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Semi-abundant Semigroups with Quasi-Ehresmann Transversals

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Abstract. Chen (Communications in Algebra 27(2), 4275-4288, 1999) introduced and investigated orthodox transversals of regular semigroups. In this paper, we initiate the investigation of quasi-Ehresmann transversals of semi-abundant semigroups which are generalizations of orthodox transversals of regular semigroups. Some interesting properties associated with quasi-Ehresmann transversals are established. Moreover, a structure theorem of semi-abundant semigroups with generalized bi-ideal strong quasi-Ehresmann transversals is obtained. Our results generalize and enrich Chen's results.

1. Introduction

The concept of inverse transversals was introduced by Blyth-McFadden [3]. From then on, inverse transversals have been extensively investigated and generalized by many authors (for example, see [1]-[7], [14]-[15] and [18]). Since orthodox semigroups can be regarded as generalizations of inverse semigroups, in 1999, Chen [4] generalized inverse transversals to *orthodox transversals* in the class of regular semigroups and gave a construction theorem for a kind of regular semigroups with orthodox transversals. Furthermore, Chen-Guo [6] explored some interesting properties associated with orthodox transversals. Most recently, Kong [14, 15] also investigated orthodox transversals and obtained some new results.

On the other hand, semi-abundant semigroups are generalized regular semigroups and have been studied by many authors, for example, see the texts [8]-[12] and [16]-[17]. In particular, Ehbal-El-Qallali [17] investigated a class of semi-abundant semigroups whose idempotents form a subsemigroup, El-Qallali-Fountain-Gould [8] and Gomes-Gould [10] studied some classes of semi-abundant semigroups by so called *"fundamental approaches"* and Lawson [16] considered some kinds of semi-abundant semigroups by *"category approaches"*. Fountain-Gomes-Gould [9] investigated this class of semigroups from the viewpoint of variety, and Gould [11] gave a survey of investigations of special semi-abundant semigroups, namely restriction semigroups and Ehresmann semigroups. Moreover, He-Shum-Wang [12] considered the representations of quasi-Ehresmann semigroups.

In this paper, we initiate the study of semi-abundant semigroups by using the idea of *"transversals"* which was firstly used to the study of regular semigroups by Blyth and McFadden in [3]. Specifically, we introduce the concept of quasi-Ehresmann transversals for semi-abundant semigroups, which is a generalization of

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the concept of orthodox transversals of regular semigroups, and give some properties associated with quasi-Ehresmann transversals. Furthermore, a structure theorem of semi-abundant semigroups with generalized bi-ideal strong quasi-Ehresmann transversals is obtained. Our results generalize and enrich the main results associated with orthodox transversals obtained in the texts Chen [4] and Chen-Guo [6].

2. Preliminaries

Let S be a semigroup. We use $E(S)$ to denote the set of idempotents of S. For $x, a \in S$, if $axa = a$ and $xax = x$, then a is called an *inverse* of x in S. We also let

$$V(x) = \{a \in S | axa = a, xax = x\}.$$

An element x in S is called *regular* if $V(x) \neq \emptyset$. A semigroup S is *regular* if every element in S is regular. A semigroup is regular if and only if each $\mathcal{L}$-class (or $\mathcal{R}$-class) of S contains idempotents. A regular semigroup S is called *orthodox* if $E(S)$ is a subsemigroup of S, an orthodox semigroup S is *inverse* if $E(S)$ is a commutative subsemigroup of S. For $\mathcal{K} \in \{\mathcal{L}, \mathcal{R}\}$ and $x \in S$, we use K_x to denote the $\mathcal{K}$-class of S containing x. On Green's relations, we also need the following results.

Lemma 2.1 ([13]). *For any semigroup S, the following statements are true:*

(1) If $e, f \in E(S)$ and $e\mathcal{D}f$ in S, then each element a of $R_e \cap L_f$ has a unique inverse a' in $R_f \cap L_e$ such that $aa' = e$ and $a'a = f$.

(2) If $a, b \in S$, then $ab \in R_a \cap L_b$ if and only if $L_a \cap R_b$ contains an idempotent.

Let S be a semigroup, S° a subsemigroup of S, $a \in S$ and $A \subseteq S$. Throughout this paper, we denote

$$V_{S^\circ}(a) = V(a) \cap S^\circ, V_{S^\circ}(A) = \bigcup_{a \in A} V_{S^\circ}(a).$$

Let S be a regular semigroup and S° a subsemigroup of S. According to Blyth-McFadden [3], S° is called an *inverse transversal* if $|V_{S^\circ}(a)| = 1$ for all $a \in S$. On the other hand, from Chen [4], a subsemigroup S° of a regular semigroup S is called *an orthodox transversal* of S if

(i) $V_{S^\circ}(a) \neq \emptyset$ for all $a \in S$;

(ii) $\{a, b\} \cap S^\circ \neq \emptyset$ implies that $V_{S^\circ}(b)V_{S^\circ}(a) \subseteq V_{S^\circ}(ab)$ for all $a, b \in S$.

On orthodox transversals, we need the following results.

Lemma 2.2 ([6]). *Let S be a regular semigroup and S° a subsemigroup of S such that $V_{S^\circ}(a) \neq \emptyset$ for all $a \in S$. Denote*

$$I = \{aa^\circ | a^\circ \in V_{S^\circ}(a), a \in S\}, \Lambda = \{a^\circ a | a^\circ \in V_{S^\circ}(a), a \in S\}.$$

(1) S is an orthodox semigroup if and only if $V_{S^\circ}(a)V_{S^\circ}(b) \subseteq V_{S^\circ}(ba)$ for all $a, b \in S$.

(2) S° is an orthodox transversal of S if and only if

$$IE(S^\circ) \subseteq I, E(S^\circ)\Lambda \subseteq \Lambda, E(S^\circ)I \subseteq E(S), \Lambda E(S^\circ) \subseteq E(S).$$

(3) If S° is an orthodox transversal of S, then the subsemigroup generated by I (resp. Λ) is a subband of S.

Let S be a semigroup and $a, b \in S$. That $a\widetilde{\mathcal{R}}b$ means that $ea = a$ if and only if $eb = b$ for all $e \in E(S)$. The relation $\widetilde{\mathcal{L}}$ can be defined dually. Denote $\widetilde{\mathcal{H}} = \widetilde{\mathcal{L}} \cap \widetilde{R}$. In general, $\widetilde{\mathcal{L}}$ is not a right congruence and $\widetilde{\mathcal{R}}$ is not a left congruence. Obviously, $\mathcal{L} \subseteq \tilde{\mathcal{L}}$ and $\mathcal{R} \subseteq \tilde{\mathcal{R}}$. If $a, b \in RegS$, the set of regular elements of S, then $a\widetilde{\mathcal{R}}b$ (resp. $a\widetilde{\mathcal{L}}b$) if and only if $a\mathcal{R}b$ (resp. $a\mathcal{L}b$). On the relation $\tilde{\mathcal{R}}$ on a semigroup S, we have the following easy but useful result.

Lemma 2.3. *Let S be a semigroup and $a \in S, e \in E(S)$. Then the following statements are equivalent:*

(1) $e\widetilde{\mathcal{R}}a$;

(2) $ea = a$ and for all $f \in E(S)$, $fa = a$ implies $fe = e$.

Now, we state the following fundamental concept of our paper.

Definition 2.4. *A semigroup S is called semi-abundant if the following conditions hold:*

(i) Each $\widetilde{\mathcal{L}}$-class and each $\widetilde{\mathcal{R}}$-class of S contains idempotents.

(ii) $\widetilde{\mathcal{L}}$ is a right congruence and $\widetilde{\mathcal{R}}$ is a left congruence on S, respectively.

A semi-abundant semigroup S is *quasi-Ehresmann* if its idempotents form a subsemigroup of S. Obviously, regular semigroups are semi-abundant, and orthodox semigroups are quasi-Ehresmann semigroups. Furthermore, a semi-abundant semigroup S is quasi-Ehresmann if and only if $RegS$ is an orthodox subsemigroup of S. Let S be a semi-abundant semigroup. For $\mathcal{K} \in \{\mathcal{L}, \mathcal{R}\}$ and $a \in S$, we use $\widetilde{K_a}$ to denote the $\widetilde{\mathcal{K}}$-class of S containing a.

Notation 2.5. *Let S be a* **quasi-Ehresmann semigroup**. *We use $a^\dagger$ and a^* to denote the typical idempotents contained in $\widetilde{R_a}$ and $\widetilde{L_a}$ for $a \in S$, respectively.*

Let S be a quasi-Ehresmann semigroup. Denote the $\mathcal{D}$-class of $E(S)$ containing the element $e \in E(S)$ by $E(e)$. Define the binary relation δ on S as follows:

$$a\delta b \text{ if and only if } b = eaf \text{ for some } e \in E(a^\dagger) \text{ and } f \in E(a^*).$$

On the relation δ on a quasi-Ehresmann semigroup S, we have the results below.

