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Total Quantum Integrals of Weak Hopf Group Coalgebras

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Abstract. In this paper, we introduce the concept of total quantum integrals in the case of weak crossed Hopf π-coalgebras and prove the affineness criterion for weak quantum Yetter-Drinfel'd π-modules, which is a generalization of the total quantum integral introduced by Chen and Wang(Filomat 26: 101-118, (2012)).

1. Introduction

The integrals for a Hopf algebra H and the more general ones were introduced by Doi [6], stating that the existence of an integral is the necessary and sufficient condition for the existence of a natural transformation between two functors linking the categories of relative Hopf-modules $\mathcal{M}_A^H$ and right H-comodules $\mathcal{M}^H$. The categorical point of view towards integrals associated to a Doi-Koppinen datum (H, A, C) was introduced by Caenepeel et al. [4] to prove separability theorems. In [8], the authors weakened the conditions for a total A-integral in the sense of Caenepeel. The integrals cover the integrals introduced by Doi[6] and the classic integral by Larson and Sweedler[7]. As a major application, the quantum integrals associated to quantum Yetter-Drinfel'd modules ${}^H\mathcal{YD}_A$ were introduced.

In [10] Turaev introduced, for a group π, the notion of a Hopf π-coalgebra, which can induce a π-category, i.e., group-category, and showed that such a category gives rise to a three-dimensional homotopy quantum field theory with target space $K(\pi, 1)$. Van Daele and Wang [11] introduced the definition of weak Hopf group coalgebras and generalized many of the properties of weak Hopf algebras in Böhm, Nill and Szláchanyi [2] to the setting of weak Hopf group coalgebras.

The main purpose of this paper is to define the more general concept of an integral associated to weak quantum Yetter-Drinfel'd π-modules, which generalizes the total integral introduced by Chen and Wang in T-coalgebras setting.

2. Preliminaries

We always work over a fixed field k and follow Sweedler's book [9] for the terminologies on coalgebras and comodules. Let C be a coalgebra with a coproduct Δ. We will use the Heyneman-Sweedler's notation, $\Delta(c) = c_1 \otimes c_2$ for all $c \in C$, for coproduct (summation understood). All algebras, linear spaces etc. will be over k. If a tensor product is written without index, then it is assumed to be taken over k.

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2.1 π-coalgebras.

A *π- coalgebra* is a family of k-spaces $C = \{C_\alpha\}_{\alpha\in\pi}$ together with a family of k-linear maps $\Delta = \{\Delta_{\alpha,\beta} : C_{\alpha\beta} \to C_\alpha \otimes C_\beta\}_{\alpha,\beta\in\pi}$ (called a *comultiplication*) and a k-linear map $\varepsilon : C_e \to k$ (called a *counit*) such that Δ is coassociative in the sense that

$$(\Delta_{\alpha,\beta} \otimes id_{C\gamma}) \circ \Delta_{\alpha\beta,\gamma} = (id_{C_\alpha} \otimes \Delta_{\beta,\gamma}) \circ \Delta_{\alpha,\beta\gamma},$$

for any $\alpha, \beta, \gamma \in \pi$ and

$$(id_{C_\alpha} \otimes \varepsilon) \circ \Delta_{\alpha,e} = id_{C_\alpha} = (\varepsilon \otimes id_{C_\alpha}) \circ \Delta_{e,\alpha},$$

for all $\alpha \in \pi$.

Remark. $(C_e, \Delta_{e,e}, \varepsilon)$ is an ordinary coalgebra in the sense of Sweedler. Following the Sweedler$'$s notation for π-coalgebras, for any $\alpha, \beta \in \pi$ and $c \in C_{\alpha\beta}$, one write

$$\Delta_{\alpha,\beta}(c) = c_{(1,\alpha)} \otimes c_{(2,\beta)}.$$

The coassociativity axiom gives that, for any $\alpha, \beta, \gamma \in \pi$ and $c \in C_{\alpha\beta\gamma}$,

$$c_{(1,\alpha\beta)(1,\alpha)} \otimes c_{(1,\alpha\beta)(2,\beta)} \otimes c_{(2,\gamma)} = c_{(1,\alpha)} \otimes c_{(2,\beta\gamma)(1,\beta)} \otimes c_{(2,\beta\gamma)(2,\gamma)},$$

which is written as $c_{(1,\alpha)} \otimes c_{(2,\beta)} \otimes c_{(3,\gamma)}$. Inductively, we can define $c_{(1,\alpha_1)} \otimes c_{(2,\alpha_2)} \otimes \cdots c_{(n,\alpha_n)}$, for any $c \in C_{\alpha_1\alpha_2\cdots\alpha_n}$. The counit axiom gives that, for any $\alpha \in \pi$ and $c \in C_\alpha$,

$$\varepsilon(c_{(1,e)})c_{(2,\alpha)} = c = c_{(1,\alpha)}\varepsilon(c_{(2,e)}).$$

2.2 π-C-comodules.

Let $C = \{C_\alpha\}_{\alpha\in\pi}$ be a π-coalgebra and a family of k-vector spaces $V = \{V_\alpha\}_{\alpha\in\pi}$. A right *$\pi$-C-comodules* is a couple $(V, \rho^V = \{\rho^V_{\alpha,\beta}\}_{\alpha,\beta\in\pi})$, where for any $\alpha, \beta \in \pi$, $\rho^V_{\alpha,\beta} : V_{\alpha\beta} \to V_\alpha \otimes C_\beta$ is a k-linear morphism, which will be called a π-comodule structure and denoted by $\rho^V_{\alpha,\beta}(v) = v_{(0,\alpha)} \otimes v_{(1,\beta)}$, satisfying the following conditions:

♦ ρ^V is coassociative in the sense that, for any $\alpha, \beta, \gamma \in \pi$, we have

$$(\rho^V_{\alpha,\beta} \otimes id_{C_\gamma}) \circ \rho^V_{\alpha\beta,\gamma} = (id_{V_\alpha} \otimes \Delta_{\beta,\gamma}) \circ \rho^V_{\alpha,\beta\gamma},$$

♦ V is counitary in the sense that

$$(\varepsilon \otimes id_{V_\alpha}) \circ \rho^V_{e,\alpha} = id_{V_\alpha}.$$

2.3 Weak Hopf π-coalgebras.

We recall from Van Daele and Wang [11] that a *weak semi-Hopf π-coalgebra* $H = \{H_\alpha, \mu_\alpha, 1_\alpha, \Delta, \varepsilon\}_{\alpha\in\pi}$ is a family of algebras $\{H_\alpha, \mu_\alpha, 1_\alpha\}_{\alpha\in\pi}$ and at the same time a π-coalgebra $\{H_\alpha, \Delta = \{\Delta_{\alpha,\beta}\}, \varepsilon\}_{\alpha,\beta\in\pi}$ such that

◊ The comultiplication $\Delta_{\alpha,\beta} : H_{\alpha\beta} \to H_\alpha \otimes H_\beta$ is a homomorphism of algebras (not necessary unit-preserving) such that

$$(\Delta_{\alpha,\beta} \otimes id_{H_\alpha})\Delta_{\alpha\beta,\gamma}(1_{\alpha\beta\gamma}) = (\Delta_{\alpha,\beta} \otimes 1_\gamma)(1_\alpha \otimes \Delta_{\beta,\gamma}(1_{\beta\gamma})),$$

$$(\Delta_{\alpha,\beta} \otimes id_{H_\alpha})\Delta_{\alpha\beta,\gamma}(1_{\alpha\beta\gamma}) = (1_\alpha \otimes \Delta_{\beta,\gamma}(1_{\beta\gamma}))(\Delta_{\alpha,\beta}(1_{\alpha,\beta}) \otimes 1_\gamma),$$

for all $\alpha, \beta, \gamma \in \pi$.

◊ The counit $\varepsilon : H_e \to k$ is a k-linear map satisfying the identity

$$\varepsilon(gxh) = \varepsilon(gx_{(2,e)})\varepsilon(x_{(1,e)}h) = \varepsilon(gx_{(1,e)})\varepsilon(x_{(2,e)}h),$$

for all $g, h, x \in H_e$.

A *weak Hopf π-coalgebra* is a weak semi-Hopf π-coalgebra $H = \{H_\alpha, \mu_\alpha, 1_\alpha, \Delta, \varepsilon\}_{\alpha\in\pi}$ endowed with a family of k-linear maps $S = \{S_\alpha : H_\alpha \to H_{\alpha^{-1}}\}$ (called *antipode*) such that the following data hold:

$$\mu_\alpha(S_{\alpha^{-1}} \otimes id_{H_\alpha})\Delta_{\alpha^{-1},\alpha}(h) = 1_{(1,\alpha)}\varepsilon(h1_{(2,e)}),$$

$$\mu_\alpha(id_{H_\alpha} \otimes S_{\alpha^{-1}})\Delta_{\alpha,\alpha^{-1}}(h) = \varepsilon(1_{(1,e)}h)1_{(2,\alpha)},$$
$$S_\alpha(g_{(1,\alpha)})g_{(2,\alpha^{-1})}S_\alpha(g_{(3,\alpha)}) = S_\alpha(g),$$

for all $h \in H_e$, $g \in H_\alpha$ and $\alpha \in \pi$.

Let H be a weak Hopf π-coalgebra. Define the family of linear maps $\varepsilon^t = \{\varepsilon^t_\alpha : H_e \to H_\alpha\}_{\alpha\in\pi}$ and $\varepsilon^s = \{\varepsilon^s_\alpha : H_e \to H_\alpha\}_{\alpha\in\pi}$ by the formulars

$$\varepsilon^s_\alpha(h) = \mu_\alpha(S_{\alpha^{-1}} \otimes id_{H_\alpha})\Delta_{\alpha^{-1},\alpha}(h) = 1_{(1,\alpha)}\varepsilon(h1_{(2,e)}),$$

$$\varepsilon^t_\alpha(h) = \mu_\alpha(id_{H_\alpha} \otimes S_{\alpha^{-1}})\Delta_{\alpha,\alpha^{-1}}(h) = \varepsilon(1_{(1,e)}h)1_{(2,\alpha)},$$

for any $h \in H_e$ and $\alpha \in \pi$, where $\varepsilon^t, \varepsilon^s$ are called the π-target and π-source counital maps. Introduce the notations $H^t = \varepsilon^t(H) = \{\varepsilon^t_\alpha(H_e)\}_{\alpha\in\pi}$ and $H^s = \varepsilon^s(H) = \{\varepsilon^s_\alpha(H_e)\}_{\alpha\in\pi}$ for their images.

