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Strong Convergence for Generalized Multiple-Set Split Feasibility Problem

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Abstract. In this paper we introduce a new algorithm based on viscosity approximation method for solving the generalized multiple-set split feasibility problem (GMSSFP)in an infinite dimensional Hilbert spaces . We establish the strong convergence for the algorithm to find a unique solution of the variational inequality which is the optimality condition for the minimization problem.

1. Introduction

The problem of finding a point in the intersection of closed and convex subsets of a Hilbert space is a frequently appearing problem in diverse areas of mathematics and physical sciences. This problem is commonly referred to as the convex feasibility problem (CFP). There is a considerable investigation on (CFP) in the framework of Hilbert spaces which captures applications in various disciplines such as image restoration, computer tomograph and radiation therapy treatment planning [11].

Let $\mathcal{H}$ and $\mathcal{K}$ be real Hilbert spaces, $\mathcal{A} : \mathcal{H} \to \mathcal{K}$, be a bounded linear operator and let $\{C_i\}_{i=1}^{p}$ be a family of nonempty closed convex subsets in $\mathcal{H}$ and $\{Q_i\}_{i=1}^{r}$ be a family of nonempty closed convex subsets in $\mathcal{K}$. The multiple-set split feasibility problem (MSSFP) was recently introduced in [**?**] and is formulated as finding a point $x^\star$ with the property:

$$x^\star \in \bigcap_{i=1}^{p} C_i \quad and \quad \mathcal{A}x^\star \in \bigcap_{i=1}^{r} Q_i.$$

The multiple-set split feasibility problem with $p = r = 1$ is known as the split feasibility problem (SEP) which is formulated as finding a point $x^\star$ with the property:

$$x^\star \in C \quad and \quad \mathcal{A}x^\star \in Q,$$

where C and Q are nonempty closed convex subsets of $\mathcal{H}$ and $\mathcal{K}$, respectively.

In 1994, the SFP was first introduced by Censor and Elfving [7], in finite-dimensional Hilbert spaces, for modeling inverse problems which arise from phase retrievals and in medical image reconstruction. A

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number of image reconstruction problems can be formulated as the SFP; see, e.g., [3] and the references therein. Recently, it has been found that the SFP can also be applied to study intensity-modulated radiation therapy; see, e.g., [6, 8, 10] and the references therein. In the recent past, a wide variety of iterative methods have been used in signal processing and image reconstruction and for solving the SFP and MSSPP; see, e.g., [1-19] and the references therein.

The original algorithm given in [7] involves the computation of the inverse A^{-1}(assuming the existence of the inverse of A) and thus has not become popular. A seemingly more popular algorithm that solves the SFP is the CQ algorithm of Byrne [2, 3] which is found to be a gradient-projection method (GPM) in convex minimization. It is also a special case of the proximal forward-backward splitting method [12]. The CQ algorithm starts with any $x_1 \in \mathcal{H}$ and generates a sequence $\{x_n\}$ through the iteration

$$x_{n+1} = P_C(I - \lambda \mathcal{A}^*(I - P_Q)\mathcal{A})x_n$$

where $\lambda \in (0, \frac{2}{\|\mathcal{A}\|^2})$, $\mathcal{A}^*$ is the adjoint of $\mathcal{A}$, P_C and P_Q are the metric projections onto C and Q respectively.

Very recently, Xu [22] gave a continuation of the study on the CQ algorithm and its convergence. Xu [22] transformed SFP to the fixed point problem of the operator $P_C(I - \lambda \mathcal{A}^*(I - P_Q))$ and shown that a point $x^\star$ solves SFP if and only if $x^\star = P_C(I - \lambda \mathcal{A}^*(I - P_Q)\mathcal{A})x^\star$. He applied Mann's algorithm to the SFP and proposed an averaged CQ algorithm which was proved to be weakly convergent to a solution of the SFP. Xu [22] also proposed the regularized method

$$x_{n+1} = P_C(I - \lambda_n(\mathcal{A}^*(I - P_Q)\mathcal{A} + \alpha_n I))x_n$$

and proved that the sequence $\{x_n\}$ converges strongly to a minimum norm solution of SFP(1) provided the parameters $\{\alpha_n\}$ and $\{\lambda_n\}$ verify some suitable conditions. Further recent work also appeared in literature, see, for example [5, 9, 16]. In [20], Wang and Xu gave a Cyclic algorithm to solve MSSFP:

$$x_{n+1} = P_{C[n]}(x_n + \gamma \mathcal{A}^*(P_{Q[n]} - I)\mathcal{A}x_n),$$

where $[n] := n(mod\, p)$, (mod function take values in $\{1, 2, ..., p\}$), and $\gamma \in (0, \frac{2}{\|\mathcal{A}\|^2})$. They show that the sequence $\{x_n\}$ convergence weakly to a solution of MSSFP whenever its solution set in nonempty. Now we consider the multiple-set split feasibility problem for a finite family of operators:

Definition 1.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be real Hilbert spaces, $\mathcal{A}_k : \mathcal{H} \to \mathcal{K}, (k = 1, 2, ..., m)$ be a family of bounded linear operators and let $\{C_i\}_{i=1}^p$ be a family of nonempty closed convex subsets in $\mathcal{H}$ and $\{Q_i\}_{i=1}^r$ be a family of nonempty closed convex subsets in $\mathcal{K}$.* Generalized multiple-set split feasibility problem (GMSSFP) is to find a point x^* such that

$$x^\star \in \bigcap_{i=1}^{p} C_i \quad and \quad \mathcal{A}_k x^\star \in \bigcap_{i=1}^{r} Q_{i}, \quad {\scriptstyle k=1,2,...,m}. \tag{1}$$

We denote Ω the solution set of GMSSFP.