Lemma 2.6. *Let S be a quasi-Ehresmann semigroup, $a, b \in S$ and $b = eaf$ for some $e \in E(a^\dagger)$ and $f \in E(a^*)$. Then*

(1) $E(a^\dagger) = E(b^\dagger)$ and $E(a^*) = E(b^*)$ for any $b^\dagger$ and b^*.

(2) δ is an equivalent relation on S.

(3) $e\widetilde{\mathcal{R}}b\widetilde{\mathcal{L}}f$.

(4) $\widetilde{\mathcal{H}} \cap \delta$ is the identity relation on S.

Proof. (1) By the hypothesis, we have $E(e) = E(a^\dagger)$ and $E(f) = E(a^*)$. Furthermore, we also obtain $eb = b$ and $bf = b$. Since $b\widetilde{\mathcal{R}}b^\dagger$ and $b\widetilde{\mathcal{L}}b^*$, it follows that $eb^\dagger = b^\dagger$ and $b^*f = b^*$. This implies that $E(b^\dagger) \leq E(e) = E(a^\dagger)$ and $E(b^*) \leq E(f) = E(a^*)$. On the other hand, we have

$$a = a^\dagger aa^* = a^\dagger ea^\dagger aa^* fa^* = a^\dagger(ea^\dagger aa^* f)a^* = a^\dagger(eaf)a^* = a^\dagger ba^* = (a^\dagger b^\dagger)b(b^* a^*).$$

Observe that $a^\dagger b^\dagger \in E(b^\dagger)$ and $b^* a^* \in E(b^*)$, by the above discussions, it follows that $E(a^\dagger) \leq E(a^\dagger b^\dagger) = E(b^\dagger)$ and $E(a^*) \leq E(b^* a^*) = E(b^*)$. Thus, $E(a^\dagger) = E(b^\dagger)$ and $E(a^*) = E(b^*)$.

(2) Since $a = a^\dagger aa^*$ for all $a \in S$, δ is reflexive. Moreover, by the proof of item (1), it follows that δ is symmetric. Finally, let $a\delta b$, $b\delta c$ and

$$b = eaf, c = gbh, e \in E(a^\dagger), f \in E(a^*), g \in E(b^\dagger), h \in E(b^*).$$

By item (1), we have $E(a^\dagger) = E(b^\dagger), E(a^*) = E(b^*)$. This implies that $c = (ge)a(fh)$ and $ge \in E(a^\dagger), fh \in E(a^*)$ whence $a\delta c$. Therefore, δ is transitive.

(3) Let $k \in E(S)$ and $kb = b$. Then $keaf = eaf$. This implies that

$$kea = keaa^* = keaa^* fa^* = keafa^* = eafa^* = eaa^* fa^* = eaa^* = ea.$$

Since $a\widetilde{\mathcal{R}}a^{\dagger}$ and $\widetilde{\mathcal{R}}$ is a left congruence, we have $ea\widetilde{\mathcal{R}}ea^{\dagger}$ and so $kea^{\dagger} = ea^{\dagger}$. Thus,

$$ke = kea^{\dagger}e = ea^{\dagger}e = e$$

by the fact that $e \in E(a^{\dagger})$. By Lemma 2.3, $e\widetilde{\mathcal{R}}b$. Dually, $b\widetilde{\mathcal{L}}f$.

(4) If $a, b \in S$ and $a(\widetilde{\mathcal{H}} \cap \delta)b$, then $b = eaf$ for some $e \in E(a^{\dagger})$ and $f \in E(a^{*})$. This implies that

$$b = a^{\dagger}ba^{*} = a^{\dagger}eafa^{*} = a^{\dagger}ea^{\dagger}aa^{*}fa^{*} = a^{\dagger}aa^{*} = a,$$

as required. $\square$

A semi-abundant subsemigroup U of a semi-abundant semigroup S is called a *$\sim$-subsemigroup* of S if

$$\widetilde{\mathcal{L}}(U) = \widetilde{\mathcal{L}}(S) \cap (U \times U), \widetilde{\mathcal{R}}(U) = \widetilde{\mathcal{R}}(S) \cap (U \times U).$$

It is easy to see that a semi-abundant subsemigroup U of a semi-abundant semigroup S is a $\sim$-subsemigroup if and only if there exist $e, f \in E(U)$ such that $e\widetilde{\mathcal{L}}x$ and $f\widetilde{\mathcal{R}}x$ in S for all $x \in U$.

Now, let S be a semi-abundant semigroup and S° a quasi-Ehresmann $\sim$-subsemigroup of S. For any $x \in S$, denote

$$\Omega_{S^{\circ}}(x) = \{(e, \overline{x}, f) \in E(S) \times S^{\circ} \times E(S) | x = e\overline{x}f, e\mathcal{L}\overline{x}^{\dagger}, f\mathcal{R}\overline{x}^{*} \text{ for some } \overline{x}^{\dagger}, \overline{x}^{*} \in E(S^{\circ})\}$$

and

$$\Gamma_{S^{\circ}}(x) = \{\overline{x} | (e, \overline{x}, f) \in \Omega_{S^{\circ}}(x)\}, I_{S^{\circ}}(x) = \{e | (e, \overline{x}, f) \in \Omega_{S^{\circ}}(x)\},$$
$$\Lambda_{S^{\circ}}(x) = \{f | (e, \overline{x}, f) \in \Omega_{S^{\circ}}(x)\}, I_{S^{\circ}} = \bigcup_{x \in S} I_{S^{\circ}}(x), \Lambda_{S^{\circ}} = \bigcup_{x \in S} \Lambda_{S^{\circ}}(x).$$

For the sake of simplicity, if no confusion, we shall use $\Omega_x, \Gamma_x, I_x, \Lambda_x, I$ and Λ to denote $\Omega_{S^{\circ}}(x), \Gamma_{S^{\circ}}(x), I_{S^{\circ}}(x), \Lambda_{S^{\circ}}(x), I_{S^{\circ}}$ and $\Lambda_{S^{\circ}}$, respectively.

Lemma 2.7. *Let S be a semi-abundant semigroup and S° a quasi-Ehresmann $\sim$-subsemigroup of S.*

(1) $I = \{e \in E | (\exists e^{\circ} \in E(S^{\circ})) e\mathcal{L}e^{\circ}\}, \Lambda = \{f \in E | (\exists f^{\circ} \in E(S^{\circ})) f\mathcal{R}f^{\circ}\}$;

(2) $I \cap \Lambda = E(S^{\circ})$, $IE(S^{\circ}) \cup E(S^{\circ})\Lambda \subseteq RegS$.

Proof. (1) Let $e \in I$. Then, there exist $x \in S, \bar{x} \in S^{\circ}$ and $f \in E(S)$ such that $(e, \bar{x}, f) \in \Omega_x$. Thus, $e\mathcal{L}\bar{x}^{\dagger}$ for some $\bar{x}^{\dagger} \in E(S^{\circ})$. Conversely, if $e \in E(S)$ and $e\mathcal{L}e^{\circ} \in E(S^{\circ})$, then $(e, e^{\circ}, e^{\circ}) \in \Omega_e$, this shows that $e \in I$. A similar argument holds for Λ.

(2) By (1), $E(S^{\circ}) \subseteq I \cap \Lambda$. If $e \in I \cap \Lambda$, again by (1), there exist $e^{\circ}, e^{*} \in E(S^{\circ})$ such that $e^{\circ}\mathcal{L}e\mathcal{R}e^{*}$, which leads to $e = e^{*}e^{\circ} \in E(S^{\circ})$ by Lemma 2.1 (2). Let $e \in I$ and $f^{\circ} \in E(S^{\circ})$. Then, there exists $e^{\circ} \in E(S^{\circ})$ such that $e\mathcal{L}e^{\circ}$. Hence, $ef^{\circ}\mathcal{L}e^{\circ}f^{\circ} \in E(S^{\circ})$. This implies that $IE(S^{\circ}) \subseteq RegS$. Dually, $E(S^{\circ})\Lambda \subseteq RegS$. $\square$

In the following three lemmas, we always assume that S is a semi-abundant semigroup and S° is a quasi-Ehresmann $\sim$-subsemigroup of S.