A weak Hopf π-coalgebra $H = \{H_\alpha, m_\alpha, 1_\alpha, \Delta, \varepsilon\}_{\alpha\in\pi}$ is called a *weak crossed Hopf π-coalgebra* if it is endowed with a family of algebra isomorphisms $\phi = \{\phi_\alpha : H_\beta \to H_{\alpha\beta\alpha^{-1}}\}_{\alpha,\beta\in\pi}$ such that $(\phi_\alpha \otimes \phi_\alpha)\Delta_{\beta,\gamma} = \Delta_{\alpha\beta\alpha^{-1},\alpha\gamma\alpha^{-1}}\phi_\alpha$, $\varepsilon\phi_\alpha = \varepsilon$ and $\phi_{\alpha\beta} = \phi_\alpha\phi_\beta$ for all $\alpha, \beta \in \pi$.

Let H be a weak Hopf π-coalgebra. Then we have the following properties, we refer to [11] for full detail.

(W1) $\Delta_{\alpha,\beta}(1_{\alpha\beta}) \in H^s_\alpha \otimes H^t_\alpha$, for all $\alpha, \beta \in \pi$;
(W2) $\varepsilon^t_\alpha(gh) = \varepsilon^t_\alpha(g\varepsilon^t_e(h))$, $\varepsilon^t_\alpha(\varepsilon^t_e(g)h) = \varepsilon^t_\alpha(g)\varepsilon^t_\alpha(h)$, for all $g, h \in H_e$;
(W3) $\Delta_{\alpha,\beta}(H^t_{\alpha\beta}) \subseteq H_\alpha \otimes H^t_\beta$, $\Delta_{\alpha,\beta}(H^s_{\alpha\beta}) \subseteq H^s_\alpha \otimes H_\beta$;
(W4) $x_{(1,\alpha)} \otimes \varepsilon^t_\beta(x_{(2,e)}) = 1_{(1,\alpha)}x \otimes 1_{(2,\beta)}$, for all $\alpha, \beta \in \pi$ and $x \in H_e$;
(W5) $\varepsilon^s_\beta(x_{(1,e)}) \otimes x_{(2,\alpha)} = 1_{(1,\beta)} \otimes x1_{(2,\alpha)}$, for all $\alpha, \beta \in \pi$ and $x \in H_e$;
(W6) $\varepsilon^t_\alpha(h)\varepsilon^s_\alpha(g) = \varepsilon^s_\alpha(g)\varepsilon^t_\alpha(h)$, for all $g, h \in H_e$;
(W7) $S_\alpha(xy) = S_\alpha(y)S_\alpha(x)$, for all $\alpha \in \pi$ and $x, y \in H_\alpha$;
(W8) $S_\alpha(1_\alpha) = 1_{\alpha^{-1}}$, for all $\alpha \in \pi$;
(W9) $\Delta_{\beta^{-1},\alpha^{-1}} \circ S_{\alpha\beta} = T_{H_{\alpha^{-1}},H_{\beta^{-1}}} \circ (S_\alpha \otimes S_\beta) \circ \Delta_{\alpha,\beta}$, for all $\alpha, \beta \in \pi$;
(W10) $\varepsilon^t_\alpha \circ S_e = \varepsilon^t_\alpha \circ \varepsilon^s_e = S_{\alpha^{-1}} \circ \varepsilon^s_{\alpha^{-1}}$;
(W11) $x_{(1,\alpha)} \otimes \varepsilon^s_\beta(x_{(2,e)}) = x1_{(1,\alpha)} \otimes S_{\beta^{-1}}(1_{(2,\beta^{-1})})$, for all $\alpha, \beta \in \pi$ and $x \in H_\alpha$;
(W12) $\varepsilon^t_\beta(x_{(1,e)}) \otimes x_{(2,\alpha)} = S_{\beta^{-1}}(1_{(1,\beta^{-1})}) \otimes 1_{(2,\alpha)}x$, for all $\alpha, \beta \in \pi$ and $x \in H_\alpha$;
(W13) If H is of finite type, then the antipode S is bijective.

2.4 Weak Left π-H-comodule Algebra.

Let H be a weak Hopf π-coalgebra over the field k. A family of k-algebra $A = \{A_\alpha, m_\alpha, 1_\alpha\}_{\alpha\in\pi}$ is called a *weak left π-H-comodule algebra*, if there exists a family of maps $\rho^A = \{\rho^A_{\alpha,\beta} : A_{\alpha\beta} \to H_\alpha \otimes A_\beta\}$ such that, for all $\alpha, \beta \in \pi$,

(1) (A, ρ^A) is a left π-H-comodule,
(2) $\rho^A_{\alpha,\beta}(1_{\alpha\beta}) = (\varepsilon^t_\alpha \otimes id_{A_\beta}) \circ \rho^A_{e,\beta}(1_\beta)$,
(3) $\rho^A_{\alpha,\beta}(ab) = \rho^A_{\alpha,\beta}(a)\rho^A_{\alpha,\beta}(b)$, for all $a, b \in A_{\alpha\beta}$.

We use the standard notation $\rho^A_{\alpha\beta}(a) = a_{<-1,\alpha>} \otimes a_{<0,\beta>}$.

3. Weak Quantum Yetter-Drinfel′d π-modules

Let H be a weak Hopf π-coalgebra. A *weak Doi-Hopf π-datum* is a triple (H, A, C), where A is a weak left π-H-comodule algebra and C a weak right π-H-module coalgebra.

A *weak Doi-Hopf π-modules* M is a right A-module which is also a left π-C-comodule with the coaction structure $\rho^M = \{\rho^M_{\alpha,\beta} : M_{\alpha\beta} \to C_\alpha \otimes M_\beta\}_{\alpha,\beta\in\pi}$ such that the following compatible condition holds:

$$\rho^M_{\alpha,\beta}(m \cdot a) = m_{<-1,\alpha>} \cdot a_{<-1,\alpha>} \otimes m_{<0,\beta>} \cdot a_{<0,\beta>},$$

for all $\alpha \in \pi$ and $m \in M_{\alpha\beta}, a \in A_{\alpha\beta}$.

The set of weak Doi-Hopf π-modules together with both a right A-module maps and a left π-C-comodule maps will form a category of weak Doi-Hopf π-modules and will be denoted by ${}^{\pi-C}\mathcal{M}_A$ (called a *weak Doi-Hopf π-modules category*).

Definition 3.1. Let H be a weak Hopf π-coalgebra and A an algebra. The algebra A is called a weak π-H-bicomodule algebra, if A is not only a right weak π-H-comodule algebra $(A, {}^r\rho^A = \{{}^r\rho^A_\alpha\}_{\alpha\in\pi})$, but also left weak π-H-comodule algebra $(A, {}^l\rho^A = \{{}^l\rho^A_\alpha\}_{\alpha\in\pi})$ such that the following condition:

$$a_{<-1,\alpha>} \otimes a_{<0,0>(0,0)} \otimes a_{<0,0>(1,\beta)} = a_{(0,0)<-1,\alpha>} \otimes a_{(0,0)<0,0>} \otimes a_{(1,\beta)} \tag{3. 1}$$

for any $\alpha, \beta \in \pi$ and $a \in A$, where we use the standard notation ${}^r\rho^A_\beta(a) = a_{(0,0)} \otimes a_{(1,\beta)}$

Definition 3.2. Let H be a weak crossed Hopf π-coalgebra and A a weak π-H-bicomodule algebra. A weak quantum Yetter-Drinfel'd π-modules M is both a right A-module and a left H-comodule with a comodule structure $\rho^M = \{\rho^M_\beta : M \to H_\beta \boxdot M\}_{\beta\in\pi}$ where $H \boxdot M = \{h1_{<-1,\beta>} \otimes m \cdot 1_{<0,0>} | h \in H_\beta, m \in M\}$ such that the following compatible condition holds:

$$m_{<-1,\beta>}a_{<-1,\beta>} \otimes m_{<0,0>} \cdot a_{<0,0>} = \phi_\alpha(a_{(1,\alpha^{-1}\beta\alpha)})(m \cdot a_{(0,0)})_{<-1,\beta>} \otimes (m \cdot a_{(0,0)})_{<0,0>}. \tag{3. 2}$$

Lemma 3.3. Let A be a weak π-H-bicomodule algebra. For all $a \in A$ and $\alpha, \beta \in \pi$, we have

$$\varepsilon^\alpha_t(a_{(1,1)}) \otimes a_{(0,0)} = 1_{(1,\alpha)} \otimes 1_{(0,0)}a, \tag{3. 3}$$

$$\varepsilon^\alpha_s(a_{(1,1)}) \otimes a_{(0,0)} = S_{\alpha^{-1}}(1_{(1,\alpha^{-1})}) \otimes a1_{(0,0)}, \tag{3. 4}$$

$$1_{<-1,\alpha>} \otimes 1_{<0,0>(0,0)} \otimes 1_{<0,0>(1,\beta)} = 1_{<-1,\alpha>} \otimes 1_{(0,0)}1_{<0,0>} \otimes 1_{(1,\beta)}. \tag{3. 5}$$

Now, we can form the category ${}^H\mathcal{YD}^\alpha_A$ of weak quantum Yetter-Drinfel'd π-modules for a fixed $\alpha \in \pi$ in which the composition of morphism of weak quantum Yetter-Drinfel'd π-modules is the standard composition of the underlying linear maps.