In this paper we introduce a new algorithm based on viscosity approximation method for solving the generalized multiple-set split feasibility problem (GMSSFP)in an infinite dimensional Hilbert spaces. We establish the strong convergence for the algorithm to find a unique solution of the variational inequality which is the optimality condition for the minimization problem.

2. Preliminaries

We use the following notion in the sequel:

• $\rightharpoonup$ for weak convergence and $\to$ for strong convergence.

It is known that a Hilbert space $\mathcal{H}$ satisfies Opial's condition, i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in \mathcal{H}$ with $y \neq x$.

Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Recall that the nearest point or metric projection from $\mathcal{H}$ onto C, denoted P_C, assigns, to each $x \in \mathcal{H}$, the unique point $P_C x \in C$ with the property

$$\|x - P_C x\| = inf\{\|x - y\| \quad \forall y \in C\}.$$

Recall that a mapping $T : \mathcal{H} \to \mathcal{H}$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \qquad \forall x, y \in \mathcal{H}.$$

It is well known that the metric projection P_C of $\mathcal{H}$ onto C has the following basic properties: • P_C is a nonexpansive,
• $\langle y - P_C(x), x - P_C(x)\rangle \leq 0, \qquad \forall x \in \mathcal{H}, y \in C.$

Definition 2.1. *A bounded linear operator $\mathcal{B}$ on $\mathcal{H}$ is called strongly positive if there exists $\overline{\gamma} > 0$ such that*

$$\langle \mathcal{B}x, x\rangle \geq \overline{\gamma}\|x\|^2, \qquad (x \in \mathcal{H}).$$

For a nonexpansive mapping T from a nonempty subset C of $\mathcal{H}$ into itself a typical problem is to minimize the quadratic function

$$\min_{x \in F(T)} \frac{1}{2}\langle \mathcal{B}x, x\rangle - \langle x, b\rangle, \tag{2}$$

over the set of all fixed points $F(T)$ of T (see [18]).

Lemma 2.2. *([18]). Let $\mathcal{B}$ be a self-adjoint strongly positive bounded linear operator on a Hilbert space $\mathcal{H}$ with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|\mathcal{B}\|^{-1}$. Then $\|I - \rho\mathcal{B}\| \leq 1 - \rho\overline{\gamma}$.*

Lemma 2.3. *[14] Let $\mathcal{H}$ be a Hilbert space and $x_i \in H$, $(1 \leq i \leq m)$. Then for any given $\{\lambda_i\}_{i=1}^m \subset (0, 1)$ with $\sum_{i=1}^m \lambda_i = 1$ and for any positive integer k, j with $1 \leq k < j \leq m$,*

$$\|\sum_{i=1}^m \lambda_i x_i\|^2 \leq \sum_{i=1}^m \lambda_i\|x_i\|^2 - \lambda_k\lambda_j\|x_k - x_j\|^2.$$

Lemma 2.4. *[21] Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \vartheta_n)a_n + \vartheta_n\delta_n, \qquad n \geq 0,$$

where $\{\vartheta_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^\infty \vartheta_n = \infty$,
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^\infty |\vartheta_n\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5. *([17]) Let $\{t_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $t_{n_i} < t_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{\tau(n)\} \subset \mathbb{N}$ such that $\tau(n) \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $n \in \mathbb{N}$:*

$$t_{\tau(n)} \leq t_{\tau(n)+1}, \qquad t_n \leq t_{\tau(n)+1}.$$

In fact

$$\tau(n) = \max\{k \leq n : t_k < t_{k+1}\}.$$

3. The Results

In this section we introduce our algorithm for solving GMSSFP (1).

Theorem 3.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be real Hilbert spaces, $\mathcal{A}_k : \mathcal{H} \to \mathcal{K}, k = 1,2$ be two bounded linear operator and let $\{C_i\}_{i=1}^r$ be a family of nonempty closed convex subsets in $\mathcal{H}$ and $\{Q_i\}_{i=1}^r$ be a family of nonempty closed convex subsets in $\mathcal{K}$. Assume that GMSSFP has a nonempty solution set Ω. Suppose h be a contraction of $\mathcal{H}$ into itself with constant $b \in (0,1)$ and $\mathcal{B}$ be a strongly positive bounded linear self-adjoint operator on $\mathcal{H}$ with coefficient $\overline{\gamma}$ and $0 < \gamma < \frac{\overline{\gamma}}{b}$. Let $\{x_n\}$ be a sequence generated by $x_0 \in \mathcal{H}$ and by*

$$\begin{cases} y_n = \alpha_n x_n + \sum_{i=1}^r \beta_{n,i} P_{C_i}(I - \lambda_{n,i}\mathcal{A}_1^*(I - P_{Q_i})\mathcal{A}_1)x_n + \sum_{i=1}^r \gamma_{n,i} P_{C_i}(I - \lambda_{n,i}\mathcal{A}_2^*(I - P_{Q_i})\mathcal{A}_2)x_n, \\ x_{n+1} = \theta_n \gamma h(x_n) + (I - \theta_n \mathcal{B})y_n, \qquad \forall n \geq 0, \end{cases} \tag{3}$$

where $\alpha_n + \sum_{i=1}^r \beta_{n,i} + \sum_{i=1}^r \gamma_{n,i} = 1$ and the sequences $\{\alpha_n\}$, $\{\beta_{n,i}\}$, $\{\gamma_{n,i}\}$, $\{\theta_n\}$ and $\{\lambda_{n,i}\}$ satisfy the following conditions:

