]_{q_k} + [\beta]_{q_k}}{[n]_{q_k} + [\beta]_{q_k}} \geq \varepsilon \right\} = 0$$

and from (35), we find

$$st - \lim_n \left\| R_{n,q}^{\alpha,\beta}(e_1, q_n; .) - e_1 \right\|_{C[0,1]} = 0.$$

Finally, for the case $\nu = 2$, we get

$$\begin{aligned} \left\| R_{n,q}^{\alpha,\beta}(e_2, q_n; .) - e_2(x) \right\|_{C[0,1]} \quad \leq \quad & \frac{[\alpha]_{q_n}^2 + [\beta]_{q_n}^2}{([n]_{q_n} + [\beta]_{q_n})^2} \\ & + (2 [\alpha]_{q_n} + 2 [\beta]_{q_n} + \frac{1}{4}) \frac{[n]_{q_n}}{([n]_{q_n} + [\beta]_{q_n})^2}. \end{aligned} \tag{36}$$

If we choose

$$\begin{aligned} \alpha_n &= \frac{[\beta]_{q_n}^2}{([n]_{q_n} + [\beta]_{q_n})^2}, \\ \beta_n &= (2 [\alpha]_{q_n} + 2 [\beta]_{q_n} + \frac{1}{4}) \frac{[n]_{q_n}}{([n]_{q_n} + [\beta]_{q_n})^2}, \\ \gamma_n &= \frac{[\alpha]_{q_n}^2}{([n]_{q_n} + [\beta]_{q_n})^2} \end{aligned}$$

then from (31), we have

$$st-\lim_n \alpha_n = st-\lim_n \beta_n = st-\lim_n \gamma_n = 0. \tag{37}$$

Now, for $\varepsilon > 0$, we define

$$U := \left\{k : \left\|R_{k,q}^{\alpha,\beta}(e_2, q_k; .) - e_2\right\|_{C[0,1]} \geq \varepsilon\right\},$$

$$U_1 := \left\{k : \alpha_k \geq \frac{\varepsilon}{3}\right\},$$

$$U_2 := \left\{k : \beta_k \geq \frac{\varepsilon}{3}\right\},$$

$$U_3 := \left\{k : \gamma_k \geq \frac{\varepsilon}{3}\right\}.$$

From the inequality (36), we observe that $U \subseteq U_1 \cup U_2 \cup U_3$. Hence, one can write

$$\delta\left\{k \leq n : \left\|R_{k,q}^{\alpha,\beta}(e_2, q_k; .) - e_2\right\|_{C[0,1]} \geq \varepsilon\right\} \leq \delta\left\{k \leq n : \alpha_k \geq \tfrac{\varepsilon}{3}\right\}$$
$$+\delta\left\{k \leq n : \beta_k \geq \frac{\varepsilon}{3}\right\} + \delta\left\{k \leq n : \gamma_k \geq \frac{\varepsilon}{3}\right\}.$$

Since the right hand side of above inequality is zero, we get

$$st-\lim_n \left\|R_{n,q}^{\alpha,\beta}(e_2, q_n; .) - e_2\right\|_{C[0,1]} = 0.$$

This gives the proof. □

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