Lemma 3.4. Let M be a left A-module and a left π-H-comodule. Then the compatibility relation (3.2) is equivalent to

$$\rho^M_\beta(m \cdot a) = S^{-1}_{\beta^{-1}}\phi_\alpha(a_{(1,\alpha^{-1}\beta^{-1}\alpha)})m_{<-1,\beta>}a_{(0,0)<-1,\beta>} \otimes m_{<0,0>} \cdot a_{(0,0)<0,0>}. \tag{3. 6}$$

Proof. Assume first that Eq.(3.6) holds. Then for $a \in A, m \in M$,

$$\begin{aligned}
&\phi_\alpha(a_{(1,\alpha^{-1}\beta\alpha)})(m \cdot a_{(0,0)})_{<-1,\beta>} \otimes (m \cdot a_{(0,0)})_{<0,0>}\\
&\overset{(3.6)}{=} \phi_\alpha(a_{(1,\alpha^{-1}\beta\alpha)})S^{-1}_{\beta^{-1}}\phi_\alpha(a_{(0,0)(1,\alpha^{-1}\beta^{-1}\alpha)})\\
&\qquad\qquad m_{<-1,\beta>}a_{(0,0)(0,0)<-1,\beta>} \otimes m_{<0,0>} \cdot a_{(0,0)(0,0)<0,0>}\\
&= \phi_\alpha(a_{(1,1)(2,\alpha^{-1}\beta\alpha)})S^{-1}_{\beta^{-1}}\phi_\alpha(a_{(1,1)(1,\alpha^{-1}\beta^{-1}\alpha)})\\
&\qquad\qquad m_{<-1,\beta>}a_{(0,0)<-1,\beta>} \otimes m_{<0,0>} \cdot a_{(0,0)<0,0>}\\
&= S^{-1}_{\beta^{-1}}(\varepsilon^{\beta^{-1}}_t(a_{(1,1)}))m_{<-1,\beta>}a_{(0,0)<-1,\beta>} \otimes m_{<0,0>} \cdot a_{(0,0)<0,0>}\\
&\overset{(3.3)}{=} S^{-1}_{\beta^{-1}}(1_{(1,\beta^{-1})})m_{<-1,\beta>}(1_{(0,0)}a)_{<-1,\beta>} \otimes m_{<0,0>} \cdot (1_{(0,0)}a)_{<0,0>}\\
&= S^{-1}_{\beta^{-1}}(1_{(1,\beta^{-1})})m_{<-1,\beta>}1_{(0,0)<-1,\beta>}a_{<-1,\beta>} \otimes m_{<0,0>} \cdot 1_{(0,0)<0,0>}a_{<0,0>}\\
&\overset{(3.6)}{=} m_{<-1,\beta>}a_{<-1,\beta>} \otimes m_{<0,0>} \cdot a_{<0,0>}.
\end{aligned}$$

Conversely, if Eq.(3.2) holds, then

$$S^{-1}_{\beta^{-1}}\phi_\alpha(a_{(1,\alpha^{-1}\beta^{-1}\alpha)})m_{<-1,\beta>}a_{(0,0)<-1,\beta>}\otimes m_{<0,0>}\cdot a_{(0,0)<0,0>}$$
$$\overset{(3.2)}{=} S^{-1}_{\beta^{-1}}\phi_\alpha(a_{(1,\alpha^{-1}\beta^{-1}\alpha)})\phi_\alpha(a_{(0,0)(1,\alpha^{-1}\beta\alpha)})(m\cdot a_{(0,0)(0,0)})_{<-1,\beta>}\otimes(m\cdot a_{(0,0)(0,0)})_{<0,0>}$$
$$= S^{-1}_{\beta^{-1}}\phi_\alpha(a_{(1,1)(2,\alpha^{-1}\beta^{-1}\alpha)})\phi_\alpha(a_{(1,1)(1,\alpha^{-1}\beta\alpha)})(m\cdot a_{(0,0)})_{<-1,\beta>}\otimes(m\cdot a_{(0,0)})_{<0,0>}$$
$$= S^{-1}_{\beta^{-1}}(\varepsilon_s^{\beta^{-1}}(a_{(1,1)}))(m\cdot a_{(0,0)})_{<-1,\beta>}\otimes(m\cdot a_{(0,0)})_{<0,0>}$$
$$\overset{(3.4)}{=} 1_{(1,\beta)}(m\cdot a1_{(0,0)})_{<-1,\beta>}\otimes(m\cdot a1_{(0,0)})_{<0,0>}$$
$$\overset{(3.2)}{=} (m\cdot a)_{<-1,\beta>}1_{<-1,\beta>}\otimes(m\cdot a)_{<0,0>}\cdot 1_{<0,0>}$$
$$= (m\cdot a)_{<-1,\beta>}\otimes(m\cdot a)_{<0,0>}$$
$$= \rho^M_\beta(m\cdot a).$$

This completes the proof. ■

An important object of ${}^H\mathcal{YD}^\alpha_A$ is the Verma structure $(A,\cdot,\tilde{\rho})$, where $\cdot$ is the multiplication on A and the left H-coaction $\tilde{\rho}$ is given by

$$\tilde{\rho}: A\to H_\beta\otimes A,\ \tilde{\rho}(a)=S^{-1}_{\beta^{-1}}\phi_\alpha(a_{(1,\alpha^{-1}\beta^{-1}\alpha)})a_{(0,0)<-1,\beta>}\otimes a_{(0,0)<0,0>}$$

for all $a\in A$ and $\beta\in\pi$.

Proposition 3.5. Let H be a weak crossed Hopf π-coalgebra. Let us fix $\alpha\in\pi$. Then we view now ${}^H\mathcal{YD}^\alpha_A$ as the category of weak Doi-Hopf π-modules associated to the weak Doi-Hopf π-datum $(H\otimes H^{op}, A, H)$, where

(1) A is a weak left π-$H\otimes H^{op}$-comodule algebra via

$$a\to(a_{<-1,\gamma>}\otimes S^{-1}_{\gamma^{-1}}\phi_\alpha(a_{<0,0>(1,\alpha^{-1}\gamma^{-1}\alpha)})\otimes a_{<0,0>(0,0)},$$

for all $a\in A$ and $\gamma\in\pi$.

(2) H is a weak right π-$H\otimes H^{op}$-module coalgebra via

$$g\cdot(h\otimes l)=lgh,$$

for all $g,h\in H_\gamma$ and $l\in H^{op}_\gamma$. Then ${}^H\mathcal{YD}^\alpha_A={}^{\pi-H}\mathcal{M}(H\otimes H^{op})_A$. Note that $H_\gamma\boxtimes A\in{}^H\mathcal{YD}^\alpha_A$ via the following structures

$$(S^{-1}_{\gamma^{-1}}\phi_\alpha(1_{(1,\alpha^{-1}\gamma^{-1}\alpha)})h1_{(0,0)<-1,\gamma>}\otimes a1_{(0,0)<0,0>})\cdot b$$
$$= S^{-1}_{\gamma^{-1}}\phi_\alpha(b_{(1,\alpha^{-1}\gamma^{-1}\alpha)})hb_{(0,0)<-1,\gamma>}\otimes ab_{(0)<0>},$$
$$\rho^{H_\gamma\boxtimes A}_\beta(S^{-1}_{\gamma^{-1}}\phi_\alpha(1_{(1,\alpha^{-1}\gamma^{-1}\alpha)})h1_{(0,0)<-1,\gamma>}\otimes a1_{(0,0)<0,0>})$$
$$= h_{(1,\beta)}\otimes S^{-1}_{\gamma^{-1}\beta}\phi_\alpha(1_{(1,\alpha^{-1}\gamma^{-1}\beta\alpha)})h_{(2,\beta^{-1}\gamma)}1_{(0,0)<-1,\beta^{-1}\gamma>}\otimes a1_{(0,0)<0,0>},$$

for all $\beta,\gamma\in\pi$ and $h\in H_\gamma, a,b\in A$.

4. Total Quantum Integrals for Weak Quantum Yetter-Drinfel'd π-modules

In the section, total quantum integrals to weak quantum Yetter-Drinfel'd π-modules are introduced. Then we prove the the affineness criterion for weak quantum Yetter-Drinfel'd π-modules.

Definition 4.1. Let H be a weak crossed Hopf π-coalgebra and A a weak π-H-bicomodule algebra. Let us fixed $\alpha\in\pi$. A family of k-linear map $\theta=\{\theta_\beta: H_\beta\to\mathrm{Hom}(H_{\beta^{-1}},A)\}_{\beta\in\pi}$ is called a quantum integral, if

$$\begin{aligned} g_{(1,\gamma)}\otimes\theta_\beta(g_{(2,\beta)})(h) &= S^{-1}_{\gamma^{-1}}\phi_\alpha(\theta_{\gamma\beta}(g)(h_{(1,\beta^{-1}\gamma^{-1})})_{(1,\alpha^{-1}\gamma^{-1}\alpha)})h_{(2,\gamma)} \\ &\quad (\theta_{\gamma\beta}(g)(h_{(1,\beta^{-1}\gamma^{-1})}))_{(0,0)<-1,\gamma>}\otimes(\theta_{\gamma\beta}(g)(h_{(1,\beta^{-1}\gamma^{-1})}))_{(0,0)<0,0>}, \end{aligned} \tag{4. 1}$$

for all $g \in H_{\gamma\beta}, h \in H_{\beta^{-1}}$ and $\gamma, \beta \in \pi$. A quantum integral $\theta = \{\theta_\beta : H_\beta \to \mathrm{Hom}(H_{\beta^{-1}}, A)\}_{\beta\in\pi}$ is called total, if

$$\theta_\beta(h_{(1,\beta)})(h_{(2,\beta^{-1})}) = \varepsilon(S_1^{-1}\phi_\alpha(1_{(1,1)})h1_{(0)<-1,1>})1_{(0,0)<0,0>}, \tag{4. 2}$$

for all $h \in H_1$.