(i) $\liminf_n \alpha_n\beta_{n,i} > 0$ *and* $\liminf_n \alpha_n\gamma_{n,i} > 0$, *for each* $1 \leq i \leq r$,

(ii) $\lim_{n\to\infty} \theta_n = 0$ *and* $\sum_{n=0}^\infty \theta_n = \infty$,

(iii) for each $1 \leq i \leq r$, $0 < \lambda_{n,i} < \min\{\frac{2}{\|\mathcal{A}_1\|^2}, \frac{2}{\|\mathcal{A}_2\|^2}\}$ *and*

$$0 < \liminf_{n\to\infty} \lambda_{n,i} \leq \limsup_{n\to\infty} \lambda_{n,i} < \min\{\frac{2}{\|\mathcal{A}_1\|^2}, \frac{2}{\|\mathcal{A}_2\|^2}\}.$$

Then, the sequences $\{x_n\}$ converges strongly to $x^\star \in \Omega$ which solves the variational inequality;

$$\langle(\mathcal{B} - \gamma h)x^\star, x - x^\star\rangle \geq 0, \qquad \forall x \in \Omega. \tag{4}$$

Proof. First, we note that the solution set Ω is closed and convex. Indeed, since $0 < \lambda_{n,i} < \min\{\frac{2}{\|\mathcal{A}_1\|^2}, \frac{2}{\|\mathcal{A}_2\|^2}\}$ we have the operators $P_{C_i}(I - \lambda_{n,i}\mathcal{A}_1^*(I - P_{Q_i})\mathcal{A}_1)$ and $P_{C_i}(I - \lambda_{n,i}\mathcal{A}_2^*(I - P_{Q_i})\mathcal{A}_2)$ are nonexpansive (see [22] for details). Note that a point $x^\star$ solves GMSSFP if and only if $x^\star = P_{C_i}(I - \lambda_{n,i}\mathcal{A}_k^*(I - P_{Q_i})\mathcal{A}_k)x^\star$ for all $1 \leq i \leq r$ and $k = 1,2$. Now since the fixed point set of nonexpansive operators is closed and convex, the solution set Ω is closed and convex. So the projection onto the solution set Ω is well defined whenever $\Omega \neq \emptyset$. Next, we assert that $P_\Omega(I - \mathcal{B} + \gamma h)$ is a contraction from $\mathcal{H}$ into itself. As a matter of fact, for any $x, y \in \mathcal{H}$ we have

$$\begin{aligned} \|P_\Omega(I - \mathcal{B} + \gamma h)(x) - P_\Omega(I - \mathcal{B} + \gamma h)(y)\| &\leq \|(I - \mathcal{B} + \gamma h)(x) - (I - \mathcal{B} + \gamma h)(y)\| \\ &\leq \|(I - \mathcal{B})x - (I - \mathcal{B})y\| + \gamma\|hx - hy\| \\ &\leq (1 - \overline{\gamma})\|x - y\| + \gamma b\|x - y\| \\ &\leq (1 - (\overline{\gamma} - \gamma b)\|x - y\|. \end{aligned}$$

So, by the Banach contraction principle there exists a unique element $x^\star \in \mathcal{H}$ such that $x^\star = P_\Omega(I - \mathcal{B} + \gamma h)x^\star$. Since $\lim_{n\to\infty} \theta_n = 0$, we can assume that $\theta_n \in (0, \|\mathcal{B}\|^{-1})$, for all $n \geq 0$. By Lemma 2.2 we have $\|I - \theta_n\mathcal{B}\| \leq 1 - \theta_n\overline{\gamma}$. Now, we show that $\{x_n\}$ is bounded. In fact, using the nonexpansive property of the operators $P_{C_i}(I - \lambda_{n,i}A^*(I - P_{Q_i})A)$ we have

$$\begin{aligned} \|y_n - x^\star\| &= \|\alpha_n x_n + \textstyle\sum_{i=1}^r \beta_{n,i} P_{C_i}(I - \lambda_{n,i}\mathcal{A}_1^*(I - P_{Q_i})\mathcal{A}_1)x_n \\ &\quad + \textstyle\sum_{i=1}^r \gamma_{n,i} P_{C_i}(I - \lambda_{n,i}\mathcal{A}_2^*(I - P_{Q_i})\mathcal{A}_2)x_n - x^\star\| \\ &\leq \alpha_n\|x_n - x^\star\| + \textstyle\sum_{i=1}^r \beta_{n,i}\|P_{C_i}(I - \lambda_{n,i}\mathcal{A}_1^*(I - P_{Q_i})\mathcal{A}_1)x_n - x^\star\| \\ &\quad + \textstyle\sum_{i=1}^r \gamma_{n,i}\|P_{C_i}(I - \lambda_{n,i}\mathcal{A}_2^*(I - P_{Q_i})\mathcal{A}_2)x_n - x^\star\| \\ &\leq \alpha_n\|x_n - x^\star\| + \textstyle\sum_{i=1}^r \beta_{n,i}\|x_n - x^\star\| + \textstyle\sum_{i=1}^r \gamma_{n,i}\|x_n - x^\star\| \\ &= \|x_n - x^\star\|, \end{aligned}$$

