On Certain Double $A$-summability Methods

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Abstract. The aim of this paper is to continue our investigations in line of our recent paper, Savas [24] and [26]. We introduce the notion of $A^I$- double statistical convergence which includes the new summability methods studied in [24] and [23] as special cases and make certain observations on this new and more general summability method.

1. Introduction

The idea of convergence of a real sequence has been extended to statistical convergence by Fast [6] and later also by Schoenberg [32] as follows: Let $K$ be a subset of $\mathbb{N}$. Then asymptotic density of $K$ is denoted by

$$\lim_{n \to \infty} \frac{1}{n} |\{k \leq n : k \in K\}|$$

where the vertical bars denoted the cardinality of the enclosed set.

A sequence $(x_k)$ of real numbers is said to be statistically convergent to $L$ if for arbitrary $\epsilon > 0$ the set $K(\epsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \epsilon\}$ has natural density zero. Statistical convergence turned out to be one of the most active areas of research in summability theory after the works of Fridy [8] and Salat [27]. More works on statistically convergence can be find from [1], [19], [30] and [33].

The notion of statistical convergence was further extended to $I$-convergence [14] using the notion of ideals of $\mathbb{N}$. Many interesting investigations using the ideals can be found in ([3], [2], [13], [15],[29], [28], [36] and [35]). In particular in [24] and [23] ideals were used to introduce new concepts of double $I$-statistical convergence, double $I$-lacunary statistical convergence and double $I_1$-statistical convergence.

Natural density was generalized by Freeman and Sember in [9] by replacing $C_1$ with a nonnegative regular summability matrix $A = (a_{m,k})$. Thus, if $K$ is a subset of $N$ then the $A$-density of $K$ is given by

$$\delta_A(K) = \lim_n \sum_{k \in K} a_{m,k}$$

if the limit exists.

On the other hand, the idea of $A$-statistical convergence was introduced by Kolk [12] using a nonnegative regular matrix $A$ (which subsequently included the ideas of statistical, lacunary statistical or $\lambda$-statistical convergence as special cases). More recent work in this line can be found in ([5],[18], [26]) and [27] where many references can be found.
In [20] the notion of convergence for double sequences was presented by A. Pringsheim. Also, in [10] and [21] the four dimensional matrix transformation $(Ax)_{m,n} = \sum_{k,l=1}^{\infty} a_{m,n,k,l} x_{k,l}$ was studied extensively by Hamilton and Robison. In their work and throughout this paper, the four dimensional matrices and double sequences have real-valued entries unless specified otherwise.

In this paper, by using the above two approaches we introduce the idea of $A^L$-double statistical convergence and make certain observations.

2. Preliminaries

Throughout the paper $\mathbb{N}$ will denote the set of all positive integers. A family $I \subset 2^\mathbb{N}$ of subsets of a nonempty set $Y$ is said to be an ideal in $Y$ if (i) $A, B \in I$ implies $A \cup B \in I$; (ii) $A \in I, B \subset A$ implies $B \in I$, while an admissible ideal $I$ of $Y$ further satisfies $|x| \in I$ for each $x \in Y$. If $I$ is a proper ideal in $Y$ (i.e. $Y \notin I, Y \neq \phi$) then the family of sets $F(I) = \{M \subset Y : \text{there exists } A \in I : M = Y \setminus A\}$ is a filter in $Y$. It is called the filter associated with the ideal $I$. Throughout $I$ will stand for a proper non-trivial admissible ideal of $\mathbb{N}$.

A sequence $\{x_{k,l}\}_{k,l=1}^{\infty}$ of real numbers is said to be $I$-convergent to $x \in \mathbb{R}$ if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - x| \geq \varepsilon\} \in I$ [14].

Before continuing with this paper we present some definitions. By the convergence in a double sequence we mean the convergence on the Pringsheim sense that is, a double sequence $(x)_{k,l}$ of complex double sequences $x_{k,l}$ of real numbers is said to be $I$-convergent to $x \in \mathbb{R}$ if for each $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $|x_{k,l} - L| < \varepsilon$ whenever $k,l > N$ [20]. We shall describe such an $x$ more briefly as "$P$-convergent".

**Definition 2.1.** Let $A = (a_{m,n,k,l})$ denote a four dimensional summability method that maps the complex double sequences $x$ into the double sequence $Ax$ where the $mn$-th term to $Ax$ is as follows:

$$(Ax)_{m,n} = \sum_{k,l=1}^{\infty} a_{m,n,k,l} x_{k,l}. $$

Such transformation is said to be non-negative if $a_{m,n,k,l}$ is nonnegative for all $m,n,k$, and $l$.