Lemma 2.8. *If $x \in S$, $(e, \bar{x}, f) \in \Omega_x$ and $e\mathcal{L}\bar{x}^{\dagger}, f\mathcal{R}\bar{x}^{*}$ for some $\bar{x}^{\dagger}$ and $\bar{x}^{*}$ in $E(S^{\circ})$, then $\bar{x} = \bar{x}^{\dagger}x\bar{x}^{*}$ and $e\widetilde{\mathcal{R}}x\widetilde{\mathcal{L}}f$. In particular, if $x \in RegS$, we have $e\mathcal{R}x\mathcal{L}f$.*

Proof. By hypothesis, $x = e\bar{x}f$. This shows that $ex = x$. Now, let $g \in E(S)$ and $gx = x$. Then $ge\bar{x}f = e\bar{x}f$ whence

$$ge\bar{x} = ge\bar{x}\bar{x}^{*} = ge\bar{x}f\bar{x}^{*} = e\bar{x}f\bar{x}^{*} = e\bar{x}\bar{x}^{*} = e\bar{x}.$$

Since $\bar{x}\widetilde{\mathcal{R}}\bar{x}^{\dagger}$ and $\widetilde{\mathcal{R}}$ is a left congruence on S, it follows that $e\bar{x}\widetilde{\mathcal{R}}e\bar{x}^{\dagger} = e$. In view of the fact that $ge\bar{x} = e\bar{x}$, we have $ge = e$. By Lemma 2.3, $e\widetilde{\mathcal{R}}x$. Dually, we have $x\widetilde{\mathcal{L}}f$. Furthermore, we have $\bar{x}^{\dagger}x\bar{x}^{*} = \bar{x}^{\dagger}e\bar{x}f\bar{x}^{*} = \bar{x}^{\dagger}\bar{x}\bar{x}^{*} = \bar{x}$. $\square$

Lemma 2.9. *If $x, y \in S^\circ$ and $z \in S$ such that $x\widetilde{\mathcal{L}}z\widetilde{\mathcal{R}}y$ and $\Gamma_z \neq \emptyset$. Then $z \in S^\circ$. In particular, if $x\widetilde{\mathcal{H}}z, \Gamma_z \neq \emptyset$ and $x \in S^\circ$, then $z \in S^\circ$.*

Proof. Let $x^*\widetilde{\mathcal{L}}x\widetilde{\mathcal{L}}z\widetilde{\mathcal{R}}y\widetilde{\mathcal{R}}y^\dagger$ for some $x^*, y^\dagger \in E(S^\circ)$. Let $(e, \bar{z}, f) \in \Omega_z$ and $\bar{z}^*\mathcal{R}f$ for some $\bar{z}^*$ in $E(S^\circ)$. Then, by Lemma 2.8, $f\widetilde{\mathcal{L}}z$. This implies that $\bar{z}^*\mathcal{R}f\mathcal{L}x^*$. By Lemma 2.1 (2), we have $\bar{z}^*\mathcal{L}x^*\bar{z}^*\mathcal{R}x^*$. Since $\bar{z}^*x^*, x^*\bar{z}^* \in E(S^\circ)$ and $f \in E(S)$, by Lemma 2.1 (2) again, $f\mathcal{H}\bar{z}^*x^*$ and so $f = \bar{z}^*x^* \in S^\circ$. Dually, $e \in S^\circ$. Hence, $z = e\bar{z}f \in S^\circ$. □

Lemma 2.10. *For any $x \in S$ and $\bar{x} \in \Gamma_x$, $x \in RegS$ if and only if $\bar{x} \in RegS^\circ$. In this case, $I_x = \{xx^\circ | x^\circ \in V_{S^\circ}(x)\}$, $\Lambda_x = \{x^\circ x | x^\circ \in V_{S^\circ}(x)\}$ and $\Gamma_x = V_{S^\circ}(V_{S^\circ}(x))$.*

Proof. Let $x \in RegS, (e, \bar{x}, f) \in \Omega_x$ and $e\mathcal{L}\bar{x}^\dagger, f\mathcal{R}\bar{x}^*$ for some $\bar{x}^\dagger, \bar{x}^* \in E(S^\circ)$. Then, by Lemma 2.8, $f\mathcal{L}x\mathcal{R}e$ and $\bar{x} = \bar{x}^\dagger x\bar{x}^*$. This deduces that there exist $x' \in V(x)$ and $x'' \in V(x')$ such that $xx' = e, x'x = f$ and $x'x'' = \bar{x}^*, x''x' = \bar{x}^\dagger$ from Lemma 2.1 (1). Moreover, by Lemma 2.1 (2), we have the following egg-box diagram:

$x = e\bar{x}f$	e	$x\bar{x}^*, e\bar{x}$
f	x'	$\bar{x}^*$
	$\bar{x}^\dagger$	$\bar{x} = \bar{x}^\dagger x\bar{x}^*,\ x''$

.

Observe that $x = xx'x''x'x = ex''f$, it follows that

$$\bar{x} = \bar{x}^\dagger x\bar{x}^* = \bar{x}^\dagger ex''f\bar{x}^* = \bar{x}^\dagger x''\bar{x}^* = x''.$$

Since $\bar{x}^*\mathcal{R}x'\mathcal{L}\bar{x}^\dagger$ and $\bar{x}^*, \bar{x}^\dagger \in S^\circ$, it follows that $x' \in S^\circ$ by Lemma 2.9. This implies that $x' \in V_{S^\circ}(\bar{x})$ and so $\bar{x} \in RegS^\circ$. Conversely, let $\bar{x} \in RegS^\circ$. By very similar method, we can see that $x \in RegS$.

On the other hand, by the discussions above, for all $x \in RegS$ and $(e, \bar{x}, f) \in \Omega_x$, we have $e = xx'$ and $f = x'x$ for some $x' \in V_{S^\circ}(x) \cap V_{S^\circ}(\bar{x})$. This implies that

$$I_x \subseteq \{xx' | x' \in V_{S^\circ}(x)\}, \Lambda_x \subseteq \{x'x | x' \in V_{S^\circ}(x)\}, \Gamma_x \subseteq V_{S^\circ}(V_{S^\circ}(x))$$

for all $x \in RegS$.

Now, let $x \in RegS$, $x' \in V_{S^\circ}(x)$ and $x'' \in V_{S^\circ}(x')$. Since

$$xx'\mathcal{L}x''x'\widetilde{\mathcal{R}}x'', x'x\mathcal{R}x'x''\widetilde{\mathcal{L}}x'', x = (xx')x''(x'x), x''x', x'x'' \in E(S^\circ),$$

it follows that $(xx', x'', x'x) \in \Omega_x$, whence $xx' \in I_x, x'x \in \Lambda_x$ and $x'' \in \Gamma_x$. Therefore,

$$\{xx' | x' \in V_{S^\circ}(x)\} \subseteq I_x, \{x'x | x' \in V_{S^\circ}(x)\} \subseteq \Lambda_x, V_{S^\circ}(V_{S^\circ}(x)) \subseteq \Gamma_x.$$

Thus, the three equalities in this lemma hold. □

3. Quasi-Ehresmann Transversals

This section will explore some properties of semi-abundant semigroups with quasi-Ehresmann transversals. We first give the following concept, which is inspired by Lemma 2.2 (2) and Lemma 2.10.

Definition 3.1. *Let S be a semi-abundant semigroup and S° a quasi-Ehresmann $\sim$-subsemigroup of S. Then S° is called a* quasi-Ehresmann transversal *of S if the following conditions hold:*

(i) $\Gamma_x \neq \emptyset$ for all $x \in S$;

(ii) $is \in I$ and "$si \in RegS \Rightarrow si \in E(S)$" for all $i \in I$ and $s \in E(S^\circ)$;

(iii) $s\lambda \in \Lambda$ and "$\lambda s \in RegS \Rightarrow \lambda s \in E(S)$" for all $\lambda \in \Lambda$ and $s \in E(S^\circ)$.

We first observe that quasi-Ehresmann transversals of semi-abundant semigroups are indeed generalizations of orthodox transversals of regular semigroups.

Theorem 3.2. *Let S be a regular semigroup and S° a subsemigroup of S. Then S° is an orthodox transversal of S if and only if S° is a quasi-Ehresmann transversal of S.*

Proof. Let S° be an orthodox transversal of S. Then S° is an orthodox subsemigroup of S and certainly a quasi-Ehresmann $\sim$-subsemigroup of S. Observe that $(xx', x'', x'x) \in \Omega_x$ for every $x \in S, x' \in V_{S^\circ}(x)$ and $x'' \in V_{S^\circ}(x')$. This shows that $\Gamma_x \neq \emptyset$ for any $x \in S$, and so the condition (i) in Definition 3.1 holds. On the other hand, by Lemma 2.10, we have

$$I = \{xx' | x' \in V_{S^\circ}(x), x \in S\}, \Lambda = \{x'x | x' \in V_{S^\circ}(x), x \in S\}.$$

By Lemma 2.2 (2), the conditions (ii) and (iii) in Definition 3.1 are satisfied, Thus, S° is a quasi-Ehresmann transversal of S.

Conversely, let S° be a quasi-Ehresmann transversal of S. By Lemma 2.10 again,

$$I_x = \{xx' | x' \in V_{S^\circ}(x)\}, \Lambda_x = \{x'x | x' \in V_{S^\circ}(x)\}, \Gamma_x = V_{S^\circ}(V_{S^\circ}(x))$$

for all $x \in RegS$. Observe that S is regular, it follows that S° is an orthodox transversal of S from Definition 3.1 and Lemma 2.2 (2). □

In the remainder of this section, we always assume that S is a semi-abundant semigroup with a quasi-Ehresmann transversal S°. In the sequel, we characterize the relations $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{R}}$ on S.

Theorem 3.3. *Let $x, y \in S$.*

(1) $x\tilde{\mathcal{R}}y$ if and only if $I_x = I_y$;

(2) $x\tilde{\mathcal{L}}y$ if and only if $\Lambda_x = \Lambda_y$.