and hence

$$\begin{aligned}
\|x_{n+1}-x^\star\| &= \|\theta_n(\gamma h(x_n)-\mathcal{B}x^\star)+(I-\theta_n\mathcal{B})(y_n-x^\star\| \\
&\le \theta_n\|\gamma h(x_n)-\mathcal{B}x^\star\|+\|I-\theta_n\mathcal{B}\|\|y_n-x^\star\| \\
&\le \theta_n\|\gamma h(x_n)-\mathcal{B}x^\star\|+(1-\theta_n\overline{\gamma})\|x_n-x^\star\| \\
&\le \theta_n\gamma\|h(x_n)-hx^\star\|+\theta_n\|\gamma hx^\star-\mathcal{B}x^\star\|+(1-\theta_n)\overline{\gamma}\|x_n-x^\star\| \\
&\le \theta_n\gamma b\|x_n-x^\star\|+\theta_n\|\gamma hx^\star-\mathcal{B}x^\star\|+(1-\theta_n)\overline{\gamma}\|x_n-x^\star\| \\
&\le (1-\theta_n(\overline{\gamma}-\gamma b))\|x_n-x^\star\|+\theta_n\|\gamma hz-\mathcal{B}z\| \\
&= (1-\theta_n(\overline{\gamma}-\gamma b))\|x_n-x^\star\|+\theta_n(\overline{\gamma}-\gamma b)\frac{\|\gamma hx^\star-\mathcal{B}x^\star\|}{\overline{\gamma}-\gamma b} \\
&\le \max\{\|x_n-x^\star\|,\frac{\|\gamma hx^\star-\mathcal{B}x^\star\|}{\overline{\gamma}-\gamma b}\} \\
&\vdots \\
&\le \max\{\|x_0-x^\star\|,\frac{\|\gamma hx^\star-\mathcal{B}x^\star\|}{\overline{\gamma}-\gamma b}\}.
\end{aligned}$$

This indicates that $\{x_n\}$ is bounded. It is easily to deduce that $\{y_n\}$ and $\{h(x_n)\}$ are also bounded. Next, we show that for each $1\le i\le r$ and $k=1,2$,

$$\lim_{n\to\infty}\|P_{C_i}(I-\lambda_{n,i}\mathcal{A}_k^*(I-P_{Q_i})\mathcal{A}_k)x_n-x_n\|=0.$$

Applying Lemma 2.3, we get that

$$\begin{aligned}
\|y_n-x^\star\|^2 &= \|\alpha_n x_n+\textstyle\sum_{i=1}^r\beta_{n,i}P_{C_i}(I-\lambda_{n,i}\mathcal{A}_1^*(I-P_{Q_i})\mathcal{A}_1)x_n \\
&+ \textstyle\sum_{i=1}^r\gamma_{n,i}P_{C_i}(I-\lambda_{n,i}\mathcal{A}_2^*(I-P_{Q_i})\mathcal{A}_2)x_n-x^\star\|^2 \\
&\le \alpha_n\|x_n-x^\star\|^2+\textstyle\sum_{i=1}^r\beta_{n,i}\|P_{C_i}(I-\lambda_{n,i}\mathcal{A}_1^*(I-P_{Q_i})\mathcal{A}_1)x_n-x^\star\|^2 \\
&+ \textstyle\sum_{i=1}^r\gamma_{n,i}\|P_{C_i}(I-\lambda_{n,i}\mathcal{A}_2^*(I-P_{Q_i})\mathcal{A}_2)x_n-x^\star\|^2 \\
&- \alpha_n\beta_{n,i}\|P_{C_i}(I-\lambda_{n,i}\mathcal{A}_1^*(I-P_{Q_i})\mathcal{A}_1)x_n-x_n\|^2 \\
&- \alpha_n\gamma_{n,i}\|P_{C_i}(I-\lambda_{n,i}\mathcal{A}_2^*(I-P_{Q_i})\mathcal{A}_2)x_n-x_n\|^2 \\
&\le \alpha_n\|x_n-x^\star\|^2+\textstyle\sum_{i=1}^r\beta_{n,i}\|x_n-x^\star\|^2+\sum_{i=1}^r\gamma_{n,i}\|x_n-x^\star\|^2 \\
&- \alpha_n\beta_{n,i}\|P_{C_i}(I-\lambda_{n,i}\mathcal{A}_1^*(I-P_{Q_i})\mathcal{A}_1)x_n-x_n\|^2 \\
&- \alpha_n\gamma_{n,i}\|P_{C_i}(I-\lambda_{n,i}\mathcal{A}_2^*(I-P_{Q_i})\mathcal{A}_2)x_n-x_n\|^2 \\
&= \|x_n-x^\star\|^2-\alpha_n\beta_{n,i}\|P_{C_i}(I-\lambda_{n,i}\mathcal{A}_1^*(I-P_{Q_i})\mathcal{A}_1)x_n-x_n\|^2 \\
&- \alpha_n\gamma_{n,i}\|P_{C_i}(I-\lambda_{n,i}\mathcal{A}_2^*(I-P_{Q_i})\mathcal{A}_2)x_n-x_n\|^2.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\|x_{n+1}-x^\star\|^2 &= \|\theta_n(\gamma h(x_n)-\mathcal{B}x^\star+(I-\theta_n\mathcal{B})(y_n-x^\star\|^2 \\
&\le \theta_n^2\|\gamma h(x_n)-\mathcal{B}x^\star\|^2+(1-\theta_n\overline{\gamma})^2\|y_n-x^\star\|^2+2\theta_n(1-\theta_n\overline{\gamma})\|\gamma h(x_n)-\mathcal{B}x^\star\|\|y_n-x^\star\| \\
&\le \theta_n^2\|\gamma h(x_n)-\mathcal{B}x^\star\|^2+(1-\theta_n\overline{\gamma})^2\|x_n-x^\star\|^2+2\theta_n(1-\theta_n\overline{\gamma})\|\gamma h(x_n)-\mathcal{B}x^\star\|\|x_n-x^\star\| \\
&- (1-\theta_n\overline{\gamma})^2\alpha_n\beta_{n,i}\|P_{C_i}(I-\lambda_{n,i}\mathcal{A}_1^*(I-P_{Q_i})\mathcal{A}_1)x_n-x_n\|^2 \\
&- (1-\theta_n\overline{\gamma})^2\alpha_n\gamma_{n,i}\|P_{C_i}(I-\lambda_{n,i}\mathcal{A}_2^*(I-P_{Q_i})\mathcal{A}_2)x_n-x_n\|^2,
\end{aligned}$$