In 1926 Robison presented a four dimensional analog of the definition of regularity for double sequences in which he added an additional assumption of boundedness. This assumption was made because a double sequence which is $P$-convergent is not necessarily bounded. In addition, to this definition we also presented a Silverman-Toeplitz type characterization of the regularity of four dimensional matrices. This characterization is called the Robison-Hamilton characterization. A double sequence $x$ is bounded if and only if there exists a positive number $M$ such that $|x_{k,l}| < M$ for all $k$ and $l$.

**Definition 2.2.** The four dimensional matrix $A$ is said to be $\textbf{RH-conservative}$ if it maps every bounded $P$-convergent sequence into a $P$-convergent sequence.

**Theorem 2.1.** (Hamilton [10], Robison [21]) The four dimensional matrix $A$ is $\textbf{RH-conservative}$ if and only if

$\textbf{RH}_1$: $P\text{-lim}_{m,n} a_{m,n,k,l} = c_{k,l}$ for each $k$ and $l$;
$\textbf{RH}_2$: $P\text{-lim}_{m,n} \sum_{k,l=1}^{\infty} a_{m,n,k,l} = a$;
$\textbf{RH}_3$: $P\text{-lim}_{m,n} \sum_{k=1}^{\infty} a_{m,n,k,l} - c_{k,l} = 0$ for each $l$;
$\textbf{RH}_4$: $P\text{-lim}_{m,n} \sum_{l=1}^{\infty} a_{m,n,k,l} - c_{k,l} = 0$ for each $k$;
$\textbf{RH}_5$: $\sum_{k,l=1}^{\infty} |a_{m,n,k,l}| < A$ for all $(m,n)$; and
$\textbf{RH}_6$: there exists finite positive integers $A$ and $B$ such that $\sum_{k,l=1}^{B} |a_{m,n,k,l}| < A$. 
When these conditions are satisfied, we have

\[ P \lim_{m,n} Y_{m,n} = \mu (a - \sum_{k,l} c_{k,l} x_{k,l}) + \sum_{k,l} c_{k,l} x_{k,l} \]

where \( \mu = P \lim_{k,l} x_{k,l} \), the double series \( \sum_{k,l} c_{k,l} (x_{k,l} - \mu) \) is always absolutely \( P \)-convergent.

**Definition 2.3.** The four dimensional matrix \( A \) is said to be RH-\textit{regular} if it maps every bounded \( P \)-convergent sequence into a \( P \)-convergent sequence with the same \( P \)-limit.

**Theorem 2.2.** (Hamilton [10], Robison [21]) The four dimensional matrix \( A \) is RH-regular if and only if

1. RH1: \( P \lim_{m,n} a_{m,n,k,l} = 0 \) for each \( k \) and \( l \);
2. RH2: \( P \lim_{m,n} \sum_{k=1}^{\infty} a_{m,n,k,l} = 1 \);
3. RH3: \( P \lim_{m,n} \sum_{k=1}^{\infty} \left| a_{m,n,k,l} \right| = 0 \) for each \( l \);
4. RH4: \( P \lim_{m,n} \sum_{l=1}^{\infty} \left| a_{m,n,k,l} \right| = 0 \) for each \( k \);
5. RH5: \( \sum_{k,l=1}^{\infty} |a_{m,n,k,l}| \) is \( P \)-convergent; and
6. RH6: there exist finite positive integers \( A \) and \( B \) such that

\[ \sum_{k,l} a_{m,n,k,l} < A. \]

Let \( K \subset N \times N \) be a two-dimensional set to positive integers and let \( K(m,n) \) be the numbers of \((i,j)\) in \( K \) such that \( i \leq n \) and \( j \leq M \). The two-dimensional analogues of natural density can be defined as follows: The lower asymptotic density of a set \( K \subset N \times N \) is defined as

\[ \delta^2(K) = \lim \inf_{m,n} \frac{K(m,n)}{mn}. \]

In case the double sequence \( \frac{K(m,n)}{mn} \) has a limit in the Pringsheim sense then we say that \( K \) has a double natural density as

\[ P \lim_{m,n} \frac{K(m,n)}{mn} = \delta^2(K). \]

Let \( K \subset N \times N \) be a two-dimensional set of positive integers, then the \( A \)-density of \( K \) is given by

\[ \delta^2_A(K) = P \lim_{m,n} \sum_{(i,j) \in K} a_{m,n,k,l} \]

provided that the limit exists. The notion of double asymptotic density for double sequence was presented by Mursaleen and Edely [18] and Tripathy [34] independently as follows:

