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Proximal Relator Spaces

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Abstract. This article introduces proximal relator spaces. The basic approach is to define a nonvoid family of proximity relations $\mathcal{R}_{\delta_\Phi}$ (called a proximal relator) on a nonempty set. The pair $(X, \mathcal{R}_{\delta_\Phi})$ (also denoted $X(\mathcal{R}_{\delta_\Phi})$) is called a proximal relator space. Then, for example, the traditional closure of a subset of the Száz relator space $(X, \mathcal{R})$ can be compared with the more recent descriptive closure of a subset of $(X, \mathcal{R}_{\delta_\Phi})$. This leads to an extension of fat and dense subsets of the relator space $(X, \mathcal{R})$ to proximal fat and dense subsets of the proximal relator space $(X, \mathcal{R}_{\delta_\Phi})$.

1. Introduction

This article introduces an extension of a Száz relator space [14–16] called a proximal relator space. A *relator* is a nonvoid family of relations $\mathcal{R}$ on a nonempty set X. The pair $(X, \mathcal{R})$ (also denoted $X(\mathcal{R})$) is called a relator space. Relator spaces are natural generalisations of ordered sets and uniform spaces [16]. With the introduction of a family of proximity relations on X, we obtain a proximal relator space $(X, \mathcal{R}_\delta)$ $(X(\mathcal{R}_\delta))$. For simplicity, we consider only two proximity relations, namely, the Efremovič proximity δ [5] and the descriptive proximity δ_Φ in defining $\mathcal{R}_{\delta_\Phi}$ [9, 11, 12]. The descriptive proximity δ_Φ results from the introduction of feature vectors that describe each point in a proximal relator space. In this paper, X denotes a metric topological space that is endowed with the relations in a proximal relator. With the introduction of $(X, \mathcal{R}_{\delta_\Phi})$, the traditional closure of a subset (*e.g.*, [4, 6]) can be compared with the more recent descriptive closure of a subset.

2. Preliminaries

In a Kovár discrete space, a non-abstract point has a location and features that can be measured [8, §3]. Let X contain non-abstract points in a proximal relator space $(X, \mathcal{R}_{\delta_\oplus})$ and let $\Phi = \left\{\phi_1, \dots, \phi_n\right\}$ be a set of probe functions that represent features of each $x \in X$. A *probe function* $\Phi : X \to \mathbb{R}$ represents a feature of a sample point in a picture. Let $\Phi(x) = (\phi_1(x), \dots, \phi_n(x))$ denote a feature vector for x, which provides a description of each $x \in X$. For example, this leads to a proximal view of sets of picture points in digital

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images [9]. To obtain a descriptive proximity relation (denoted by δ_Φ), one first chooses a set of probe functions. Let $A, B \in 2^X$ and $\mathcal{Q}(A), \mathcal{Q}(B)$ denote sets of descriptions of points in A, B, respectively. For example, $\mathcal{Q}(A) = \{\Phi(a) : a \in A\}$. The expression $A\ \delta_\Phi\ B$ reads *A is descriptively near B*. Similarly, $A\ \underline{\delta}_\Phi\ B$ reads *A is descriptively far from B*. The descriptive proximity of A and B is defined by

$$A\ \delta_\Phi\ B \Leftrightarrow \mathcal{Q}(A) \cap \mathcal{Q}(B) \neq \emptyset.$$

The EF-proximity of $A, B \subset X$ (denoted $A\ \delta\ B$) is defined by

$$A \cap B \neq \emptyset \Rightarrow A\ \delta\ B.$$

In an ordinary metric closure space [1, §14A.1] X, the closure of $A \subset X$ (denoted by $\mathrm{cl}(A)$) is defined by

$$\mathrm{cl}(A) = \{x \in X : D(x, A) = 0\}, \text{ where}$$
$$D(x, A) = inf\,\{d(x, a) : a \in A\},$$

i.e., $\mathrm{cl}(A)$ is the set of all points x in X that are close to A ($D(x, A)$ is the Hausdorff distance [7, §22, p. 128] between x and the set A and $d(x, a) = |x - a|$ (standard distance)). Subsets $A, B \in 2^X$ are spatially near (denoted by $A\ \delta\ B$), provided the intersection of the closure of A and the closure of B is nonempty, i.e., $\mathrm{cl}(A) \cap \mathrm{cl}(B) \neq \emptyset$. That is, nonempty sets are spatially near, provided the sets have at least one point in common.