Proof. (1) Assume that $I_x = I_y$ and $e \in I_x = I_y$. By Lemma 2.8, we have $x\tilde{\mathcal{R}}e\tilde{\mathcal{R}}y$ and so $x\tilde{\mathcal{R}}y$. Now, let $x\tilde{\mathcal{R}}y$, $(e, \bar{x}, f) \in \Omega_x$ and $(g, \bar{y}, h) \in \Omega_y$. Then $e\mathcal{L}\bar{x}^\dagger, f\mathcal{R}\bar{x}^*$ and $g\mathcal{L}\bar{y}^\dagger, h\mathcal{R}\bar{y}^*$ for some $\bar{x}^\dagger, \bar{x}^*$ and $\bar{y}^\dagger, \bar{y}^*$ in $E(S^\circ)$. By Lemma 2.8, $e\tilde{\mathcal{R}}x\tilde{\mathcal{R}}y\tilde{\mathcal{R}}g$ and so $e\mathcal{R}g$. Then, by Definition 3.1 (ii) and Lemma 2.1 (2), we have the following graph:

$e = g\bar{x}^\dagger \in E(S)$	$g = e\bar{y}^\dagger \in E(S)$
$\bar{x}^\dagger$	$\bar{x}^\dagger\bar{y}^\dagger = \bar{x}^\dagger g \in E(S)$
$\bar{y}^\dagger\bar{x}^\dagger = \bar{y}^\dagger e \in E(S)$	$\bar{y}^\dagger$

.

Hence,

$$y = g\bar{y}h = (e\bar{y}^\dagger)\bar{y}h = e(\bar{y}^\dagger\bar{y})h = e\bar{y}h = (e\bar{x}^\dagger)\bar{y}h = e(\bar{x}^\dagger\bar{y})h.$$

We assert that $\bar{x}^\dagger\tilde{\mathcal{R}}\bar{x}^\dagger\bar{y}\tilde{\mathcal{L}}\bar{y}^*$. In fact, let $m \in E(S)$ and $m\bar{x}^\dagger\bar{y} = \bar{x}^\dagger\bar{y}$. Observe that $\bar{y}\tilde{\mathcal{R}}\bar{y}^\dagger$ and $\tilde{\mathcal{R}}$ is a left congruence on S, it follows that $\bar{x}^\dagger\bar{y}\tilde{\mathcal{R}}\bar{x}^\dagger\bar{y}^\dagger$. By Lemma 2.3, we have $m\bar{x}^\dagger\bar{y}^\dagger = \bar{x}^\dagger\bar{y}^\dagger$ whence $m\bar{x}^\dagger = m\bar{x}^\dagger\bar{y}^\dagger\bar{x}^\dagger = \bar{x}^\dagger\bar{y}^\dagger\bar{x}^\dagger = \bar{x}^\dagger$. Observe that $\bar{x}^\dagger(\bar{x}^\dagger\bar{y}) = \bar{x}^\dagger\bar{y}$, it follows that $\bar{x}^\dagger\tilde{\mathcal{R}}\bar{x}^\dagger\bar{y}$ by Lemma 2.3 again. On the other hand, if $n \in E(S)$ and $\bar{x}^\dagger\bar{y} = \bar{x}^\dagger\bar{y}n$, then

$$\bar{y} = \bar{y}^\dagger\bar{x}^\dagger(\bar{y}^\dagger\bar{y}) = \bar{y}^\dagger\bar{x}^\dagger(\bar{y}^\dagger\bar{y})n = \bar{y}n.$$

By $\bar{y}\tilde{\mathcal{L}}\bar{y}^*$ and the dual of Lemma 2.3, we have $\bar{y}^* = \bar{y}^*n$. Observe that $\bar{x}^\dagger\bar{y}\bar{y}^* = \bar{x}^\dagger\bar{y}$, by the dual of Lemma 2.3 again, $\bar{x}^\dagger\bar{y}\tilde{\mathcal{L}}\bar{y}^*$. By the above discussions, we have $e\mathcal{L}\bar{x}^\dagger\tilde{\mathcal{R}}\bar{x}^\dagger\bar{y}$ and $\bar{x}^\dagger\bar{y}\tilde{\mathcal{L}}\bar{y}^*\mathcal{R}h$. This implies that $(e, \bar{x}^\dagger\bar{y}, h) \in \Omega_y$ and so $e \in I_y$. Hence, $I_x \subseteq I_y$. Dually, $I_y \subseteq I_x$.

(2) The is the dual of (1). □

Now, we investigate some properties of Γ_x for $x \in S$.

Theorem 3.4. *Let $x \in S$ and $(e, \overline{x}, f) \in \Omega_x$.*

(1) $\Gamma_x = \{\overline{y} \in S^\circ | \overline{y}\delta\overline{x}\}$.

(2) $\Gamma_{x_1} = \Gamma_{x_2}$ if and only if $\Gamma_{x_1} \cap \Gamma_{x_2} \neq \emptyset$ for all $x_1, x_2 \in S$.

(3) $\Gamma_x \cap E(S^\circ) \neq \emptyset$ implies that $\Gamma_x \subseteq E(S^\circ)$ and $V_{S^\circ}(x) \subseteq E(S^\circ)$.

Proof. (1) Let $(e_1, \overline{y}, f_1) \in \Omega_x$. By Lemma 2.8, we can let

$$\bar{x}^\dagger \mathcal{L} e \mathcal{R} e_1 \mathcal{L} \bar{y}^\dagger, \;\; \bar{x}^* \mathcal{R} f \mathcal{L} f_1 \mathcal{R} \bar{y}^*$$

for some $\bar{x}^\dagger, \bar{y}^\dagger, \bar{x}^*$ and $\bar{y}^*$ in $E(S^\circ)$. In view of Lemma 2.1, $\bar{x}^\dagger e_1 \mathcal{L} e_1$ and $f_1 \bar{x}^* \mathcal{L} \bar{x}^*$.

e	e_1
$\overline{x}^\dagger$	$\overline{x}^\dagger e_1$
$\overline{y}^\dagger \bar{x}^\dagger$	$\overline{y}^\dagger$

f	$\overline{x}^*$	$f\overline{y}^*$
f_1	$f_1\overline{x}^*$	$\overline{y}^*$

By Definition 3.1 (ii),(iii), we can obtain that $\bar{x}^\dagger e_1, f_1 \bar{x}^* \in E(S)$. Again by Lemma 2.1, $\bar{x}^\dagger e_1 = \bar{x}^\dagger \bar{y}^\dagger$ and $f_1 \bar{x}^* = \bar{y}^* \bar{x}^*$. Thus, by Lemma 2.8,

$$\bar{x} = \bar{x}^\dagger x \bar{x}^* = \bar{x}^\dagger e_1 \bar{y} f_1 \bar{x}^* = \bar{x}^\dagger \bar{y}^\dagger \bar{y} \bar{y}^* \bar{x}^* = \bar{x}^\dagger \cdot \bar{y} \cdot \bar{x}^*,$$

where $\bar{x}^\dagger \in E(\bar{y}^\dagger)$ and $\bar{x}^* \in E(\bar{y}^*)$. This implies that $\bar{x} \delta \bar{y}$.

On the other hand, if $\overline{y} \in S^\circ$, $\overline{y} \delta \overline{x}$ and $e \mathcal{L} \bar{x}^\dagger$ for some $\bar{x}^\dagger$ in $E(S^\circ)$, then there exist $i \in E(\overline{y}^\dagger), \lambda \in E(\overline{y}^*)$ such that $\overline{x} = i\overline{y}\lambda$ for some (all) $\overline{y}^\dagger$ and $\overline{y}^*$ in $E(S^\circ)$ (Notice that $i, \lambda \in E(S^\circ)$). By Lemma 2.6, $E(\overline{x}^\dagger) = E(\overline{y}^\dagger)$. According to Lemma 2.1 (1), we have

e	$ei\overline{y}^\dagger$	ei
$\overline{x}^\dagger$	$\overline{x}^\dagger \overline{y}^\dagger$	
$\overline{y}^\dagger \overline{x}^\dagger$	$\overline{y}^\dagger$	$\overline{y}^\dagger i$
$i\overline{y}^\dagger \overline{x}^\dagger$	$i\overline{y}^\dagger$	i

.

Since $e \in I$ and $i\bar{y}^\dagger \in E(S^\circ)$, $ei\bar{y}^\dagger \in I \subseteq E(S)$ by Definition 3.1 (ii). Thus, $ei\overline{y}^\dagger \mathcal{L} \overline{y}^\dagger$. Dually, we can obtain $\overline{y}^* \lambda f \in E(S)$ and $\overline{y}^* \mathcal{R} \overline{y}^* \lambda f$. Observe that

$$x = e\bar{x}f = ei\overline{y}\lambda f = (ei\overline{y}^\dagger)\overline{y}(\overline{y}^* \lambda f),$$

$(ei\overline{y}^\dagger, \overline{y}, \overline{y}^* \lambda f) \in \Omega_x$ and $\overline{y} \in \Gamma_x$.

(2) This is a direct consequence of item (1) and Lemma 2.6 (2).