which hence implies that

$$(1-\theta_n\overline{\gamma})^2\alpha_n\beta_{n,i}\|P_{C_i}(I-\lambda_{n,i}\mathcal{A}_1^*(I-P_{Q_i})\mathcal{A}_1)x_n-x_n\|^2 \le \|x_n-x^\star\|^2-\|x_{n+1}-x^\star\|^2 + 2\theta_n(1-\theta_n\overline{\gamma})\|\gamma h(x_n)-\mathcal{B}x^\star\|\|x_n-x^\star\|+\theta_n^2\|\gamma h(x_n)-\mathcal{B}x^\star\|^2. \quad (5)$$

We finally analyze the inequality (5) by considering the following two cases.

Case 1. Assume that $\{\|x_n-x^\star\|\}$ is a monotone sequence. In other words, for n_0 large enough, $\{\|x_n-x^\star\|\}_{n\ge n_0}$ is either nondecreasing or nonincreasing. Since $\{\|x_n-x^\star\|\}$ is bounded, it is convergent. Since $\lim_{n\to\infty}\theta_n=0$ and $\{h(x_n)\}$ and $\{x_n\}$ are bounded, from (5) we deduce

$$\lim_{n\to\infty}\alpha_n\beta_{n,i}\|P_{C_i}(I-\lambda_{n,i}\mathcal{A}_1^*(I-P_{Q_i})\mathcal{A}_1)x_n-x_n\|^2=0.$$

By our assumption that $\liminf_n \alpha_n\beta_{n,i}>0$, we get that

$$\lim_{n\to\infty}\|P_{C_i}(I-\lambda_{n,i}\mathcal{A}_1^*(I-P_{Q_i})\mathcal{A}_1)x_n-x_n\|=0, \qquad 1\le i\le r. \quad (6)$$

By similar argument we can obtain that

$$\lim_{n\to\infty}\|P_{C_i}(I-\lambda_{n,i}\mathcal{A}_2^*(I-P_{Q_i})\mathcal{A}_2)x_n-x_n\|=0, \qquad 1\le i\le r. \quad (7)$$

Next, we show that

$$\lim sup_{n\to\infty}\langle(\mathcal{B}-\gamma h)x^\star, x^\star-x_n\rangle\le 0.$$

To show this inequality, We can choose a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that

$$\lim_{l\to\infty}(\langle(\mathcal{B}-\gamma h)x^\star, x^\star-x_{n_l}\rangle = \lim sup_{n\to\infty}\langle(\mathcal{B}-\gamma h)x^\star, x^\star-x_n\rangle.$$

Since $\{x_{n_l}\}$ is bounded, there exists a subsequence $\{x_{n_{l_j}}\}$ of $\{x_{n_l}\}$ which converges weakly to z. Without loss of generality, we can assume that $x_{n_l}\rightharpoonup z$ and $\lambda_{n,i}\to\lambda_i\in(0,\min\{\frac{2}{\|\mathcal{A}_1\|^2},\frac{2}{\|\mathcal{A}_2\|^2}\})$ for each $1\le i\le r$. From (7) for $k=1,2$, we have

$$\begin{aligned}\|P_{C_i}(I-\lambda_i\mathcal{A}_k^*(I-P_{Q_i})\mathcal{A}_k)x_n-x_n\| &\le \|P_{C_i}(I-\lambda_i\mathcal{A}_k^*(I-P_{Q_i})\mathcal{A}_k)x_n-P_{C_i}(I-\lambda_{n,i}\mathcal{A}_k^*(I-P_{Q_i})\mathcal{A}_k)x_n\|\\ &+\|P_{C_i}(I-\lambda_{n,i}\mathcal{A}_k^*(I-P_{Q_i})\mathcal{A}_k)x_n-x_n\|\\ &\le \|(I-\lambda_i\mathcal{A}_k^*(I-P_{Q_i})\mathcal{A}_2)x_n-(I-\lambda_{n,i}\mathcal{A}_k^*(I-P_{Q_i})\mathcal{A}_k)x_n\|\\ &+\|P_{C_i}(I-\lambda_{n,i}\mathcal{A}_k^*(I-P_{Q_i})\mathcal{A}_k)-x_n\|\\ &\le|\lambda_i-\lambda_{n,i}|\,\|\mathcal{A}_k^*(I-P_{Q_i})\mathcal{A}_k)x_n\|\\ &+\|P_{C_i}(I-\lambda_{n,i}\mathcal{A}_k^*(I-P_{Q_i})\mathcal{A}_k)x_n-x_n\|\to 0 \quad as \quad n\to\infty.\end{aligned}$$