Proposition 4.2. Let H be a weak crossed Hopf π-coalgebra and A a weak H-bicomodule algebra. Let us fixed $\alpha \in \pi$. Assume that there exists $\theta = \{\theta_\beta : H_\beta \to \mathrm{Hom}(H_{\beta^{-1}}, A)\}_{\beta\in\pi}$ a total quantum integral. Then $\tilde{\rho}^A = \bigoplus_{\gamma\in\pi}\tilde{\rho}_\gamma : A \to \bigoplus_{\beta\in\pi} H_\beta \boxtimes A$ splits in ${}^H\mathcal{YD}_A^\alpha$.

Proof. We can prove that the map

$$ł_A : \bigoplus_{\beta\in\pi} H_\beta \boxtimes A \to A,$$

$$\begin{aligned} & ł_A(\bigoplus_{\beta\in\pi} S_{\beta^{-1}}^{-1}\phi_\alpha(1_{(1,\alpha^{-1}\beta^{-1}\alpha)})h1_{(0,0)<-1,\beta>} \otimes a1_{(0,0)<0,0>}) \\ = \ & a_{(0,0)<0,0>}\theta_\beta(1'_{(1,\beta)}h1_{(1,\beta)})(1'_{(2,\beta^{-1})}S_\beta^{-1}\phi_\alpha(a_{(1,\alpha^{-1}\beta\alpha)})a_{(0,0)<-1,\beta^{-1}>}1_{(2,\beta^{-1})}), \end{aligned}$$

for all $\beta, \gamma \in \pi$ and $h \in H_\beta, a \in A$. Then $ł_A$ is a left π-H-colinear retraction of $\tilde{\rho}$. In particular,

$$ł_A(\bigoplus_{\beta\in\pi} S_{\beta^{-1}}^{-1}\phi_\alpha(a_{(1,\alpha^{-1}\beta^{-1}\alpha)})1_{(0,0)<-1,\beta>} \otimes 1_{(0,0)<0,0>}) = 1_A.$$

Since

$$\begin{aligned} & ł_A(\bigoplus_{\beta\in\pi} S_{\beta^{-1}}^{-1}\phi_\alpha(1_{(1,\alpha^{-1}\beta^{-1}\alpha)})1_{(0,0)<-1,\beta>} \otimes 1_{(0,0)<0,0>}) \\ = \ & 1_{(0,0)<0,0>}\theta_\beta(1'_{(1,\beta)}1_{(1,\beta)})(1'_{(2,\beta^{-1})}S_\beta^{-1}\phi_\alpha(1_{(1,\alpha^{-1}\beta\alpha)})1_{(0,0)<-1,\beta^{-1}>}1_{(2,\beta^{-1})}) \\ = \ & 1_{(0,0)<0,0>}\theta_\beta(1'_{(1,\beta)}1_{(1,\beta)})(1'_{(2,\beta^{-1})}1_{(2,\beta^{-1})}S_\beta^{-1}\phi_\alpha(1_{(1,\alpha^{-1}\beta\alpha)})1_{(0,0)<-1,\beta^{-1}>}) \\ = \ & 1_{(0,0)<0,0>}\theta_\beta(1_{(1,\beta)})(1_{(2,\beta^{-1})}S_\beta^{-1}\phi_\alpha(1_{(1,\alpha^{-1}\beta\alpha)})1_{(0,0)<-1,\beta^{-1}>}) \\ = \ & 1_{(0,0)<0,0>}\theta_1((S_1^{-1}\phi_\alpha(1_{(1,1)})1_{(0,0)<-1,\beta^{-1}>})_{1,1})((S_1^{-1}\phi_\alpha(1_{(1,1)})1_{(0,0)<-1,1>})_{2,1}) \\ = \ & 1_{(0,0)<0,0>}\varepsilon(S_1^{-1}\phi_\alpha(1'_{(1,1)})S^{-1}\phi_\alpha(1_{(1,1)})1_{(0,0)<-1,1>}1'_{(0,0)<-1,1>})1'_{(0,0)<0,0>} \\ = \ & 1_{(0,0)<0,0>}\varepsilon(S_1^{-1}\phi_\alpha(1_{(1,1)}1'_{(1,1)})1_{(0,0)<-1,1>}1'_{(0,0)<-1>})1'_{(0,0)<0,0>} \\ = \ & 1_{(0,0)<0,0>}\varepsilon(S_1^{-1}\phi_\alpha(1_{(1,1)})1_{(0,0)<-1,1>}), \end{aligned}$$

we have

$$1_{(0,0)<0,0>}\varepsilon(S_1^{-1}\phi_\alpha(1_{(1,1)})1_{(0,0)<-1,1>}) = 1_A.$$

By $ł_A$ is a π-H-colinear map, we have

$$\begin{aligned} & \bigoplus_{\beta\in\pi} g_{(1,\gamma)} \otimes ł(S_{\beta^{-1}\gamma}^{-1}\phi_\alpha(1_{(1,\alpha^{-1}\beta^{-1}\gamma\alpha)})g_{(2,\gamma^{-1}\beta)}1_{(0,0)<-1,\gamma^{-1}\beta>} \otimes a1_{(0,0)<0,0>}) \\ & = S_{\gamma^{-1}}^{-1}\phi_\alpha(ł_A(\bigoplus_{\beta\in\pi} S^{-1}(1_{(1)})g1_{(0)<-1>} \otimes a1_{(0,0)<0,0>})_{(1,\alpha^{-1}\gamma^{-1}\alpha)}) \\ & \qquad ł_A(\bigoplus_{\beta\in\pi} S_{\beta^{-1}}^{-1}\phi_\alpha(1_{(1,\alpha^{-1}\beta^{-1}\alpha)})g1_{(0,0)<-1,\beta>} \otimes a1_{(0,0)<0,0>})_{(0,0)<-1,\gamma>} \\ & \qquad \otimes ł_A(\bigoplus_{\beta\in\pi} S_{\beta^{-1}}^{-1}\phi_\alpha(1_{(1,\alpha^{-1}\beta^{-1}\alpha)})g1_{(0,0)<-1,\beta>} \otimes a1_{(0,0)<0,0>})_{(0,0)<0,0>} \end{aligned}$$

for all $\gamma \in \pi$ and $g \in H_\beta, a \in A$. We define now

$$\tau_A : \bigoplus_{\beta\in\pi} H_\beta \boxtimes A \to A,$$

$$\begin{aligned}
&\tau_A(\bigoplus_{\beta\in\pi} S^{-1}_{\beta^{-1}}\phi_\alpha(1_{(1,\alpha^{-1}\beta^{-1}\alpha)})h1_{(0,0)<-1,\beta>}\otimes a1_{(0,0)<0,0>})\\
=\ & \text{Ł}_A(S^{-1}_{\beta^{-1}}\phi_\alpha(1_{(1,\alpha^{-1}\beta^{-1}\alpha)})S^{-1}_{\beta^{-1}}(S^{-1}_{\beta}\phi_\alpha(a_{(1,\alpha^{-1}\beta\alpha)}))hS_{\beta^{-1}}(a_{(0)<-1,\beta^{-1}>})1_{(0,0)<-1,\beta>}\otimes 1_{(0,0)<0,0>})a_{(0,0)<0,0>},
\end{aligned}$$