(3) Let $\bar{x} \in \Gamma_x$ and $e^\circ \in \Gamma_x \cap E(S^\circ)$. Then, $e^\circ \delta \bar{x}$ by (1). Hence, there exist $k, l \in E(e^\circ)$ such that $\bar{x} = k e^\circ l$, which implies that $\bar{x} \in E(S^\circ)$. On the other hand, by Lemma 2.10, in this case,

$$\Gamma_x = V_{S^\circ}(V_{S^\circ}(x)) \subseteq E(S^\circ).$$

Since $RegS^\circ$ is orthodox, we have $V_{S^\circ}(x) \subseteq E(S^\circ)$. $\square$

The following theorem shows that quasi-Ehresmann transversal have transitivity.

Theorem 3.5. *Let S be a semi-abundant semigroup with a quasi-Ehresmann transversal S° and S^* a quasi-Ehresmann transversal of S°. Then S^* is a quasi-Ehresmann transversal of S.*

Proof. By Lemma 2.7, $I_{S^\circ} = \{e \in E(S) | (\exists e^\circ \in E(S^\circ)) e \mathcal{L} e^\circ\}$. Let $x \in S$ and $(e_1, x_1, f_1) \in \Omega_{S^\circ}(x)$ with $e_1 \mathcal{L} x_1^\dagger \tilde{\mathcal{R}} x_1$ and $x_1^\dagger \in E(S^\circ)$. Let $(e_2, x_2, f_2) \in \Omega_{S^*}(x_1)$ such that (In view of Lemma 2.8)

$$x_1^\dagger \tilde{\mathcal{R}} x_1 \tilde{\mathcal{R}} e_2 \mathcal{L} x_2^\dagger \tilde{\mathcal{R}} x_2, \; x_1 \tilde{\mathcal{L}} f_2 \mathcal{R} x_2^* \tilde{\mathcal{L}} x_2, \; x_2^\dagger, x_2^* \in E(S^*), \; e_2, f_2 \in E(S^\circ).$$

Then $e_1 \mathcal{L} x_1^\dagger \mathcal{R} e_2 \mathcal{L} x_2^\dagger$. By Lemma 2.1, $e_1 e_2 \mathcal{L} x_2^\dagger$. On the other hand, since $e_1 \in I_{S^\circ}$ and $e_2 \in E(S^\circ)$, $e_1 e_2 \in I_{S^\circ} \subseteq E(S)$ by Definition 3.1 (ii). Dually, we can obtain that $f_2 f_1 \in E(S)$ and $f_2 f_1 \mathcal{R} x_2^*$. Observe that $x = e_1 x_1 f_1 = (e_1 e_2) x_2 (f_2 f_1)$, it follows that $(e_1 e_2, x_2, f_2 f_1) \in \Omega_{S^*}(x)$. This implies that $\Gamma_{S^*}(x) \neq \emptyset$ for all $x \in S$.

On the other hand, by Lemma 2.7 again, we have

$$I_{S^*} = \{e \in E(S) | (\exists e^* \in E(S^*)) e\mathcal{L}e^*\}, \Lambda_{S^*} = \{f \in E(S) | (\exists f^* \in E(S^*)) f\mathcal{R}f^*\}.$$

Let $s, e^* \in E(S^*) \subseteq E(S^\circ)$ and $e^*\mathcal{L}e \in I_{S^*} \subseteq I_{S^\circ}$. Apply Definition 3.1 (ii) to I_{S°, $es \in I_{S^\circ} \subseteq E(S)$. Observe that $es\mathcal{L}e^*s \in E(S^*)$, it follows that $es \in I_{S^*}$. On the other hand, let $se \in RegS$. Apply Definition 3.1 (ii) to I_{S° again, $se \in E(S)$. Hence, Definition 3.1 (ii) for I_{S^*} is satisfied. Dually, we can prove Definition 3.1 (iii) for Λ_{S^*} also holds. Thus, S^* is a quasi-Ehresmann transversal of S. □

From Lemma 2.7, we have $IE(S^\circ) \cup E(S^\circ)\Lambda \subseteq RegS$. In the following, we shall give some equivalent conditions such that $E(S^\circ)I \cup \Lambda E(S^\circ) \subseteq RegS$. We give the lemma below firstly.

Lemma 3.6. *Let $a, b \in RegS, e, f \in I$ and $g, h \in \Lambda$. Then*

(1) If $a^\circ \in V_{S^\circ}(a)$, then $V_{S^\circ}(a) = V_{S^\circ}(a^\circ a)a^\circ V_{S^\circ}(aa^\circ)$;

(2) If $e\mathcal{L}f$, then $V_{S^\circ}(e) = V_{S^\circ}(f)$;

(3) If $g\mathcal{R}h$, then $V_{S^\circ}(g) = V_{S^\circ}(h)$;

(4) If $V_{S^\circ}(a) \cap V_{S^\circ}(b) \neq \emptyset$, then $V_{S^\circ}(a) = V_{S^\circ}(b)$.

Proof. (1) Let $a^* \in V_{S^\circ}(a)$ and $a^{\circ\circ} \in V_{S^\circ}(a^\circ)$. Then, by Lemma 2.1 (2)

$$a^{\circ\circ}a^\circ\mathcal{R}a^{\circ\circ}a^\circ aa^*\mathcal{L}a^*\mathcal{R}a^*aa^\circ a^{\circ\circ}\mathcal{L}a^\circ a^{\circ\circ}.$$

By Lemma 2.9, $a^{\circ\circ}a^\circ aa^*, a^*aa^\circ a^{\circ\circ} \in S^\circ$. The remainder is similar to the proof of Lemma 2.4 in Chen [4].

(2) Let $t \in V_{S^\circ}(e)$. By Lemma 2.7, we may let $e\mathcal{L}f\mathcal{L}h$ for some $h \in E(S^\circ)$. Then, $(e, h, h) \in \Omega_e$ and so $h \in \Gamma_e \cap E(S^\circ)$. By (3) of Theorem 3.4, $t \in V_{S^\circ}(e) \subseteq E(S^\circ)$. In view of Definition 3.1 (ii), we have $ft \in I$. Observe that $t\mathcal{R}te\mathcal{L}e\mathcal{L}f$, it follows that $f\mathcal{R}ft\mathcal{L}t$ by Lemma 2.1. Since $ft \in I \subseteq E(S)$, by Lemma 2.1 again, $tf\mathcal{H}te \in E(S)$. This implies that $tf \in RegS$. By Definition 3.1 (ii), $tf \in E(S)$. Hence, $tf = te$.

e	et
f	$ft \in I$
$tf = te$	t
h	

This implies that $tft = tet = t$ and $ftf = (ft)f = f$. Therefore, $t \in V_{S^\circ}(f)$ and so $V_{S^\circ}(e) \subseteq V_{S^\circ}(f)$. Dually, $V_{S^\circ}(f) \subseteq V_{S^\circ}(e)$.

(3) This is the dual of (2).

(4) Let $x \in V_{S^\circ}(a) \cap V_{S^\circ}(b)$. Then $ax\mathcal{L}bx$ and $xa\mathcal{R}xb$. In view of Lemma 2.10, we have $ax, bx \in I$ and $xa, xb \in \Lambda$. By (1), (2) and (3), we have

$$V_{S^\circ}(a) = V_{S^\circ}(xa)xV_{S^\circ}(ax) = V_{S^\circ}(xb)xV_{S^\circ}(bx) = V_{S^\circ}(b),$$

as required. □

Theorem 3.7. *The following conditions on S are equivalent:*

(1) $(\forall u, v \in I \cup \Lambda)$ "$\{u, v\} \cap E(S^\circ) \neq \emptyset \Rightarrow \Gamma_u\Gamma_v \subseteq \Gamma_{uv}$";

(2) $E(S^\circ)I \subseteq E(S), \Lambda E(S^\circ) \subseteq E(S)$;

(3) $(\forall a, b \in RegS)$ "$\{a, b\} \cap S^\circ \neq \emptyset \Rightarrow V_{S^\circ}(b)V_{S^\circ}(a) \subseteq V_{S^\circ}(ab)$".

Proof. (1)*implies* (2). Let $i \in I, \lambda \in \Lambda$ and $s \in E(S^\circ)$. By Definition 3.1 (ii) and (iii), it suffices to show $si, \lambda s \in RegS$. In fact, by Lemma 2.7, there exist $i^\circ, \lambda^\circ \in E(S^\circ)$ such that $i^\circ \mathcal{L} i$ and $\lambda^\circ \mathcal{R} \lambda$. This implies that $(i, i^\circ, i^\circ) \in \Omega_i$ and $(\lambda^\circ, \lambda^\circ, \lambda) \in \Omega_\lambda$. Hence, $i^\circ \in \Gamma_i$ and $\lambda^\circ \in \Gamma_\lambda$. Clearly, $s \in \Gamma_s$. By (1), $si^\circ \in \Gamma_{si} \cap E(S^\circ)$ and $\lambda^\circ s \in \Gamma_{\lambda s} \cap E(S^\circ)$. In view of Lemma 2.10, we have $si, \lambda s \in RegS$.