Notice that, since $\lambda_i\in(0,\min\{\frac{2}{\|\mathcal{A}_1\|^2},\frac{2}{\|\mathcal{A}_2\|^2}\})$ we have $P_{C_i}(I-\lambda_{n,i}\mathcal{A}_k^*(I-P_{Q_i})\mathcal{A}_k)$ is nonexpansive. Thus

$$\begin{aligned}\|x_{n_l}-P_{C_i}(I-\lambda_i\mathcal{A}_k^*(I-P_{Q_i})\mathcal{A}_k)z\| &\le \|x_{n_l}-P_{C_i}(I-\lambda_i\mathcal{A}_k^*(I-P_{Q_i})\mathcal{A}_k)x_{n_l}\|\\ &+\|P_{C_i}(I-\lambda_i\mathcal{A}_k^*(I-P_{Q_i})\mathcal{A}_k)x_{n_l}-P_{C_i}(I-\lambda_i\mathcal{A}_k^*(I-P_{Q_i})\mathcal{A}_k)z\|\\ &\le\|x_{n_l}-P_{C_i}(I-\lambda_i\mathcal{A}_k^*(I-P_{Q_i})\mathcal{A}_k)x_{n_l}\|+\|x_{n_l}-z\|.\end{aligned}$$

This implies that

$$\limsup_{n\to\infty}\|x_{n_l}-P_{C_i}(I-\lambda_i\mathcal{A}_k^*(I-P_{Q_i})\mathcal{A}_k)z\|\le\limsup_{n\to\infty}\|x_{n_l}-z\|.$$

By the Opial property of the Hilbert space $\mathcal{H}$ we obtain that $P_{C_i}(I-\lambda_i\mathcal{A}_k^*(I-P_{Q_i})\mathcal{A}_k)z=z$, for all$1\le i\le r$ and $k=1,2$, hence $z\in\Omega$. Since $x^\star=P_\Omega(I-\mathcal{B}+\gamma h)x^\star$ and $z\in\Omega$, we have

$$\lim sup_{n\to\infty}\langle(\mathcal{B}-\gamma h)x^\star, x^\star-x_n\rangle=\lim_{i\to\infty}(\langle(\mathcal{B}-\gamma h)x^\star, x^\star-x_{n_l}\rangle=(\langle(\mathcal{B}-\gamma h)x^\star, x^\star-z\rangle\le 0.$$

It is known that in a Hilbert space $\mathcal{H}$

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \qquad \forall x, y \in \mathcal{H}.$$

From this and since

$$x_{n+1} - x^\star = \theta_n(\gamma h(x_n) - \mathcal{B}x^\star) + (I - \theta_n\mathcal{B})(y_n - x^\star),$$

we conclude that

$$\begin{aligned}
\|x_{n+1} - x^\star\|^2 &\le \|(I-\theta_n\mathcal{B})(y_n - x^\star)\|^2 + 2\theta_n\langle \gamma h(x_n) - \mathcal{B}x^\star, x_{n+1} - x^\star\rangle \\
&\le (1-\theta_n\overline{\gamma})^2\|x_n - x^\star\|^2 + 2\theta_n\gamma\langle h(x_n) - h(x^\star), x_{n+1} - x^\star\rangle \\
&+ 2\theta_n\langle \gamma h(x^\star) - \mathcal{B}x^\star, x_{n+1} - x^\star\rangle \\
&\le (1-\theta_n\overline{\gamma})^2\|x_n - x^\star\|^2 + 2\theta_n b\gamma\|x_n - x^\star\|\|x_{n+1} - x^\star\| \\
&+ 2\theta_n\langle \gamma h(x^\star) - \mathcal{B}x^\star, x_{n+1} - x^\star\rangle \\
&\le (1-\theta_n\overline{\gamma})^2\|x_n - x^\star\|^2 + \theta_n b\gamma(\|x_n - x^\star\|^2 + \|x_{n+1} - x^\star\|^2) \\
&+ 2\theta_n\langle \gamma h(x^\star) - \mathcal{B}x^\star, x_{n+1} - x^\star\rangle \\
&\le ((1-\theta_n\overline{\gamma})^2 + \theta_n b\gamma)\|x_n - x^\star\|^2 + \theta_n\gamma b\|x_{n+1} - x^\star\|^2 \\
&+ 2\theta_n\langle \gamma h(x^\star) - \mathcal{B}x^\star, x_{n+1} - x^\star\rangle.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|x_{n+1} - x^\star\|^2 &\le \tfrac{1-2\theta_n\overline{\gamma}+(\theta_n\overline{\gamma})^2+\theta_n\gamma b}{1-\theta_n\gamma b}\|x_n - x^\star\|^2 + \tfrac{2\theta_n}{1-\theta_n\gamma b}\langle \gamma h(x^\star) - \mathcal{B}x^\star, x_{n+1} - x^\star\rangle \\
&= (1 - \tfrac{2(\overline{\gamma}-\gamma b)\theta_n}{1-\theta_n\gamma b})\|x_n - x^\star\|^2 + \tfrac{(\theta_n\overline{\gamma})^2}{1-\theta_n\gamma b}\|x_n - x^\star\|^2 \\
&+ \tfrac{2\theta_n}{1-\theta_n\gamma b}\langle \gamma h(x^\star) - \mathcal{B}x^\star, x_{n+1} - x^\star\rangle \\
&\le (1 - \tfrac{2(\overline{\gamma}-\gamma b)\theta_n}{1-\theta_n\gamma b})\|x_n - x^\star\|^2 + \tfrac{2(\overline{\gamma}-\gamma b)\theta_n}{1-\theta_n\gamma b}(\tfrac{(\theta_n\overline{\gamma}^2)L}{2(\overline{\gamma}-\gamma b)} \\
&+ \tfrac{1}{\overline{\gamma}-\gamma b}\langle \gamma h(x^\star) - \mathcal{B}x^\star, x_{n+1} - x^\star\rangle) \\
&= (1-\vartheta_n)\|x_n - x^\star\|^2 + \vartheta_n\delta_n,
\end{aligned}$$