for all $h\in H_\beta, a\in A$. Then, for $a\in A$, we have

$$\begin{aligned}
(\tau_A\circ\tilde{\rho})(a) &= \tau_A(\bigoplus_{\beta\in\pi} S^{-1}_{\beta^{-1}}\phi_\alpha(a_{(1,\alpha^{-1}\beta^{-1}\alpha)})a_{(0,0)<-1,\beta>}\otimes a_{(0,0)<0,0>})\\
&= \text{Ł}_A(\bigoplus_{\beta\in\pi} S^{-1}_{\beta^{-1}}\phi_\alpha(1_{(1,\alpha^{-1}\beta^{-1}\alpha)})S^{-1}_{\beta^{-1}}(S^{-1}_{\beta}\phi_\alpha(a_{(0,0)<0,0>(1,\alpha^{-1}\beta\alpha)}))S^{-1}_{\beta^{-1}}\phi_\alpha(a_{(1,\alpha^{-1}\beta^{-1}\alpha)})\\
&\quad a_{(0,0)<-1,\beta>}S_{\beta^{-1}}(a_{(0,0)<0,0>(0,0)<-1,\beta^{-1}>})1_{(0,0)<-1,\beta>}\otimes 1_{(0,0)<0,0>})a_{<0,0>(0,0)(0,0)<0,0>}\\
&\overset{(3.1)}{=} \text{Ł}_A(\bigoplus_{\beta\in\pi} S^{-1}_{\beta^{-1}}\phi_\alpha(1_{(1,\alpha^{-1}\beta^{-1}\alpha)})S^{-1}_{\beta^{-1}}(S^{-1}_{\beta}\phi_\alpha(a_{<0,0>(0,0)(1,\alpha^{-1}\beta\alpha)}))S^{-1}_{\beta^{-1}}\phi_\alpha(a_{(0,0)(1,\alpha^{-1}\beta^{-1}\alpha)})\\
&\quad a_{<-1,\beta>}S_{\beta^{-1}}(a_{<0,0>(0,0)(0,0)<-1,\beta^{-1}>})1_{(0,0)<-1,\beta>}\otimes 1_{(0,0)<0,0>})a_{<0,0>(0,0)(0,0)<0,0>}\\
&= \text{Ł}_A(\bigoplus_{\beta\in\pi} S^{-1}_{\beta^{-1}}\phi_\alpha(1_{(1,\alpha^{-1}\beta^{-1}\alpha)})S^{-1}_{\beta^{-1}}(S^{-1}_{\beta}\phi_\alpha(a_{<0,0>(1,1)(1,\alpha^{-1}\beta\alpha)}))S^{-1}_{\beta^{-1}}\phi_\alpha(a_{<0,0>(1,1)(2,\alpha^{-1}\beta^{-1}\alpha)})\\
&\quad a_{<-1,\beta>}S_{\beta^{-1}}(a_{<0,0>(0,0)<-1,\beta^{-1}>})1_{(0,0)<-1,\beta>}\otimes 1_{(0,0)<0,0>})a_{<0,0>(0,0)<0,0>}\\
&= \text{Ł}_A(\bigoplus_{\beta\in\pi} S^{-1}_{\beta^{-1}}\phi_\alpha(1_{(1,\alpha^{-1}\beta^{-1}\alpha)})S^{-1}_{\beta^{-1}}(S^{-1}_{\beta}\varepsilon^{\beta}_t(\phi_\alpha(a_{<0,0>(1,1)})))a_{<-1,\beta>}S_{\beta^{-1}}\\
&\qquad (a_{<0,0>(0,0)<-1,\beta^{-1}>})1_{(0,0)<-1,\beta>}\otimes 1_{(0,0)<0,0>})a_{<0,0>(0,0)<0,0>}\\
&\overset{(3.3)}{=} \text{Ł}(\bigoplus_{\beta\in\pi} S^{-1}_{\beta^{-1}}\phi_\alpha(1_{(1,\alpha^{-1}\beta^{-1}\alpha)})S^{-1}_{\beta^{-1}}\phi_\alpha(1'_{(1,\alpha^{-1}\beta^{-1}\alpha)})a_{<-1,\beta>}S_{\beta^{-1}}\\
&\qquad ((1'_{(0,0)}a_{<0,0>})_{<-1,\beta^{-1}>})1_{(0,0)<-1,\beta>}\otimes 1_{(0,0)<0,0>})(1'_{(0,0)}a_{<0,0>})_{<0,0>}\\
&= \text{Ł}_A(\bigoplus_{\beta\in\pi} S^{-1}_{\beta^{-1}}\phi_\alpha(1_{(1,\alpha^{-1}\beta^{-1}\alpha)})S^{-1}_{\beta^{-1}}\phi_\alpha(1'_{(1,\alpha^{-1}\beta^{-1}\alpha)})a_{<-1,\beta>}S_{\beta^{-1}}\\
&\qquad (1'_{(0,0)<-1,\beta^{-1}>}a_{<0,0><-1,\beta^{-1}>})1_{(0,0)<-1,\beta>}\otimes 1_{(0,0)<0,0>})1'_{(0,0)<0,0>}a_{<0,0><0,0>}\\
&= \text{Ł}_A(\bigoplus_{\beta\in\pi} S^{-1}_{\beta^{-1}}\phi_\alpha(1_{(1,\alpha^{-1}\beta^{-1}\alpha)})S^{-1}_{\beta^{-1}}\phi_\alpha(1'_{(1,\alpha^{-1}\beta^{-1}\alpha)})a_{<-1,1>(1,\beta)}\\
&\quad S_{\beta^{-1}}(a_{<-1,1>(1,\beta^{-1})})S_{\beta^{-1}}(1'_{(0,0)<-1,\beta^{-1}>})1_{(0,0)<-1,\beta>}\otimes 1_{(0,0)<0,0>})1'_{(0,0)<0,0>}a_{<0,0><0,0>}\\
&= \text{Ł}_A(\bigoplus_{\beta\in\pi} S^{-1}_{\beta^{-1}}\phi_\alpha(1_{(1,\alpha^{-1}\beta^{-1}\alpha)})S^{-1}_{\beta^{-1}}\phi_\alpha(1'_{(1,\alpha^{-1}\beta^{-1}\alpha)})\varepsilon^{\beta}_t(a_{<-1,1>})\\
&\qquad S_{\beta^{-1}}(1'_{(0,0)<-1,\beta^{-1}>})1_{(0,0)<-1,\beta>}\otimes 1_{(0,0)<0,0>})1'_{(0,0)<0,0>}a_{<0,0>}\\
&= \text{Ł}_A(\bigoplus_{\beta\in\pi} S^{-1}_{\beta^{-1}}\phi_\alpha(1_{(1,\alpha^{-1}\beta^{-1}\alpha)})S^{-1}_{\beta^{-1}}\phi_\alpha(1'_{(1,\alpha^{-1}\beta^{-1}\alpha)})S_{\beta^{-1}}(1_{<-1,\beta^{-1}>})\\
&\qquad S_{\beta^{-1}}(1'_{(0,0)<-1,\beta^{-1}>})1_{(0,0)<-1,\beta>}\otimes 1_{(0,0)<0,0>})1'_{(0,0)<0,0>}1_{<0,0>}a\\
&= \text{Ł}_A(\bigoplus_{\beta\in\pi} S^{-1}_{\beta^{-1}}\phi_\alpha(1_{(1,\alpha^{-1}\beta^{-1}\alpha)})S^{-1}_{\beta^{-1}}\phi_\alpha(1'_{(1,\alpha^{-1}\beta^{-1}\alpha)})S_{\beta^{-1}}(1'_{(0,0)<-1,\beta^{-1}>}\\
&\qquad 1_{<-1,\beta^{-1}>})1_{(0,0)<-1,\beta>}\otimes 1_{(0,0)<0,0>})1'_{(0,0)<0,0>}1_{<0,0>}a\\
&= \text{Ł}_A(\bigoplus_{\beta\in\pi} S^{-1}_{\beta^{-1}}\phi_\alpha(1_{(1,\alpha^{-1}\beta^{-1}\alpha)})S^{-1}_{\beta^{-1}}\phi_\alpha(1'_{(1,\alpha^{-1}\beta^{-1}\alpha)})S_{\beta^{-1}}(1'_{(0,0)<-1,\beta^{-1}>})1_{(0,0)<-1,\beta>}\otimes 1_{(0,0)<0,0>})1'_{(0,0)<0,0>}a\\
&= 1_{(0,0)<0,0>}\theta_\beta(1_{(1,\beta)}S^{-1}_{\beta^{-1}}(1'_{(0,0)<-1,\beta^{-1}>})1_{(1,\beta)})(1_{(2,\beta^{-1})}S^{-1}_{\beta}\phi_\alpha(1_{(1,\alpha^{-1}\beta\alpha)}))1_{(0,0)<-1,\beta^{-1}>}1_{(2,\beta^{-1})})1'_{(0,0)<0,0>}a
\end{aligned}$$

$$= \quad 1_{(0,0)<0,0>}\varepsilon(S^{-1}(1'_{(1,1)})S^{-1}(1_{(1,1)})1_{(0,0)<-1,1>}1'_{(0,0)<-1,1>})1'_{(0,0)<0,0>}a$$
$$= \quad 1_{(0,0)<0,0>}\varepsilon(S^{-1}(1_{(1,1)})1_{(0,0)<-1,1>})a = a,$$

i.e., λ is still a retraction of $\tilde{\rho}$. Now, for $\beta \in \pi$ and $h \in H_\beta, a, b \in A$, we have