(2) *implies* (3). Let $a \in RegS^\circ$ and $b \in RegS$. Take $a^\circ \in V_{S^\circ}(a)$ and $b^\circ \in V_{S^\circ}(b)$. Then $a^\circ a \in E(S^\circ)$ and $bb^\circ \in I$ by Lemma 2.10. By (2) and Definition 3.1 (ii), we have

$$abb^\circ a^\circ ab = a(a^\circ abb^\circ)(a^\circ abb^\circ)b = aa^\circ abb^\circ b = ab$$

and

$$b^\circ a^\circ abb^\circ a^\circ = b^\circ(bb^\circ a^\circ a)(bb^\circ a^\circ a)a^\circ = b^\circ bb^\circ a^\circ aa^\circ = b^\circ a^\circ.$$

Dually, we can prove the case for $a \in RegS$ and $b \in RegS^\circ$.

(3) *implies* (1). Let $u \in E(S^\circ)$ and $v \in I \cup \Lambda$. Clearly, $u, v \in RegS$. Take

$$u^\circ \in V_{S^\circ}(u), u^{\circ\circ} \in V_{S^\circ}(u^\circ), v^\circ \in V_{S^\circ}(v), v^{\circ\circ} \in V_{S^\circ}(v^\circ).$$

Then by (3), $v^\circ u^\circ \in V_{S^\circ}(uv)$. Since $RegS^\circ$ is orthodox, we have

$$u^{\circ\circ}v^{\circ\circ} \in V_{S^\circ}(u^\circ)V_{S^\circ}(v^\circ) \subseteq V_{S^\circ}(v^\circ u^\circ) \subseteq V_{S^\circ}(V_{S^\circ}(uv)).$$

Hence, by Lemma 2.10, $\Gamma_u\Gamma_v \subseteq \Gamma_{uv}$. Similarly, we can show the case for $v \in E(S^\circ)$ and $u \in I \cup \Lambda$. This implies that (1) holds. □

The following Theorem 3.8 yields that if Condition (1) of Theorme 3.7 is strengthened by removing $\{u, v\} \cap E(S^\circ) \neq \emptyset$, then S itself is quasi-Ehreshmann.

Theorem 3.8. *The following conditions on S are equivalent:*

(1) $(\forall u, v \in I \cup \Lambda)\ \Gamma_u\Gamma_v \subseteq \Gamma_{uv}$*;*

(2) $\Lambda I, I\Lambda \subseteq E(S)$*;*

(3) S is quasi-Ehreshmann.

Proof. (1) *implies* (2). Let $i \in I$ and $\lambda \in \Lambda$. Then, by Lemma 2.7 (1), there exist $i^\circ, \lambda^\circ \in E(S^\circ)$ such that $i\mathcal{L}i^\circ$ and $\lambda\mathcal{R}\lambda^\circ$. This shows that $(i, i^\circ, i^\circ) \in \Omega_i$ and $(\lambda^\circ, \lambda^\circ, \lambda) \in \Omega_\lambda$. Hence, $i^\circ \in \Gamma_i$ and $\lambda^\circ \in \Gamma_\lambda$. By (1), $\lambda^\circ i^\circ \in \Gamma_{\lambda i} \cap E(S^\circ)$. In view of Lemma 2.10, $\lambda i \in RegS$ and $\Gamma_{\lambda i} = V_{S^\circ}(V_{S^\circ}(\lambda i))$ whence $\lambda^\circ i^\circ \in V_{S^\circ}(V_{S^\circ}(\lambda i))$. Hence, there exists $(\lambda i)^\circ \in V_{S^\circ}(\lambda i) \cap V_{S^\circ}(\lambda^\circ i^\circ)$. By Lemma 3.6 (4), $V_{S^\circ}(\lambda i) = V_{S^\circ}(\lambda^\circ i^\circ)$. Noticing that $i^\circ\lambda^\circ, \lambda^\circ i^\circ \in V_{S^\circ}(\lambda^\circ i^\circ)$, we have $\lambda^\circ i^\circ, i^\circ\lambda^\circ \in V_{S^\circ}(\lambda i)$. Thus,

$$\lambda i = \lambda i i^\circ \lambda^\circ \lambda i = \lambda i \lambda i \in E(S).$$

On the other hand, by similar arguments, we can obtain $\lambda^\circ i^\circ \in V_{S^\circ}(i\lambda)$. Hence,

$$i\lambda = i\lambda\lambda^\circ i^\circ i\lambda = i\lambda^\circ i^\circ \lambda.$$

Since $\lambda^\circ i^\circ \in V_{S^\circ}(\lambda i)$, this implies that

$$i\lambda i\lambda = i(\lambda^\circ i^\circ \lambda i \lambda^\circ i^\circ)\lambda = i\lambda^\circ i^\circ \lambda = i\lambda \in E(S).$$

(2) *implies* (3). Let $a, b \in RegS$. Then, we can take $a^\circ \in V_{S^\circ}(a)$ and $b^\circ \in V_{S^\circ}(b)$ by Lemma 2.10. We assert $b^\circ a^\circ \in V_{S^\circ}(ab)$. In fact, since $bb^\circ \in I$ and $a^\circ a \in \Lambda$ by Lemma 2.10, by (2),

$$b^\circ a^\circ abb^\circ a^\circ = b^\circ(bb^\circ a^\circ a)(bb^\circ a^\circ a)a^\circ = b^\circ a^\circ$$

and

$$abb^\circ a^\circ ab = a(a^\circ abb^\circ)(a^\circ abb^\circ)b = ab.$$

Hence, $RegS$ is a regular subsemigroup of S and

$$V_{RegS^\circ}(b)V_{RegS^\circ}(a) \subseteq V_{RegS^\circ}(ab)$$

for each $a, b \in RegS$. It is clear that $RegS^\circ$ is a subsemigroup of $RegS$ and $V_{RegS^\circ}(a) = V_{S^\circ}(a) \neq \emptyset$ for each $a \in RegS$. In view of Lemma 2.2 (1), $RegS$ is orthodox. Thus, S is quasi-Ehreshmann.

(3) *implies* (1). Let $u, v \in I \cup \Lambda$. Then $u, v \in RegS$. Take

$$u^\circ \in V_{S^\circ}(u), u^{\circ\circ} \in V_{S^\circ}(u^\circ), v^\circ \in V_{S^\circ}(v), v^{\circ\circ} \in V_{S^\circ}(v^\circ).$$

By (3), $RegS$ is orthodox. This implies $v^\circ u^\circ \subseteq V_{S^\circ}(uv)$. Hence,

$$u^{\circ\circ}v^{\circ\circ} \in V_{S^\circ}(u^\circ)V_{S^\circ}(v^\circ) \subseteq V_{S^\circ}(v^\circ u^\circ) \subseteq V_{S^\circ}(V_{S^\circ}(uv)).$$

In view of Lemma 2.10, $\Gamma_u\Gamma_v \subseteq \Gamma_{uv}$. $\square$

Let S be a semi-abundant semigroup and S° a quasi-Ehreshmann transversal of S. We shall say that S° is *strong* if one (equivalently, all) of the conditions in Theorem 3.7 holds. Obviously, orthodox transversals are strong quasi-Ehresmann transversals by Theorem 3.7 (3). However, quasi-Ehresmann transversals may not be strong in general. The following result illustrates this situation.

Example 3.9. (Example 2.7 in [5]) *Let $S = \{e, g, h, w, f\}$ with the following multiplication table*

	e	g	h	w	f
e	e	g	e	g	g
g	g	g	g	g	g
h	h	g	h	g	g
w	w	g	w	g	g
f	g	g	w	w	f

.

Then, it is routine to check that S is a semi-abundant semigroup with a quasi-Ehresmann transversal $S^\circ = \{w, e, f, g\}$. In this case, $I = \{e, h, f, g\}$ and $f \in E(S^\circ)$, but $fh = w \notin E(S)$.

Theorem 3.10. *Let S be a semi-abundant semigroup with a strong quasi-Ehreshmann transversal S° and $\bar{I}$ the subsemigroup generated by I. Then*

(1) For $i_k \in I$ and $i_k^\circ \in E(S^\circ)$ such that $i_k \mathcal{L} i_k^\circ$, where $k = 1, 2, \cdots, n$, we have $i_n^\circ i_{n-1}^\circ \cdots i_1^\circ \in V_{S^\circ}(i_1 i_2 \cdots i_n)$.

(2) $E(S^\circ)$ is an orthodox transversal of $\bar{I}$ and $\bar{I}$ is a subband of S.

Dually, we have a symmetrical result for Λ.