A real double sequence \( x = (x_{k,l}) \) is said to be \( P \)-statistically convergent to \( L \) provided that for each \( \epsilon > 0 \)

\[ P \lim_{m,n} \frac{1}{mn} |(k,l) : k < m \text{ and } k < n, |x_{k,l} - L| \geq \epsilon | = 0. \]

In this case we write \( S_{\ell_2} \)-\( \lim_{m,n} x_{k,l} = L \) and denote the set of all statistical convergent double sequences by \( S_{\ell_2} \). It is clear that a convergent double sequence is also \( S_{\ell_2} \)-convergent but the converse is not true, in general. Also \( S_{\ell_2} \)-convergent double sequence need not be bounded.

Throughout \( \epsilon \) will denote a sequence all of whose elements are 1. Also as usual,

\[ \ell_\infty = \left\{ x = (x_{k,l}) : \|x\|_\infty = \sup_{k,l} |x_{k,l}| < \infty \right\}. \]
3. Main Results

Now we introduce the main concept of this paper, namely the notion of $A^I_2$-statistical convergence.

**Definition 3.1.** Let $A = (a_{m,n,k,l})$ be a non-negative RH-regular four dimensional matrix. A sequence $(x_{k,l})$ is said to be $A^I$-double statistically convergent to $L$ if for any $\varepsilon > 0$ and $\delta > 0$,

$$\left\{ m, n \in \mathbb{N} \times \mathbb{N} : \sum_{k,l \in K_2(\varepsilon \to \infty)} a_{m,n,k,l} \geq \delta \right\} \in I$$

where

$$K_2(x - Le, \varepsilon) = \left\{ k, l \in \mathbb{N} \times \mathbb{N} : |x_{k,l} - L| \geq \varepsilon \right\}.$$  In this case we write $x_{k,l} \xrightarrow{A^I_-st} L$. We denote the class of all $A^I_2$-statistically convergent sequences by $S^2_A(I)$.

1. If we take $A = (C, 1, 1)$, i.e., the double Cesàro matrix then $A^I_2$-statistical convergence becomes $I$-double statistical convergence [23].

2. Let us consider the following notations and definitions. The double sequence $\theta_{rs} = \{(k_r, l_s)\}$ is called double lacunary if there exist two increasing sequences of integers such that

$$k_0 = 0, h_s = k_s - k_{s-1} \to \infty \text{ as } r \to \infty,$$

$$l_0 = 0, h_s = l_s - l_{s-1} \to \infty \text{ as } s \to \infty,$$

and let $\theta_{rs} = A_{rs} \theta_{rs}$ is determine by $I_{rs} = \{(i, j) : k_{r-1} < i \leq k_r \land l_{s-1} < j \leq l_s\}$. If we take

$$a_{rs,kl} = \begin{cases} \frac{1}{h_s}, & \text{if } (k, l) \in I_{rs}; \\ 0, & \text{otherwise.} \end{cases}$$

then $A^I_2$-statistical convergence coincides with $I$-double lacunary statistical convergence [23].

4. As a final illustration let

$$a_{ij,kl} = \begin{cases} \frac{1}{\lambda_i}, & \text{if } k \in I_i = [i - \lambda_i + 1, i] \text{ and } l \in L_j = [j - \lambda_j + 1, j] \\ 0, & \text{otherwise} \end{cases}$$

where we shall denote $\lambda_{ij}$ by $\lambda_i \mu_j$. Let $\lambda = (\lambda_i)$ and $\mu = (\mu_j)$ be two non-decreasing sequences of positive real numbers such that each tending to $\infty$ and $\lambda_{i+1} \leq \lambda_i + 1, \lambda_1 = 0$ and $\mu_{j+1} \leq \mu_j + 1, \mu_1 = 0$. Then $A^I_2$-statistical convergence coincides with $I_1$-double statistical convergence [24].

Non-trivial examples of such sequences can be seen from [24], [23].

Also note that for $I = I_{m,n}$, $A^I_2$-statistical convergence becomes $A$-double statistical convergence [25].

We now prove the following result which establishes the topological character of the space $S^2_A(I)$.