$$\tau((\bigoplus_{\beta\in\pi} S^{-1}_{\beta^{-1}}\phi_\alpha(1_{(1,\alpha^{-1}\beta^{-1}\alpha)})h1_{(0,0)<-1,\beta>} \otimes a1_{(0,0)<0,0>}) \cdot b)$$
$$= \quad \tau(\bigoplus_{\beta\in\pi} S^{-1}_{\beta^{-1}}\phi_\alpha(b_{(1,\alpha^{-1}\beta^{-1}\alpha)})hb_{(0,0)<-1,\beta>} \otimes ab_{(0,0)<0,0>})$$
$$= \quad \text{ł}_A(\bigoplus_{\beta\in\pi} S^{-1}_{\beta^{-1}}\phi_\alpha(1_{(1,\alpha^{-1}\beta^{-1}\alpha)})S^{-1}_{\beta^{-1}}(S^{-1}_{\beta}\phi_\alpha((ab_{(0,0)<0,0>})_{(1,\alpha^{-1}\beta\alpha)}))S^{-1}_{\beta^{-1}}\phi_\alpha(b_{(1,\alpha^{-1}\beta^{-1}\alpha)})$$
$$hb_{(0,0)<-1,\beta>}S_{\beta^{-1}}((ab_{(0,0)<0,0>})_{(0,0)<-1,\beta^{-1}>})1_{(0,0)<-1,\beta>} \otimes 1_{(0,0)<0,0>})(ab_{(0,0)<0,0>})_{(0,0)<0,0>}$$
$$= \quad \text{ł}_A(\bigoplus_{\beta\in\pi} S^{-1}_{\beta^{-1}}\phi_\alpha(1_{(1,\alpha^{-1}\beta^{-1}\alpha)})S^{-1}_{\beta^{-1}}(S^{-1}_{\beta}\phi_\alpha(a_{(1,\alpha^{-1}\beta\alpha)}))S^{-1}_{\beta^{-1}}(S^{-1}_{\beta}\phi_\alpha(b_{<0,0>(1,1)(1,\alpha^{-1}\beta\alpha)}))S^{-1}_{\beta^{-1}}\phi_\alpha(b_{<0,0>(1,1)(2,\alpha^{-1}\beta^{-1}\alpha)})$$
$$hb_{<-1,\beta>}S_{\beta^{-1}}(b_{<0>(0)<-1,\beta^{-1}>})S_{\beta^{-1}}(a_{(0,0)<-1,\beta^{-1}>})1_{(0,0)<-1,\beta>} \otimes 1_{(0,0)<0,0>})a_{(0,0)<0,0>}b_{<0,0>(0,0)<0,0>}$$
$$= \quad \text{ł}_A(\bigoplus_{\beta\in\pi} S^{-1}_{\beta^{-1}}\phi_\alpha(1_{(1,\alpha^{-1}\beta^{-1}\alpha)})S^{-1}_{\beta^{-1}}(S^{-1}_{\beta}\phi_\alpha(a_{(1,\alpha^{-1}\beta\alpha)}))S^{-1}_{\beta^{-1}}(S^{-1}_{\beta}\varepsilon^{\beta}_{t}\phi_\alpha(b_{<0,0>(1,1)}))hb_{<-1,\beta>}S_{\beta^{-1}}(b_{<0>(0)<-1,\beta^{-1}>})$$
$$S_{\beta^{-1}}(a_{(0,0)<-1,\beta^{-1}>})1_{(0,0)<-1,\beta>} \otimes 1_{(0,0)<0,0>})a_{(0,0)<0,0>}b_{<0,0>(0,0)<0,0>}$$
$$= \quad \text{ł}_A(\bigoplus_{\beta\in\pi} S^{-1}_{\beta^{-1}}\phi_\alpha(1_{(1,\alpha^{-1}\beta^{-1}\alpha)})S^{-1}_{\beta^{-1}}(S^{-1}_{\beta}\phi_\alpha(a_{(1,\alpha^{-1}\beta\alpha)}))S^{-1}_{\beta^{-1}}(S^{-1}_{\beta}\phi_\alpha(1'_{(1,\alpha^{-1}\beta\alpha)}))hb_{<-1,\beta>}$$
$$S_{\beta^{-1}}((1'_{(0,0)}b_{<0,0>})_{<-1,\beta^{-1}>})S_{\beta^{-1}}(a_{(0,0)<-1,\beta^{-1}>})1_{(0,0)<-1,\beta>} \otimes 1_{(0,0)<0,0>})a_{(0,0)<0,0>}(1'_{(0,0)}b_{<0,0>})_{<0,0>}$$
$$= \quad \text{ł}_A(\bigoplus_{\beta\in\pi} S^{-1}_{\beta^{-1}}\phi_\alpha(1_{(1,\alpha^{-1}\beta^{-1}\alpha)})S^{-1}_{\beta^{-1}}(S^{-1}_{\beta}\phi_\alpha(a_{(1,\alpha^{-1}\beta\alpha)}))S^{-1}_{\beta^{-1}}(S^{-1}_{\beta}\phi_\alpha(1'_{(1,\alpha^{-1}\beta\alpha)}))hb_{<-1,1>(1,\beta)}$$
$$S_{\beta^{-1}}(b_{<-1,1>(2,\beta^{-1})}S_{\beta^{-1}}(1'_{(0,0)<-1,\beta^{-1}>})S_{\beta^{-1}}(a_{(0,0)<-1,\beta^{-1}>})1_{(0,0)<-1,\beta>} \otimes 1_{(0,0)<0,0>})a_{(0,0)<0,0>}1'_{(0,0)<0,0>}b_{<0,0>}$$
$$= \quad \text{ł}_A(\bigoplus_{\beta\in\pi} S^{-1}_{\beta^{-1}}\phi_\alpha(1_{(1,\alpha^{-1}\beta^{-1}\alpha)})S^{-1}_{\beta^{-1}}(S^{-1}_{\beta}\phi_\alpha(a_{(1,\alpha^{-1}\beta\alpha)}))S^{-1}_{\beta^{-1}}(S^{-1}_{\beta}\phi_\alpha(1'_{(1,\alpha^{-1}\beta\alpha)}))$$
$$h\varepsilon^{\beta}_{t}(b_{<-1,1>})S_{\beta^{-1}}(1'_{(0,0)<-1,\beta^{-1}>})S_{\beta^{-1}}(a_{(0,0)<-1,\beta^{-1}>})1_{(0,0)<-1,\beta>} \otimes 1_{(0,0)<0,0>})a_{(0,0)<0,0>}1'_{(0,0)<0,0>}b_{<0,0>}$$
$$= \quad \text{ł}_A(\bigoplus_{\beta\in\pi} S^{-1}_{\beta^{-1}}\phi_\alpha(1_{(1,\alpha^{-1}\beta^{-1}\alpha)})S^{-1}_{\beta^{-1}}(S^{-1}_{\beta}\phi_\alpha(a_{(1,\alpha^{-1}\beta\alpha)}))S^{-1}_{\beta^{-1}}(S^{-1}_{\beta}\phi_\alpha(1'_{(1,\alpha^{-1}\beta\alpha)}))$$
$$hS_{\beta^{-1}}(1''_{<-1,\beta^{-1}>})S_{\beta^{-1}}(1'_{(0,0)<-1,\beta^{-1}>})S_{\beta^{-1}}(a_{(0,0)<-1,\beta^{-1}>})1_{(0,0)<-1,\beta>} \otimes 1_{(0,0)<0,0>})a_{(0,0)<0,0>}1'_{(0,0)<0,0>}1''_{<0,0>}b$$
$$= \quad \text{ł}_A(\bigoplus_{\beta\in\pi} S^{-1}_{\beta^{-1}}\phi_\alpha(1_{(1,\alpha^{-1}\beta^{-1}\alpha)})S^{-1}_{\beta^{-1}}(S^{-1}_{\beta}\phi_\alpha(a_{(1,\alpha^{-1}\beta\alpha)}))hS_{\beta^{-1}}(1''_{<-1,\beta^{-1}>})$$
$$S_{\beta^{-1}}(a_{(0,0)<-1,\beta^{-1}>})1_{(0,0)<-1,\beta>} \otimes 1_{(0,0)<0,0>})a_{(0,0)<0,0>}1''_{<0,0>}b$$
$$= \quad \text{ł}_A(\bigoplus_{\beta\in\pi} S^{-1}_{\beta^{-1}}\phi_\alpha(1_{(1,\alpha^{-1}\beta^{-1}\alpha)})S^{-1}_{\beta^{-1}}(S^{-1}_{\beta}\phi_\alpha(a_{(1,\alpha^{-1}\beta\alpha)}))hS_{\beta^{-1}}(a_{(0,0)<-1,\beta^{-1}>})1_{(0,0)<-1,\beta>} \otimes 1_{(0,0)<0,0>})a_{(0,0)<0,0>}b$$
$$= \quad \text{ł}_A(\bigoplus_{\beta\in\pi} S^{-1}_{\beta^{-1}}\phi_\alpha(1_{(1,\alpha^{-1}\beta^{-1}\alpha)})h1_{(0,0)<-1,\beta>} \otimes a1_{(0,0)<0,0>})b.$$

So we finish the proof. ■

We can define now the coinvariants of A as

$$B = A^{coH} = \{a \in A | S^{-1}_{\beta^{-1}}\phi_\alpha(a_{(1,\alpha^{-1}\beta^{-1}\alpha)})a_{(0,0)<-1,\beta>} \otimes a_{(0,0)<0,0>}$$
$$= S^{-1}_{\beta^{-1}}\phi_\alpha(1_{(1,\alpha^{-1}\beta^{-1}\alpha)})1_{(0,0)<-1,\beta>} \otimes a1_{(0,0)<0,0>}\},$$

then B is a subalgebra of A and will be called the subalgebra of quantum coinvariants.

Now, we shall construct functors connecting ${}^H\mathcal{YD}_A^\alpha$ and $\mathcal{M}_B$. First, if $M \in {}^H\mathcal{YD}_A^\alpha$, then

$$M^{coH} = \{m \in M | m_{<-1,\beta>} \otimes m_{<0>} = S_{\beta^{-1}}^{-1}\phi_\alpha(1_{(1,\alpha^{-1}\beta^{-1}\alpha)})1_{(0,0)<-1,\beta>} \otimes m \cdot 1_{(0,0)<0,0>}\}$$

is the right B-module of the coinvariants of M. Furthermore, we have a covariant functor

$$(-)^{coH} : {}^H\mathcal{YD}_A^\alpha \to \mathcal{M}_B.$$

Now, for $N \in \mathcal{M}_B$, $N \otimes_B A \in {}^H\mathcal{YD}_A^\alpha$ via the structures

$$(n \otimes_B a) \cdot a' = n \otimes_B aa',$$

$$\rho_\beta^{N\otimes_B A}(n \otimes_B a) = S_{\beta^{-1}}^{-1}\phi_\alpha(a_{(1,\alpha^{-1}\beta^{-1}\alpha)})a_{(0,0)<-1,\beta>} \otimes n \otimes_B a_{(0,0)<0,0>},$$

for all $n \in N, a, a' \in A$ and $\beta \in \pi$. In this way, we have constructed a covariant functor called the induction functor

$$- \otimes_B A : \mathcal{M}_B \to {}^H\mathcal{YD}_A^\alpha.$$

We shall prove now that the above functors are an adjoint pair.

Proposition 4.3. Let H be a weak crossed Hopf π-coalgebra and A a weak π-H-bicomodule algebra. Let fixed $\alpha \in \pi$. Then the induction functor $- \otimes_B A : \mathcal{M}_B \to {}^H\mathcal{YD}_A^\alpha$ is a left adjoint of the coinvariant functor $(-)^{coH} : {}^H\mathcal{YD}_A^\alpha \to \mathcal{M}_B$.