Proof. (1) Clearly, the result holds for the case $n = 1$. Now, we assume that the result holds for $n = t - 1$ and prove that it is also true for $n = t$. Let

$$i_1, i_2, \cdots, i_t \in I, \ x = i_1 i_2 \cdots i_t.$$

Then, by hypothesis, $i_t^\circ i_{t-1}^\circ \cdots i_2^\circ \in V_{S^\circ}(i_2 i_3 \cdots i_t)$, which shows that $i_2 i_3 \cdots i_t \in RegS$. Clearly, $i_1^\circ \in RegS^\circ$. By (3) of Theorem 3.7, we have $i_t^\circ i_{t-1}^\circ \cdots i_1^\circ \in V_{S^\circ}(i_1^\circ i_2 \cdots i_t)$. This yields $i_t^\circ i_{t-1}^\circ \cdots i_1^\circ \in V_{S^\circ}(x)$. Indeed, observe that $i_k \mathcal{L} i_k^\circ, k = 1, 2, 3, \cdots, t$, it follows that

$$i_t^\circ i_{t-1}^\circ \cdots i_1^\circ x i_t^\circ i_{t-1}^\circ \cdots i_1^\circ = i_t^\circ i_{t-1}^\circ \cdots i_1^\circ (i_1^\circ i_1 i_2 \cdots i_t) i_t^\circ i_{t-1}^\circ \cdots i_1^\circ =$$

$$i_t^\circ i_{t-1}^\circ \cdots i_1^\circ (i_1^\circ i_2 \cdots i_t) i_t^\circ i_{t-1}^\circ \cdots i_1^\circ = i_t^\circ i_{t-1}^\circ \cdots i_1^\circ$$

and

$$x(i_t^\circ i_{t-1}^\circ \cdots i_1^\circ)x = i_1(i_1^\circ i_2 \cdots i_t)(i_t^\circ i_{t-1}^\circ \cdots i_1^\circ)(i_1^\circ i_1 i_2 \cdots i_t) = i_1 i_2 \cdots i_t = x.$$

(2) By Lemma 2.7,

$$I = \{e \in E(S) | (\exists e^\circ \in E(S^\circ)) e\mathcal{L}e^\circ\}, \Lambda = \{f \in E(S) | (\exists f^\circ \in E(S^\circ)) f\mathcal{R}f^\circ\}.$$

In view of item (1), $\bar{I}$ is a regular semigroup and $V_{E(S^\circ)}(x) \neq \emptyset$ for all $x \in \bar{I}$. Denote

$$I^{E(S^\circ)} = \{xx^\circ | x \in \bar{I}, x^\circ \in V_{E(S^\circ)}(x)\}, \Lambda^{E(S^\circ)} = \{x^\circ x | x \in \bar{I}, x^\circ \in V_{E(S^\circ)}(x)\}.$$

Then by Lemma 2.7 and Lemma 2.10,

$$I^{E(S^\circ)} = \{e \in E(\bar{I}) | (\exists e^\circ \in E(S^\circ)) e\mathcal{L}e^\circ\}, \Lambda^{E(S^\circ)} = \{f \in E(\bar{I}) | (\exists f^\circ \in E(S^\circ)) f\mathcal{R}f^\circ\}.$$

It is easy to see that $I = I^{E(S^\circ)}$ and $\Lambda^{E(S^\circ)} = \Lambda \cap \bar{I}$. Hence by Definition 3.1 and Theorem 3.7 (2), we have

$$I^{E(S^\circ)}E(S^\circ) = IE(S^\circ) \subseteq I = I^{E(S^\circ)}, E(S^\circ)I^{E(S^\circ)} = E(S^\circ)I \subseteq E(S) \cap \bar{I} = E(\bar{I})$$

and

$$E(S^\circ)\Lambda^{E(S^\circ)} = E(S^\circ)(\Lambda \cap \bar{I}) \subseteq E(S^\circ)\Lambda \cap E(S^\circ)\bar{I} \subseteq \Lambda \cap \bar{I} = \Lambda^{E(S^\circ)},$$
$$\Lambda^{E(S^\circ)}E(S^\circ) = (\Lambda \cap \bar{I})E(S^\circ) \subseteq \Lambda E(S^\circ) \cap \bar{I}E(S^\circ) \subseteq E(S) \cap \bar{I} = E(\bar{I}).$$

By Lemma 2.2 (2), $E(S^\circ)$ is an orthodox transversal of $\bar{I}$. According to Lemma 2.2 (3), the subsemigroup generated by $I^{E(S^\circ)} = I$ in $\bar{I}$ is a subband of $\bar{I}$. This implies that $\bar{I}$ itself is a subband of S. By dual arguments, we can obtain a symmetrical result for Λ. $\square$

In the end of this section, we give some properties of semi-abundant semigroups with generalized bi-ideal quasi-Ehreshmann transversals, which will be used in the next section. Recall that a subset T of a semigroup S is called a *generalized bi-ideal* if $TST \subseteq T$.

Lemma 3.11. *Let S be a semi-abundant semigroup with a strong quasi-Ehreshmann transversal S° which is also a generalized bi-ideal of S. Then I and Λ are subbands of S. In this case, $E(S^\circ)I \subseteq E(S^\circ)$ and $\Lambda E(S^\circ) \subseteq E(S^\circ)$.*

Proof. Let $e, f \in I$. Then, by Lemma 2.7, there exist $e^\circ, f^\circ \in E(S^\circ)$ such that $e\mathcal{L}e^\circ$ and $f\mathcal{L}f^\circ$. Since S° is a generalized bi-ideal of S, $e^\circ f = e^\circ f f^\circ \in S^\circ$. By (2) of Theorem 3.7, we have $e^\circ f \in E(S)$. This implies that $e^\circ f \in E(S^\circ)$. In virtue of condition (ii) of Definition 3.1, we have

$$ef = e(e^\circ f) \in IE(S^\circ) \subseteq I.$$

This shows that I is a subband of S. Dually, Λ is also a subband of S.

Now, let $s \in E(S^\circ)$ and $i \in I$. Then, by Lemma 2.7, $i\mathcal{L}i^\circ$ for some $i^\circ \in E(S^\circ)$. Since I is a subband and S° is a generalized bi-ideal of S, we have

$$si = sii^\circ \in I \cap S^\circ = E(S^\circ).$$

This yields that $E(S^\circ)I \subseteq E(S^\circ)$. Dually, $\Lambda E(S^\circ) \subseteq E(S^\circ)$. $\square$

Lemma 3.12. *Let $x, y \in S^\circ, e, g \in I, f, h \in \Lambda$ and $e\mathcal{L}x^\dagger, f\mathcal{R}x^*, g\mathcal{L}y^\dagger, h\mathcal{R}y^*$. Then*

$$e\mathcal{R}g, x\delta y, f\mathcal{L}h \Leftrightarrow exf = gyh.$$

Proof. Necessity. By hypothesis, we have $x = kyl$, where $k \in E(y^\dagger)$ and $l \in E(y^*)$(Notice that $k, l \in E(S^\circ)$!). By lemma 2.6, $E(x^\dagger) = E(y^\dagger)$. By Lemma 2.1, we have

e	$g, eky^\dagger$	ek
$x^\dagger$	$x^\dagger y^\dagger$	
$y^\dagger x^\dagger$	$y^\dagger$	$y^\dagger k$
$ky^\dagger x^\dagger$	$ky^\dagger$	k

.

Since $ky^\dagger \in E(S^\circ)$ and $e \in I$, by condition (i) of Definition 3.1, we have $eky^\dagger \in I$ whence $eky^\dagger = g$. Dually, $y^* lf = h$. Therefore,

$$exf = ekylf = eky^\dagger \cdot y \cdot y^* lf = gyh.$$

Sufficiency. Let $exf = gyh$. Then,

$$(e, x, f), (g, y, h) \in \Omega_{exf} = \Omega_{gyh}.$$

By Lemma 2.8,

$$e\mathcal{R}^* exf = gyh\mathcal{R}^* g, f\mathcal{L}^* exf = gyh\mathcal{L}^* h.$$

The fact that $x\delta y$ follows from Theorem 3.4 (1). □

Lemma 3.13. *The following statements are equivalent:*

(1) S° is a generalized bi-ideal of S;

(2) $(\forall x, y \in S)(\forall (e, \overline{x}, f) \in \Omega_x)(\forall (g, \overline{y}, h) \in \Omega_y) \quad \overline{x} f g \overline{y} \in \Gamma_{xy}$;

(3) $(\forall f \in \Lambda)(\forall g \in I) \quad fg \in S^\circ$.

Proof. (1)⇒ (2). By (1), $\bar{x} f g \bar{y} \in S^\circ$. Let $e\mathcal{L}\bar{x}^\dagger$ and $h\mathcal{R}\bar{y}^*$. Then, for any $(\bar{x} f g \bar{y})^\dagger$ and $(\bar{x} f g \bar{y})^*$, by Lemma 2.3 and its dual, we have

$$e(\bar{x} f g \bar{y})^\dagger \mathcal{L}\bar{x}^\dagger (\bar{x} f g \bar{y})^\dagger = (\bar{x} f g \bar{y})^\dagger, (\bar{x} f g \bar{y})^* h\mathcal{R}(\bar{x} f g \bar{y})^* \bar{y}^* = (\bar{x} f g \bar{y})^*.$$

Observe that $xy = e(\bar{x} f g \bar{y})^\dagger \bar{x} f g \bar{y} (\bar{x} f g \bar{y})^* h$, it follows that $\overline{x} f g \overline{y} \in \Gamma_{xy}$.