**Theorem 3.1.** $S^2_A(l) \cap \ell_{\infty}$ is a closed subset of $\ell_{\infty}$ endowed with the superior norm.

**Proof.** Suppose that $(x^{mn}) \subset S^2_A(l) \cap \ell_{\infty}$ is a convergent sequence and it converges to $x \in \ell_{\infty}$. We have to show that $x \in S^2_A(l) \cap \ell_{\infty}$. Let $x^{mn} \xrightarrow{A^I_-st} L_{mn}$ for all $(m, n) \in \mathbb{N} \times \mathbb{N}$. Take a sequence $(\varepsilon_{mn})$ where $\varepsilon_{mn} = \frac{1}{4N_{mn}} \forall (m, n) \in \mathbb{N} \times \mathbb{N}$. We can find a positive integer $N_{mn}$ such that $||x - x^{mn}||_\infty < \frac{\varepsilon_{mn}}{4}, \forall mn \geq N_{mn}$. Choose $0 < \delta < \frac{1}{4}$. Now
shall prove that forms a closed linear subspace of for any given δ > M.

Let Let A completes the proof of the result.

We can say that the set of all bounded Remark 1: A A -summable to L if the sequence \( (a_{mn}) \) is A -summable reduces to statistical double convergence. We define another related summability method and establish its relation with A -summability, [5].

**Definition 3.2.** Let A = \( (a_{mn}) \) be a non-negative RH-regular four dimensional matrix. Then we say that x is A -summable to L if the sequence \( (a_{mn}(x)) \) L -converges to L.

For I = I M, A -summability reduces to statistical double A-summability, [5].
Theorem 3.2. If a sequence is bounded and $A_2^I$-statistically convergent to $L$ then it is $A_2^I$-summable to $L$.

Proof. Let $x = (x_{kl})$ be bounded and $A_2^I$-statistically convergent to $L$ and for $\varepsilon > 0$, let as before $K_2(\frac{\varepsilon}{2}) := \{(k,l) \in \mathbb{N} \times \mathbb{N} : |x_{kl} - L| \geq \frac{\varepsilon}{2}\}$. Then

$$|A_{mn}(x) - L| \leq \left| \sum_{(k,l) \in K_2(\frac{\varepsilon}{2})} a_{mnkl} (x_{kl} - L) \right| + \left| \sum_{(k,l) \in K_2(\frac{\varepsilon}{2})} a_{mnkl} (x_{kl} - L) \right| \leq \frac{\varepsilon}{2} \sum_{k,l \in K_2(\frac{\varepsilon}{2})} a_{mnkl} + \sum_{k,l \in K_2(\frac{\varepsilon}{2})} a_{mnkl} |x_{kl} - L| \leq \frac{\varepsilon}{2} + B \sum_{k,l \in K_2(\frac{\varepsilon}{2})} a_{mnkl},$$

where $B = \sup_{x_{kl} \in x} |x_{kl} - L|$. It now follows that

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : |A_{mn}(x) - L| \geq \varepsilon\} \subset \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \sum_{k,l \in K_2(\frac{\varepsilon}{2})} a_{mnkl} \geq \frac{\varepsilon}{2B}\right\}.$$

Since $x$ is $A_2^I$-statistically convergent to $L$, so the set on the right hand side belongs to $I$ and this consequently implies that $x$ is $A_2^I$-summable to $L$. \hfill \Box

The converse of the above result is not generally true.

Example 2. If $A = (a_{mnkl}) = (C,1,1)$, double Cesàro matrix and let

$$x_{kl} = \begin{cases} 1 & \text{if } k,l \text{ are odd} \\ 0 & \text{if } k,l \text{ are even}. \end{cases}$$

Then $x = (x_{kl})$ is $A_2$-summable to $1/2$ and so is $A_2^I$-summable to $1/2$ for any admissible ideal $I$. But note that for any $L \in \mathbb{R}$ and for $0 < \varepsilon < \frac{1}{2}$, $K_2(\varepsilon) = \{(k,l) \in \mathbb{N} \times \mathbb{N} : |x_{kl} - L| \geq \varepsilon\}$ contains either the set of all even integers or the set of all odd integers or both. Consequently $\sum_{k,l \in K_2(\varepsilon)} a_{mnkl} = \infty$ for any $(k,l) \in \mathbb{N} \times \mathbb{N}$ and so for any $\delta > 0$

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \sum_{k,l \in K_2(\varepsilon)} a_{mnkl} \geq \delta\right\} \notin I.$$

This shows that $x = (x_{kl})$ is not $A_2^I$-statistically convergent for any non-trivial ideal $I$.

We conclude this paper with the following theorem which shall give that continuity preserves the $A_2^I$-statistical convergence.

Theorem 3.3. If for a sequence $x = (x_{kl})$, $x_{kl} \xrightarrow{A_2^I-\text{st}} L$ and $g$ is a real valued function which is continuous then $g(x_{kl}) \xrightarrow{A_2^I-\text{st}} g(L)$.

Proof. The proof can be established using standard techniques, so omitted. \hfill \Box

References