Proof. The unit and the counit of the adjointness are given by

$$\eta_N : N \to (N \otimes_B A)^{coH}, \ \eta_N(n) = n \otimes_B 1_A,$$

for all $N \in \mathcal{M}_B, n \in N$, and

$$\psi_M : M^{coH} \otimes_B A \to M, \ \beta_M(m \otimes_B a) = ma,$$

for all $M \in {}^H\mathcal{YD}_A^\alpha$, $m \in M^{coH}$, and $a \in A$. So the proof is finished. ∎

Lemma 4.4. Let A be a weak π-H-bicomodule algebra. Then

$$(H \boxtimes A)^{coH} \cong A.$$

Proof. We can construct the desired map as follows

$$\delta : A \to (\bigoplus_{\gamma\in\pi} H_\gamma \boxtimes A)^{coH}, \ \ \delta(a) = S_{\gamma^{-1}}^{-1}\phi_\alpha(1_{(1,\alpha^{-1}\gamma^{-1}\alpha)})1_{(0,0)<-1,\gamma>} \otimes a1_{(0,0)<0,0>}.$$

Notice that $\delta(a) \in (\bigoplus_{\gamma\in\pi} H_\gamma \boxtimes A)^{coH}$, we check it as follows

$$\begin{aligned}
& \rho_\beta^{\bigoplus_{\gamma\in\pi} H_\gamma \boxtimes A}(\delta(a)) \\
= \ & \rho_\beta^{\bigoplus_{\gamma\in\pi} H_\gamma \boxtimes A}(S_{\gamma^{-1}}^{-1}\phi_\alpha(1_{(1,\alpha^{-1}\gamma^{-1}\alpha)})1_{(0,0)<-1,\gamma>} \otimes a1_{(0,0)<0,0>}) \\
= \ & 1_{(1,\beta)} \otimes S_{\gamma^{-1}\beta}^{-1}\phi_\alpha(1_{(1,\alpha^{-1}\gamma^{-1}\beta\alpha)})1_{(2,\beta^{-1}\gamma)}1_{(0,0)<-1,\beta^{-1}\gamma>} \otimes a1_{(0,0)<0,0>},
\end{aligned}$$

and

$$\begin{aligned}
& S^{-1}(1'_{(1)})1'_{(0)<-1>} \otimes (S^{-1}(1_{(1)})1_{(0)<-1>} \otimes a1_{(0)<0>}) \cdot 1_{(0)<0>} \\
& = S^{-1}('_{(1)})1'_{(0)<-1>} \otimes S^{-1}(1'_{(0)<0>(1)})1'_{(0)<0>(0)<-1>} \otimes a1'_{(0)<0>(0)<0>} \\
& = S^{-1}(1'_2)1_1 \otimes S^{-1}(1'_1 1_{<0>(1)})1_2 1_{<-1>} \otimes a1_{<0>(0)} \\
& = 1_1 \otimes S^{-1}(1_{<0>(1)})1_2 1_{<-1>} \otimes a1_{<0>(0)} \\
& = \rho_{H\boxtimes A}(\delta(a)).
\end{aligned}$$

This finishes the proof. ■

From the lemma 4.4, the adjunction map $\psi_{H\boxtimes A}$ can be viewed as a map in ${}^{H}\mathcal{YD}_{A}^{\alpha}$ via

$$\psi_{\bigoplus_{\gamma\in\pi} H_\gamma\boxtimes A} : A\otimes_B A \to \bigoplus_{\gamma\in\pi} H_\gamma \boxtimes A,$$

$$\psi(a\otimes_B b) = \bigoplus_{\gamma\in\pi} S^{-1}_{\gamma^{-1}}\phi_\alpha(b_{(1,\alpha^{-1}\gamma^{-1}\alpha)})b_{(0,0)<-1,\gamma>}\otimes ab_{(0,0)<0,0>},$$

for all $a, b\in A$. Here $A\otimes_B A\in {}^{H}\mathcal{YD}_{A}^{\alpha}$ via the structures

$$(a\otimes_B b)\cdot a' = a\otimes_B ba',$$

$$\rho_\gamma^{A\otimes_B A}(a\otimes_B b) = \bigoplus_{\gamma\in\pi} S^{-1}_{\gamma^{-1}}\phi_\alpha(1_{(1,\alpha^{-1}\gamma^{-1}\alpha)})1_{(0,0)<-1,\gamma>}\otimes a\otimes_B 1_{(0,0)<0,0>},$$

for all $a, b, a'\in A$ and $\gamma\in\pi$.

We shall prove now the main result of this section, that is, the affineness criterion for weak quantum Yetter-Drinfel'd π-modules.

Proposition 4.5. Let H be a weak crossed Hopf π-coalgebra and A a weak π-H-bicomodule algebra. Let fixed $\alpha\in\pi$, and $B = A^{coH}$. Assume that there exists a total quantum integral $\theta = \{\theta_\beta : H_\beta \to \mathrm{Hom}(H_{\beta^{-1}}, A)\}_{\beta\in\pi}$, and the canonical map

$$\psi : A\otimes_B A \to \bigoplus_{\beta\in\pi} H_\beta\boxtimes A,$$

$$\psi(a\otimes_B b) = \bigoplus_{\beta\in\pi} S^{-1}_{\beta^{-1}}\phi_\alpha(b_{(1,\alpha^{-1}\beta^{-1}\alpha)})b_{(0,0)<-1,\beta>}\otimes ab_{(0,0)<0,0>}$$

for all $a, b\in A$ is surjective. Then the induction functor $-\otimes_B A : \mathcal{M}_B \to {}^{H}\mathcal{YD}_{A}^{\alpha}$ is an equivalence of categories.

Proof. In Proposition 4.3 we have shown that the adjunction map $\eta_N : N \to (N\otimes_B A)^{coH}$ is an isomorphism for all $N\in\mathcal{M}_B$. It remains to prove that the other adjunction map, namely $\psi_M : M^{coH}\otimes_B A\to M$ is also an isomorphism for all $M\in {}^{H}\mathcal{YD}_{A}^{\alpha}$.

Let V be a k-module. Then $A\otimes V\in {}^{H}\mathcal{YD}_{A}^{\alpha}$ via the structures induced by A, i.e.,

$$(a\otimes v)\cdot b = ab\otimes v,$$
$$\rho_\gamma^{A\otimes V}(a\otimes v) = \bigoplus_{\gamma\in\pi} S^{-1}_{\gamma^{-1}}\phi_\alpha(a_{(1,\alpha^{-1}\gamma^{-1}\alpha)})a_{(0,0)<-1,\gamma>}\otimes a_{(0,0)<0,0>}\otimes v,$$

for all $a, b\in A$ and $v\in V$. In particular, for $V = A$, $A\otimes A\in {}^{H}\mathcal{YD}_{A}$ via

$$(a\otimes a')\cdot b = ab\otimes v, \tag{4. 3}$$
$$\rho_\gamma^{A\otimes A}(a\otimes a') = \bigoplus_{\gamma\in\pi} S^{-1}_{\gamma^{-1}}\phi_\alpha(a_{(1,\alpha^{-1}\gamma^{-1}\alpha)})a_{(0,0)<-1,\gamma>}\otimes a_{(0,0)<0,0>}\otimes a', \tag{4. 4}$$

for all $a, b, a'\in A$ and $\gamma\in\pi$. We will prove first that the adjunction map $\beta_{A\otimes V} : (A\otimes V)^{coH}\otimes_B A\to A\otimes V$ is an isomorphism for any k-module V.

First, $V\otimes B$ and $B\otimes V\in\mathcal{M}_B$ via the usual B-actions $(v\otimes a)\cdot b = v\otimes ab$, and $a'\cdot(b'\otimes v') = a'b'\otimes v'$ for all $a, b, a', b'\in B$ and $v, v'\in V$. The flip map $\tau : V\otimes B\to B\otimes V$, $\tau(v\otimes b) = b\otimes v$, for all $b\in B$ and $v\in V$, is an isomorphism in $\mathcal{M}_B$. On the other hand $V\otimes A\in {}^{H}\mathcal{YD}_{A}^{\alpha}$ via the structures induced by A, i.e.

$$(v\otimes a)\cdot b = v\otimes ab,$$
$$\rho_\gamma^{V\otimes A}(v\otimes a) = \bigoplus_{\gamma\in\pi} S^{-1}_{\gamma^{-1}}\phi_\alpha(a_{(1,\alpha^{-1}\gamma^{-1}\alpha)})a_{(0,0)<-1,\gamma>}\otimes a_{(0,0)<0,0>},$$

It is easy to see that the flip map $\tau : A\otimes V\to V\otimes A$, $\tau(a\otimes v) = v\otimes a$ is an isomorphism in ${}^{H}\mathcal{YD}_{A}^{\alpha}$.

Applying Proposition 3.5 for $N = V \otimes B \cong B \otimes V$, we obtain the following isomorphisms in $\mathcal{M}_B$:

$$B \otimes V \cong V \otimes B \cong (V \otimes B \otimes_B A)^{coH} \cong (V \otimes A)^{coH} \cong (A \otimes V)^{coH}.$$

Hence, $(A \otimes V)^{coH} \otimes_B A \cong A \otimes V$.

Let us define

$$\widetilde{\psi} : A \otimes_B A \to \bigoplus_{\gamma\in\pi} H_\gamma \boxtimes A,$$
$$\widetilde{\psi}(a \otimes_B b) = \bigoplus_{\gamma\in\pi} S^{-1}_{\gamma^{-1}}\phi_\alpha(b_{(1,\alpha^{-1}\gamma^{-1}\alpha)})b_{(0,0)<-1,\gamma>} \otimes ab_{(0,0)<0,0>},$$

for all $a, b \in A$. As ψ is surjective, $\widetilde{\psi}$ is surjective, because $\widetilde{\psi} = \psi can$, where $can : A \otimes A \to A \otimes_B A$ is the canonical surjection.

Let us define now

$$\xi : A \otimes A \to \bigoplus_{\gamma\in\pi} H_\gamma \boxtimes A,$$
$$\xi(a \otimes b) = (\widetilde{\psi} \circ \tau)(a \otimes b) = \bigoplus_{\gamma\in\pi} S^{-1}_{\gamma^{-1}}\phi_\alpha(a_{(1,\alpha^{-1}\gamma^{-1}\alpha)})a_{(0,0)<-1,\gamma>} \otimes ba_{(0,0)<0,0>}$$

for all $a, b \in A$. The map ξ is surjective, as $\widetilde{\psi}$ and τ are. We will prove that ξ is a morphism in ${}^H\mathcal{YD}^\alpha_A$. where $A \otimes A$ and $\bigoplus_{\gamma\in\pi} H_\gamma \boxtimes A$ are quantum Yetter-Drinfel'd modules. Indeed,

$$\begin{aligned}
& \xi((a \otimes b)c) = \xi(ac \otimes b) \\
= & \bigoplus_{\gamma\in\pi} S^{-1}_{\gamma^{-1}}\phi_\alpha(c_{(1,\alpha^{-1}\gamma^{-1}\alpha)})S^{-1}_{\gamma^{-1}}\phi_\alpha(a_{(1,\alpha^{-1}\gamma^{-1}\alpha)})a_{(0,0)<-1,\gamma>}c_{(0,0)<-1,\gamma>} \otimes ba_{(0,0)<0,0>}c_{(0,0)<0,0>} \\
= & (\bigoplus_{\gamma\in\pi} S^{-1}_{\gamma^{-1}}\phi_\alpha(a_{(1,\alpha^{-1}\gamma^{-1}\alpha)})a_{(0)<-1>} \otimes ba_{(0,0)<0,0>})c \\
= & \xi(a \otimes b)c
\end{aligned}$$