The Efremovič nearness relation δ (called a *discrete* proximity [5]) is defined by

$$\delta = \left\{(A, B) \in 2^X \times 2^X : \mathrm{cl}(A)\ \cap\ \mathrm{cl}(B) \neq \emptyset\right\}.$$

The pair (X, δ) is called an EF-proximity space. In a proximity space X, the closure of A in X coincides with the intersection of all closed sets that contain A.

Theorem 2.1. [13] *The closure of any set A in the proximity space X is the set of points $x \in X$ that are close to A.*

The expression $A\ \delta_\Phi\ B$ reads *A is descriptively near B*. The relation δ_Φ is called a *descriptive proximity relation*. Similarly, $A\ \underline{\delta}_\Phi\ B$ denotes that A is descriptively far (remote) from B. The descriptive proximity of A and B is defined by

$$A\ \delta_\Phi\ B \Leftrightarrow \mathcal{Q}(\mathrm{cl}(A)) \cap \mathcal{Q}(\mathrm{cl}(B)) \neq \emptyset.$$

The *descriptive intersection* $\underset{\Phi}{\cap}$ of A and B is defined by

$$A \underset{\Phi}{\cap} B = \{x \in A \cup B : \Phi(x) \in \mathcal{Q}(\mathrm{cl}(A)) \text{ and } \Phi(x) \in \mathcal{Q}(\mathrm{cl}(B))\}.$$

That is, $x \in A \cup B$ is in $\mathrm{cl}(A) \underset{\Phi}{\cap} \mathrm{cl}(B)$, provided $\Phi(x) = \Phi(a) = \Phi(b)$ for some $a \in \mathrm{cl}(A), b \in \mathrm{cl}(B)$.
The descriptive proximity relation δ_Φ is defined by

$$\delta_\Phi = \left\{(A, B) \in 2^X \times 2^X : \mathrm{cl}(A) \underset{\Phi}{\cap} \mathrm{cl}(B) \neq \emptyset\right\}.$$

The pair (X, δ_Φ) is called a descriptive EF-proximity space. In a proximal relator space X, the descriptive closure of A in X contains all points in X that are descriptively close to the closure of A. The *descriptive closure of a set A* (denoted by $\mathrm{cl}_\Phi(A)$) is defined by

$$\mathrm{cl}_\Phi(A) = \{x \in X : \Phi(x) \in \mathcal{Q}(\mathrm{cl}(A))\}.$$

That is, $x \in X$ is in the descriptive closure of A, provided $\Phi(x)$ (description of x) matches $\Phi(a) \in \mathcal{Q}(\mathrm{cl}(A))$ for at least one $a \in \mathrm{cl}(A)$.

Theorem 2.2. [10] *The descriptive closure of any set A in the proximal relator space X is the set of points $x \in X$ that are descriptively close to A.*

3. Main Results

In a proximal relator space, EF-proximity δ leads to the following results for descriptive proximity δ_Φ.

Theorem 3.1. *Let $(X, \mathcal{R}_{\delta_\Phi})$ be a proximal relator space, $A, B, C \subset X$. Then*
1^o $A \cap B \neq \emptyset$ implies $A\ \delta_\Phi\ B$.
2^o $(A \cup B) \cap\ C \neq \emptyset$implies $(A \cup B)\ \delta_\Phi\ C$.
3^o $clA\ \cap\ clB \neq \emptyset$ implies $clA\ \delta_\Phi\ clB$.