where

$$L = sup\{\|x_n - x^\star\|^2 : n \ge 0\}, \qquad \vartheta_n = \frac{2(\overline{\gamma} - \gamma k)\theta_n}{1-\theta_n\gamma b},$$

and

$$\delta_n = \frac{(\theta_n\overline{\gamma}^2)L}{2(\overline{\gamma}-\gamma b)} + \frac{1}{\overline{\gamma}-\gamma b}\langle \gamma hx^\star - Bx^\star, x_{n+1} - x^\star\rangle.$$

It is easy to see that $\vartheta_n \to 0, \sum_{n=1}^{\infty}\vartheta_n = \infty$ and $\limsup_{n\to\infty}\delta_n \le 0$. Hence, all conditions of Lemma 2.4 are satisfied. Therefore, we immediately deduce that $x_n \to x^\star$.

Case 2. Assume that $\{\|x_n - x^\star\|\}$ is not a monotone sequence. Then, we can define an integer sequence $\{\tau(n)\}$ for all $n \ge n_0$ (for some n_0 large enough) by

$$\tau(n) := \max\{k \in \mathbb{N}; k \le n : \|x_k - x^\star\| < \|x_{k+1} - x^\star\|\}.$$

Clearly, τ is a nondecreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and for all $n \ge n_0$,

$$\|x_{\tau(n)} - x^\star\| \le \|x_{\tau(n)+1} - x^\star\|.$$

From (5) we obtain that

$$\lim_{n\to\infty}\|P_{C_i}(I - \lambda_{\tau(n),i}\mathcal{A}_1^*(I - P_{Q_i})\mathcal{A}_1)x_{\tau(n)} - x_{\tau(n)}\| = 0, \qquad 1 \le i \le r. \tag{8}$$

Following an argument similar to that in *Case* 1 we have

$$\|x_{\tau(n)+1} - x^\star\|^2 \le (1 - \vartheta_{\tau(n)})\|x_{\tau(n)} - x^\star\|^2 + \vartheta_{\tau(n)}\delta_{\tau(n)},$$

where $\vartheta_{\tau(n)} \to 0, \sum_{n=1}^{\infty} \vartheta_{\tau(n)} = \infty$ and $\limsup_{n\to\infty} \delta_{\tau(n)} \le 0$. Hence, by Lemma 2.4, we obtain $\lim_{n\to\infty} \|x_{\tau(n)} - x^\star\| = 0$ and $\lim_{n\to\infty} \|x_{\tau(n)+1} - x^\star\| = 0$. Now Lemma 2.5 implies

$$0 \le \|x_n - x^\star\| \le \max\{\|x_{\tau(n)} - x^\star\|, \|x_n - x^\star\|\} \le \|x_{\tau(n)+1} - x^\star\|.$$

Therefore $\{x_n\}$ converges strongly to $x^\star = P_\Omega(I - \mathcal{B} + \gamma h)x^\star$. This complete the proof. □

Setting $\mathcal{B} = I$ and $\gamma = 1$ in Theorem 3.1 we obtain the following result.

Theorem 3.2. *Let $\mathcal{H}$ and $\mathcal{K}$ be real Hilbert spaces, $\mathcal{A}_k : \mathcal{H} \to \mathcal{K}, k = 1,2$ be two bounded linear operator and let $\{C_i\}_{i=1}^r$ be a family of nonempty closed convex subsets in $\mathcal{H}$ and $\{Q_i\}_{i=1}^r$ be a family of nonempty closed convex subsets in $\mathcal{K}$. Assume that GMSSFP has a nonempty solution set Ω. Suppose h be a contraction of $\mathcal{H}$ into itself with constant $b \in (0,1)$. Let $\{x_n\}$ be a sequence generated by $x_0 \in \mathcal{H}$ and by*

$$\begin{cases} y_n = \alpha_n x_n + \sum_{i=1}^r \beta_{n,i} P_{C_i}(I - \lambda_{n,i}\mathcal{A}_1^*(I - P_{Q_i})\mathcal{A}_1)x_n + \sum_{i=1}^r \gamma_{n,i} P_{C_i}(I - \lambda_{n,i}\mathcal{A}_2^*(I - P_{Q_i})\mathcal{A}_2)x_n, \\ x_{n+1} = \theta_n h(x_n) + (1 - \theta_n)y_n, \qquad \forall n \ge 0, \end{cases}$$

where $\alpha_n + \sum_{i=1}^r \beta_{n,i} + \sum_{i=1}^r \gamma_{n,i} = 1$ and the sequences $\{\alpha_n\}$, $\{\beta_{n,i}\}$, $\{\gamma_{n,i}\}$, $\{\theta_n\}$ and $\{\lambda_{n,i}\}$ satisfy the following conditions:

(i) $\liminf_n \alpha_n\beta_{n,i} > 0$ *and* $\liminf_n \alpha_n\gamma_{n,i} > 0$, *for each* $1 \le i \le r$,