and

$$\begin{aligned}
& \rho_\beta^{\bigoplus_{\gamma\in\pi} H_\gamma\boxtimes A}\xi(a \otimes b) \\
= & \rho_\beta^{\bigoplus_{\gamma\in\pi} H_\gamma\boxtimes A}(\bigoplus_{\gamma\in\pi} S^{-1}_{\gamma^{-1}}\phi_\alpha(a_{(1,\alpha^{-1}\gamma^{-1}\alpha)})a_{(0,0)<-1,\gamma>} \otimes ba_{(0,0)<0,0>}) \\
= & \bigoplus_{\gamma\in\pi} S^{-1}_{\beta^{-1}}\phi_\alpha(a_{(1,\alpha^{-1}\gamma^{-1}\sigma)(2,\alpha^{-1}\beta^{-1}\alpha)})a_{(0,0)<-1,\gamma>(1,\beta)} \otimes S^{-1}_{\gamma^{-1}\beta}\phi_\alpha(a_{(1,\alpha^{-1}\gamma^{-1}\alpha)(1,\alpha^{-1}\gamma^{-1}\beta\alpha)}) \\
& \qquad a_{(0,0)<-1,\gamma>(2,\beta^{-1}\gamma)} \otimes ba_{(0,0)<0,0>} \\
= & (id \otimes \xi)(S^{-1}_{\beta^{-1}}\phi_\alpha(a_{(1,\alpha^{-1}\beta^{-1}\alpha)})a_{(0,0)<-1,\beta>} \otimes ba_{(0,0)<0,0>} \otimes b) \\
= & (id \otimes \xi)\rho_\beta^{A\otimes A}(a \otimes b).
\end{aligned}$$

Hence, ξ is a surjective morphism in ${}^H\mathcal{YD}^\alpha_A$. $\bigoplus_{\gamma\in\pi} H_\gamma \boxtimes A$ is projective as a right A-module, where $\bigoplus_{\gamma\in\pi} H_\gamma \boxtimes A$ is a right A-module in the usual way, i.e.

$$\begin{aligned}
(\bigoplus_{\gamma\in\pi} h \boxtimes a)b &= (\bigoplus_{\gamma\in\pi} S^{-1}_{\gamma^{-1}}\phi_\alpha(1_{(1,\alpha^{-1}\gamma^{-1}\alpha)})ha_{(0,0)<-1,\gamma>} \otimes a1_{(0,0)<0,0>})b \\
&= \bigoplus_{\gamma\in\pi} S^{-1}_{\gamma^{-1}}\phi_\alpha(1_{(1,\alpha^{-1}\gamma^{-1}\alpha)})ha_{(0,0)<-1,\gamma>} \otimes ab1_{(0,0)<0,0>} \\
&= \bigoplus_{\gamma\in\pi} h \boxtimes ab,
\end{aligned}$$

for all $h \in H$ and $a, b \in A$. On the other hand, the map

$$u : \bigoplus_{\gamma\in\pi} H_\gamma \boxtimes A \to A \boxtimes \bigoplus_{\gamma\in\pi} H_\gamma,$$
$$u(\bigoplus_{\gamma\in\pi} h \otimes a) = S^{-1}_{\gamma^{-1}}\phi_\alpha(a_{(1,\alpha^{-1}\gamma^{-1}\alpha)})a_{(0,0)<-1,\gamma>} \otimes a_{(0,0)<0,0>}$$

is a splitting surjection of right A-module, where the first $\bigoplus_{\gamma\in\pi} H_\gamma \boxtimes A$ has the usual right A-module structure and the second $\bigoplus_{\gamma\in\pi} H_\gamma \boxtimes A$ has the right A-module given in Eq.(3.9). The right inverse of u is given by

$$v : \bigoplus_{\gamma\in\pi} H_\gamma \boxtimes A \to A \boxtimes \bigoplus_{\gamma\in\pi} H_\gamma,$$
$$v(\bigoplus_{\gamma\in\pi} h \otimes a) = \bigoplus_{\gamma\in\pi} S^{-1}_{\gamma^{-1}}(S^{-1}_\gamma\phi_\alpha(a_{(1,\alpha^{-1}\gamma\alpha)}))hS_{\gamma^{-1}}(a_{(0,0)<-1,\gamma^{-1}>}) \otimes a_{(0,0)<0,0>}$$

Hence, we can view the second $\bigoplus_\gamma H_\gamma \boxtimes A$ as a right A-module direct summand of the first $\bigoplus_\gamma H_\gamma \boxtimes A$. So we obtain that $\bigoplus_\gamma H_\gamma \boxtimes A$, with the right A-module structure given in Eq.(4.3), is still projective as a right A-module. It follows that there exits $\zeta : \bigoplus_{\gamma\in\pi} H_\gamma \boxtimes A \to A \otimes A$ such that $\xi\zeta = id_{\bigoplus_{\gamma\in\pi} H_\gamma\boxtimes A}$ since $A \otimes A \to \bigoplus_{\gamma\in\pi} H_\gamma \boxtimes A$ is surjective. Hence, ξ splits in the category of right A-modules. In particular ξ is a k-split epimorphism in ${}^H\mathcal{YD}_A$.

Let now $M \in {}^H\mathcal{YD}^\alpha_A$. Then $A \otimes A \otimes M \in {}^H\mathcal{YD}^\alpha_A$ via the structures arising from $A \otimes A$, that is,

$$(a \otimes b \otimes m) \cdot c = ac \otimes b \otimes m; \tag{4. 5}$$
$$\rho^{A\otimes A\otimes M}_\gamma(a \otimes b \otimes m) = S^{-1}_{\gamma^{-1}}\phi_\alpha(a_{(1,\alpha^{-1}\gamma^{-1}\alpha)})a_{(0,0)<-1,\gamma>} \otimes a_{(0,0)<0,0>} \otimes b \otimes m \tag{4. 6}$$

for all $a, b, c \in A$ and $m \in M$. On the other hand, $H \boxtimes A \otimes M \in {}^H\mathcal{YD}_A$ via the structures arising from $H \boxtimes A$, that is,

$$(\bigoplus_{\gamma\in\pi} h \boxtimes a \otimes m) \cdot b = S^{-1}_{\gamma^{-1}}\phi_\alpha(b_{(1,\alpha^{-1}\gamma^{-1}\alpha)})hb_{(0,0)<-1,\gamma>} \otimes ab_{(0,0)<0,0>} \otimes m; \tag{4. 7}$$
$$\rho^{\bigoplus_{\gamma\in\pi} H_\gamma\boxtimes A\otimes M}_\beta(h \boxtimes a \otimes m) = \bigoplus_{\gamma\in\pi} h_{(1,\beta)} \otimes h_{(2,\beta^{-1}\gamma)} \otimes a \otimes m, \tag{4. 8}$$

for all $a, b \in A, h \in H_\gamma, m \in M$ and $\beta \in \pi$. We obtain that

$$\xi \otimes id_M : A \otimes A \otimes M \to \bigoplus_{\gamma\in\pi} H_\gamma \boxtimes A \otimes M$$

is a k-split epimorphism in ${}^H\mathcal{YD}^\alpha_A$.

Applying ${}^H\mathcal{YD}^\alpha_A = {}^{\pi-H}\mathcal{M}(H \otimes H^{op})_A$ we obtain that the A-A map

$$f : \bigoplus_{\gamma\in\pi} H_\gamma \boxtimes A \otimes M \to M$$
$$f(\bigoplus_{\gamma\in\pi} h \boxtimes a \otimes m) = m_{<0,0>}\theta_\gamma(S^{-1}_{\gamma^{-1}}S^{-1}_\gamma\phi_\alpha(a_{(1,\alpha^{-1}\gamma\alpha)})hS_{\gamma^{-1}}(a_{(0,0)<-1,\gamma^{-1}>}))(m_{<-1,\gamma^{-1}>})a_{(0,0)}$$

is a k-split epimorphism in ${}^H\mathcal{YD}^\alpha_A$. Hence, the composition

$$g = f \circ (\xi \otimes id_M) : A \otimes A \otimes M \to M,$$
$$g(a \otimes b \otimes m) = m_{<0,0>}\theta_\gamma(S^{-1}_{\gamma^{-1}}S^{-1}_\gamma\phi_\alpha(b_{(1,\alpha^{-1}\gamma\alpha)})hS_{\gamma^{-1}}(b_{(0,0)<-1,\gamma^{-1}>}))(m_{<-1,\gamma^{-1}>})b_{(0,0)}a$$

is a k-split epimorphism in ${}^H\mathcal{YD}_A^\alpha$. We note that the structure of $A \otimes A \otimes M$ as an object in ${}^H\mathcal{YD}_A^\alpha$ is of the form $A \otimes V$, for the k-module $V = A \otimes M$.

To conclude, we have constructed a k-split epimorphism in ${}^H\mathcal{YD}_A^\alpha$

$$A \otimes A \otimes M = M_1 \xrightarrow{g} M \longrightarrow 0$$

such that the adjunction map ψ_{M_1} for M_1 is bijective. As g is k-split and there exists a total quantum integral $\gamma : H \to Hom(H, A)$, we obtain that g also splits in ${}^H\mathcal{M}$. In particular, the sequence

$$M_1^{coH} \xrightarrow{g^{coH}} M^{coH} \longrightarrow 0$$

is exact. Continuing the resolution with $Ker(g)$. instead of M, we obtain an exact sequence in ${}^H\mathcal{YD}_A^\alpha$

$$M_2 \longrightarrow M_1 \longrightarrow M \longrightarrow 0$$

which splits in ${}^H\mathcal{M}$ and the adjunction maps for M_1 and M_2 are bijective. Using the Five lemma we obtain that the adjunction map for M is bijective. ■

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