Proof.
1^o: For $x \in A \cap B, \Phi(x) \in \mathcal{Q}(A)$ and $\Phi(x) \in \mathcal{Q}(B)$. Consequently, $A\ \delta_\Phi\ B$.
$1^o \Rightarrow 2^o$.
3^o: cl$A\ \cap$ cl$B \neq \emptyset$ implies that cl$A$ and cl$A$ have at least one point in common. Hence, $1^o \Rightarrow 3^o$. □

In a pseudometric proximal relator space X, the neighbourhood of a point $x \in X$ (denoted by $N_{x,\varepsilon}$), for $\varepsilon > 0$, is defined by

$$N_{x,\varepsilon} = \{y \in X : d(x, y) < \varepsilon\}.$$

The interior of a set A (denoted by int(A)) and boundary of A (denoted by bdy(A)) in a proximal relator space X are defined by

$$\text{int}(A) = \{x \in X : N_{x,\varepsilon} \subseteq A\}.$$
$$\text{bdy}(A) = \text{cl}(A) \setminus \text{int}(A).$$

The descriptive interior of a set A (denoted by $\text{int}_\Phi(A)$) and descriptive boundary of A (denoted by $\text{bdy}_\Phi(A)$) in a proximal relator space X are defined by

$$\text{int}_\Phi(A) = \{x \in X : \Phi(x) \in \mathcal{Q}(\text{int}(A))\}.$$
$$\text{bdy}_\Phi(A) = \{x \in X : \Phi(x) \in \mathcal{Q}(\text{cl}(A) \setminus \text{int}(A))\}.$$

A set A has a *natural strong inclusion* in a set B associated with δ [2, 3] (denoted by $A \ll_\delta B$), provided $A \subset$ int(cl(int$B$)), *i.e.*, $A\ \underline{\delta}\ X \setminus$ cl(int$B$) (A is far from the complement of cl(int$B$)). Correspondingly, a set A has a *descriptive strong inclusion* in a set B associated with δ_Φ (denoted by $A \ll_{\delta_\Phi} B$), provided $\mathcal{Q}(A) \subset \mathcal{Q}(\text{int}(\text{cl}_\Phi(\text{int}B)))$, *i.e.*, $A\ \underline{\delta}_\Phi\ X \setminus \text{cl}_\Phi(\text{int}B)$ ($\mathcal{Q}(A)$ is far from the complement of $\text{cl}_\Phi(\text{int}B)$)). This leads to the following results.

Theorem 3.2. *Let $(X, \mathcal{R}_{\delta_\Phi})$ be a proximal relator space, $A, B \subset X$. Then*
1^o $A \ll_\delta B \Leftrightarrow A \ll_{\delta_\Phi} B$.
2^o $clA \ll_\delta clB \Leftrightarrow clA \ll_{\delta_\Phi} clB$.

Proof.
1^o: $A \ll_\delta B \Leftrightarrow \Phi(x) \in \mathcal{Q}(\text{int}(\text{cl}_\Phi(\text{int}B)))$ for each $\Phi(x) \in \mathcal{Q}(A) \Leftrightarrow A \ll_{\delta_\Phi} B$.
$1^o \Rightarrow 2^o$. □

Theorem 3.3. *Let $(X, \mathcal{R}_{\delta_\Phi})$ be a proximal relator space, $A \subset X$. Then $cl(A) \subseteq cl_\Phi(A)$.*

Proof. Let $\Phi(x) \in \mathcal{Q}(X \setminus \text{cl}(A))$ such that $\Phi(x) = \Phi(a)$ for some $a \in$ cl$A$. Consequently, $\Phi(x) \in \mathcal{Q}(\text{cl}_\Phi(A))$. Hence, $\text{cl}(A) \subseteq \text{cl}_\Phi(A)$. □

Theorem 3.4. *Let $(X, \mathcal{R}_{\delta_\Phi})$ be a proximal relator space, $A \subset X$. Then*
1^0 $int(A) \subseteq int_\Phi(A)$.
2^0 $bdy(A) \subseteq bdy_\Phi(A)$.