(ii) $\lim_{n\to\infty} \theta_n = 0$ *and* $\sum_{n=0}^{\infty} \theta_n = \infty$,

(iii) for each $1 \le i \le r$, $0 < \lambda_{n,i} < \min\{\frac{2}{\|\mathcal{A}_1\|^2}, \frac{2}{\|\mathcal{A}_2\|^2}\}$ *and*

$$0 < \liminf_{n\to\infty} \lambda_{n,i} \le \limsup_{n\to\infty} \lambda_{n,i} < \min\{\frac{2}{\|\mathcal{A}_1\|^2}, \frac{2}{\|\mathcal{A}_2\|^2}\}.$$

Then, the sequences $\{x_n\}$ converges strongly to $x^\star \in \Omega$ which solves the variational inequality;

$$\langle (x^\star - hx^\star, x - x^\star\rangle \ge 0, \qquad \forall x \in \Omega.$$

Putting $f(x) = u$ and similar argument as in Theorem 3.1, we can obtain the following result.

Theorem 3.3. *Let $\mathcal{H}$ and $\mathcal{K}$ be real Hilbert spaces, $\mathcal{A}_k : \mathcal{H} \to \mathcal{K}, k = 1,2$ be two bounded linear operator and let C be a nonempty closed convex subsets in $\mathcal{H}$ and Q be a nonempty closed convex subsets in $\mathcal{K}$. Assume that GSFP has a nonempty solution set Ω. Let $u \in \mathcal{H}$ and $\{x_n\}$ be a sequence generated by $x_0 \in C$ and by*

$$x_{n+1} = \alpha_n x_n + \beta_n u + \gamma_n P_C(I - \lambda_n\mathcal{A}_1^*(I - P_Q)\mathcal{A}_1)x_n + \theta_n P_C(I - \lambda_n\mathcal{A}_2^*(I - P_Q)\mathcal{A}_2)x_n,$$

where $\alpha_n + \beta_n + \gamma_n + \theta_n = 1$ and the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\theta_n\}$ and $\{\lambda_n\}$ satisfy the following conditions:

(i) $\liminf_n \alpha_n\theta_n > 0$ *and* $\liminf_n \alpha_n\gamma_n > 0$,

(ii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$,

(iii) $0 < \lambda_n < \min\{\frac{2}{\|\mathcal{A}_1\|^2}, \frac{2}{\|\mathcal{A}_2\|^2}\}$ *and* $0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < \min\{\frac{2}{\|\mathcal{A}_1\|^2}, \frac{2}{\|\mathcal{A}_2\|^2}\}$.

Then, the sequences $\{x_n\}$ converges strongly to $P_\Omega u$.

When the point u in the above theorem is taken to be 0, we see that the limit point $x^\star$ of the sequence $\{x_n\}$ is the unique minimum norm solution of GSFP, that is,

$$\|x^\star\| = \min\{\|x\| : x \in \Omega\}$$

.

Corollary 3.4. *Let $\mathcal{H}$ and $\mathcal{K}$ be real Hilbert spaces, $\mathcal{A} : \mathcal{H} \to \mathcal{K}$ be a bounded linear operator and let $\{C_i\}_{i=1}^r$ be a family of nonempty closed convex subsets in $\mathcal{H}$ and $\{Q_i\}_{i=1}^r$ be a family of nonempty closed convex subsets in $\mathcal{K}$. Assume that MSSFP has a nonempty solution set Ω. Suppose h be a contraction of $\mathcal{H}$ into itself with constant $b \in (0, 1)$ and $\mathcal{B}$ be a strongly positive bounded linear self-adjoint operator on $\mathcal{H}$ with coefficient $\overline{\gamma}$ and $0 < \gamma < \frac{\overline{\gamma}}{b}$. Let $\{x_n\}$ be a sequence generated by $x_0 \in \mathcal{H}$ and by*

$$\begin{cases} y_n = \alpha_n x_n + \sum_{i=1}^r \beta_{n,i} P_{C_i}(I - \lambda_{n,i}\mathcal{A}^*(I - P_{Q_i})\mathcal{A})x_n, \\ x_{n+1} = \theta_n \gamma h(x_n) + (I - \theta_n \mathcal{B})y_n, \qquad \forall n \geq 0, \end{cases}$$

where $\alpha_n + \sum_{i=1}^r \beta_{n,i} = 1$ and the sequences $\{\alpha_n\}$, $\{\beta_{n,i}\}$, $\{\theta_n\}$ and $\{\lambda_{n,i}\}$ satisfy the following conditions:

(i) $\liminf_n \alpha_n \beta_{n,i} > 0$,

(ii) $\lim_{n\to\infty} \theta_n = 0$ *and* $\sum_{n=0}^\infty \theta_n = \infty$,

(iii) *for each* $1 \leq i \leq r$, $0 < \lambda_{n,i} < \frac{2}{\|\mathcal{A}\|^2}$ *and* $0 < \liminf_{n\to\infty} \lambda_{n,i} \leq \limsup_{n\to\infty} \lambda_{n,i} < \frac{2}{\|\mathcal{A}\|^2}$.

Then, the sequences $\{x_n\}$ converges strongly to $x^\star \in \Omega$ which solves the variational inequality;

$$\langle (\mathcal{B} - \gamma h)x^\star, x - x^\star \rangle \geq 0, \qquad \forall x \in \Omega.$$

Competing interests:
The authors declare that they have no competing interests.

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