Proof. Immediate from the definition of $\text{int}(A), \text{int}_\Phi(A), \text{bdy}(A), \text{bdy}_\Phi(A)$. □

Theorem 3.5. *Let $(X, \mathcal{R}_{\delta_\Phi})$ be a proximal relator space, $A \subset X$. Then $cl(A) = int(A) \cup bdy(A)$.*

Theorem 3.6. *Let $(X, \mathcal{R}_{\delta_\Phi})$ be a proximal closure relator space, $A \subset X$. Then $cl_\Phi(A) = int_\Phi(A) \cup bdy_\Phi(A)$.*

Proof.

$$\begin{aligned}
\mathrm{cl}(A) \subseteq \mathrm{cl}_\Phi(A)[\text{Theorem 3.3}] &\Rightarrow \mathrm{int}(A) \cup \mathrm{bdy}(A) \subseteq \mathrm{cl}_\Phi(A) \\
&\Rightarrow \mathrm{bdy}_\Phi(A) \subset \mathrm{cl}_\Phi(A), \text{since } \mathrm{bdy}(A) \subset \mathrm{cl}(A), \text{and} \\
&\quad\ \mathrm{int}_\Phi(A) \subset \mathrm{cl}_\Phi(A), \text{since } \mathrm{int}(A) \subset \mathrm{cl}(A) \\
&\Rightarrow \mathrm{int}_\Phi(A) \cup \mathrm{bdy}_\Phi(A) \subseteq \mathrm{cl}_\Phi(A).
\end{aligned}$$

Similarly, $\mathrm{cl}_\Phi(A) \subseteq \mathrm{int}_\Phi(A) \cup \mathrm{bdy}_\Phi(A)$. $\square$

If $\mathcal{R}_{\delta_\Phi}$ is a proximal relator on X, members of the families

$$\mathcal{E}_{\mathcal{R}_{\delta_\Phi}} = \left\{A \subset X : \mathrm{int}_{\mathcal{R}_{\delta_\Phi}}(A) \neq \emptyset\right\} \text{ and } \mathcal{D}_{\mathcal{R}_{\delta_\Phi}} = \left\{A \subset X : \mathrm{cl}_{\mathcal{R}_{\delta_\Phi}}(A) = X\right\}$$

are called *fat* and *dense* subsets of the proximal relator space $(X, \mathcal{R}_{\delta_\Phi})$.

Theorem 3.7. *Let $(X, \mathcal{R}_{\delta_\Phi})$ be a proximal relator space, $A, B \subset X$. Then*
1^0 $\mathcal{E}_{\mathcal{R}_{\delta_\Phi}} = \left\{A \subset X : \forall B \in \mathcal{D}_{\mathcal{R}_{\delta_\Phi}} : A \cap B \neq \emptyset\right\}$.
2^0 $\mathcal{D}_{\mathcal{R}_{\delta_\Phi}} = \left\{A \subset X : \forall B \in \boldsymbol{\varepsilon}_{\mathcal{R}_{\delta_\Phi}} : A \cap B \neq \emptyset\right\}$.
3^0 $\mathcal{E}_{\mathcal{R}_{\delta_\Phi}} = \left\{A \subset X : \forall B \in \mathcal{D}_{\mathcal{R}_{\delta_\Phi}} : A \underset{\Phi}{\cap} B \neq \emptyset\right\}$.
4^0 $\mathcal{D}_{\mathcal{R}_{\delta_\Phi}} = \left\{A \subset X : \forall B \in \boldsymbol{\varepsilon}_{\mathcal{R}_{\delta_\Phi}} : A \underset{\Phi}{\cap} B \neq \emptyset\right\}$.

Proof. Immediate from the definition of $\mathcal{E}_{\mathcal{R}_{\delta_\Phi}}, \mathcal{D}_{\mathcal{R}_{\delta_\Phi}}$. $\square$

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