(2) ⇒ (3). Let $f \in \Lambda$ and $g \in I$. Then, by Lemma 2.7, there exist $f^\circ, g^\circ \in E(S^\circ)$ such that $f\mathcal{R}f^\circ$ and $g\mathcal{L}g^\circ$. Hence, $(f^\circ, f^\circ, f) \in \Omega_f$ and $(g, g^\circ, g^\circ) \in \Omega_g$. By (2), $fg = f^\circ f g g^\circ \in \Gamma_{fg}$. Therefore, $fg \in S^\circ$.

(3) ⇒ (1). Let $x, z \in S^\circ$, $y \in S$ and $(g, \bar{y}, h) \in \Omega_y$. Then, by (3) we have $xyz = x(\bar{x}^* g)\bar{y}(h\bar{z}^\dagger)z \in S^\circ$. This shows that S° is a generalized bi-ideal of S. □

4. A Structure Theorem

In this section, a structure theorem of semi-abundant semigroups with a generalized bi-ideal strong quasi-Ehreshmann transversal is established by using so-called *$\mathcal{QSQE}$-systems* which are defined as follows.

Definition 4.1. *Let I and Λ be two bands, S° be a quasi-Ehreshmann semigroup such that*

$$E(S^\circ) = I \cap \Lambda, E(S^\circ)I \subseteq E(S^\circ), \Lambda E(S^\circ) \subseteq E(S^\circ)$$

and P be a $\Lambda \times I$-matrix over S°. Then (I, Λ, S°, P) is called a $\mathcal{QSQE}$-system if for all $i, j \in E^\circ, e \in I$ and $f \in \Lambda$,

$$(\mathcal{QSQE}) \qquad iP_{f,e} = P_{if,e},\ P_{f,e}j = P_{f,ej},\ P_{f,i} = fi,\ P_{j,e} = je.$$

Let (I, Λ, S°, P) be a $\mathcal{QSQE}$-system and denote $E^\circ = E(S^\circ)$. Write

$$Q = Q(I, \Lambda, S^\circ, P) = \{(R_e, \delta(x), L_f) \in I/\mathcal{R} \times S^\circ/\delta \times \Lambda/\mathcal{L} | e\mathcal{L}x^\dagger, f\mathcal{R}x^* \text{ for some } x^\dagger, x^* \in E^\circ\}.$$

The following result shows that the above set Q is well-defined.

Lemma 4.2. *Let $(R_e, \delta(x), L_f) \in Q$ and $g \in I, y \in S^\circ, h \in \Lambda$. If $e\mathcal{R}g, x\delta y$ and $f\mathcal{L}h$, then there exist $y^\dagger, y^* \in E^\circ$ such that $g\mathcal{L}y^\dagger$ and $h\mathcal{R}y^*$.*

Proof. Let $(R_e, \delta(x), L_f) \in Q, g \in I, y \in S^\circ, h \in \Lambda$ and

$$e\mathcal{R}g, x\delta y, f\mathcal{L}h, e\mathcal{L}x^\dagger, f\mathcal{R}x^*$$

for some $x^\dagger, x^* \in E^\circ$. Then, there exist $i \in E(x^\dagger), \lambda \in E(x^*)$ such that $y = ix\lambda$. Let $\alpha = ix^\dagger g, \beta = hx^*\lambda$. Since I and Λ are bands and $E^\circ I \subseteq E^\circ, \Lambda E^\circ \subseteq E^\circ$, we have $\alpha, \beta \in E^\circ$. Since $i \in E(x^\dagger), e, g \in I$ and $x^\dagger \mathcal{L} e \mathcal{R} g$, it follows that $e, g, i, x^\dagger$ in the same $\mathcal{D}$-class of I(**This method will be used in the rest of this section frequently**). Hence,

$$g\alpha = gix^\dagger g = g.$$

Clearly, $\alpha g = \alpha$. Therefore, $g\mathcal{L}\alpha$. On the other hand, if $k \in E^\circ$ and $ky = y$, then $kix\lambda = ix\lambda$.This implies that

$$kix = kixx^*\lambda x^* = kix\lambda x^* = ix\lambda x^* = ixx^*\lambda x^* = ixx^* = ix.$$

Since $x\widetilde{\mathcal{R}}x^\dagger$, we have $ix\widetilde{\mathcal{R}}ix^\dagger$, whence $kix^\dagger = ix^\dagger$ by Lemma 2.3, and so $k\alpha = kix^\dagger g = ix^\dagger g = \alpha$. But $\alpha y = (ix^\dagger gi)x\lambda = ix\lambda = y$, again by Lemma 2.3, $y\widetilde{\mathcal{R}}\alpha$. Therefore, $g\mathcal{L}\alpha\widetilde{\mathcal{R}}y$. Dually, we have $h\mathcal{R}\beta\widetilde{\mathcal{L}}y$. □

Lemma 4.3. *Define a multiplication on Q by the rule*

$$(R_e, \delta(x), L_f)(R_g, \delta(y), L_h) = (R_{ea^\dagger}, \delta(a), L_{a^*h}),$$

where $a = xP_{f,g}y$. Then the following statements are true:

(1) $(R_{ea^\dagger}, \delta(a), L_{a^*h}) \in Q$ dose not depend on the choice of a^* and $a^\dagger$;

(2) the above multiplication dose not depend on the choice of e, x, f and g, y, h;

(3) Q becomes a semigroup with the above multiplication.

Proof. (1) Let $(R_e, \delta(x), L_f), (R_g, \delta(y), L_h) \in Q$ and $e\mathcal{L}x^\dagger, h\mathcal{R}y^*$ for some $x^\dagger, y^* \in E^\circ$. Then, by Lemma 2.3 and its dual, $x^\dagger a^\dagger = a^\dagger$ and $a^*y^* = a^*$. Therefore, $ea^\dagger\mathcal{L}x^\dagger a^\dagger = a^\dagger$ and $a^*h\mathcal{R}a^*y^* = a^*$. This implies that $(R_{ea^\dagger}, \delta(a), L_{a^*h}) \in Q$. If $a^{\dagger\dagger}, a^{**} \in E^\circ$ and $a^{**}\widetilde{\mathcal{L}}a\widetilde{\mathcal{R}}a^{\dagger\dagger}$, then $a^\dagger\mathcal{R}a^{\dagger\dagger}$ and $a^*\mathcal{L}a^{**}$, whence $ea^\dagger\mathcal{R}ea^{\dagger\dagger}$ and $a^*h\mathcal{L}a^{**}h$. This proves that $(R_{ea^\dagger}, \delta(a), L_{a^*h})$ dose not depend on the choice of a^* and $a^\dagger$.

(2) Let $(R_e, \delta(x), L_f) = (R_k, \delta(z), L_l), (R_g, \delta(y), L_h) = (R_p, \delta(w), L_q) \in Q$ and

$$e\mathcal{L}x^\dagger, k\mathcal{L}z^\dagger, g\mathcal{L}y^\dagger, p\mathcal{L}w^\dagger, f\mathcal{R}x^*, l\mathcal{R}z^*, h\mathcal{L}y^*, q\mathcal{L}w^*.$$

Then,

$$e\mathcal{R}k, x\delta z, f\mathcal{L}l, g\mathcal{R}p, y\delta w, h\mathcal{L}q.$$

By Lemma 2.6 (1), there exist

$$i \in E(z^\dagger) = E(x^\dagger), \lambda \in E(z^*) = E(x^*), j \in E(w^\dagger) = E(y^\dagger), \mu \in E(w^*) = E(y^*)$$

such that $x = iz\lambda$ and $y = jw\mu$. Let $a = xP_{f,g}y$ and $b = zP_{l,p}w$. Then,

$$\begin{aligned}
a &= xP_{f,g}y = iz\lambda P_{f,g}jw\mu = izP_{\lambda f,gj}w\mu \quad ((QSQ\mathcal{E}),\ \lambda, j \in E^\circ) \\
&= izP_{\lambda fz^*l, pw^\dagger gj}w\mu \quad (f\mathcal{L}l, g\mathcal{R}p, l\mathcal{R}z^*, p\mathcal{L}w^\dagger) \\
&= izz^*\lambda fz^*P_{l,p}w^\dagger gjw^\dagger w\mu \quad ((QSQ\mathcal{E}), \lambda fz^*, w^\dagger gj \in E^\circ, zz^* = z, w^\dagger w = w) \\
&= izP_{l,p}w\mu \quad (\lambda \in E(z^*) = E(x^*), f\mathcal{R}x^*, j \in E(w^\dagger) = E(y^\dagger), g\mathcal{L}y^\dagger, zz^* = z, w^\dagger w = w) \\
&= (ib^\dagger)b(b^*\mu).
\end{aligned}$$

Noticing that $i \in E(z^\dagger)$ and $z^\dagger b = b$, we have $z^\dagger b^\dagger = b^\dagger$ and $ib^\dagger \in E(b^\dagger)$. Dually, $b^*\mu \in E(b^*)$. Thus, $\delta(a) = \delta(b)$. By lemma 2.6 (3), we have $ib^\dagger \mathcal{R} a^\dagger, b^*\mu \mathcal{L} a^*$ and $E(a^\dagger) = E(b^\dagger)$. Therefore,

$$\begin{aligned} ea^\dagger kb^\dagger &= kz^\dagger ex^\dagger a^\dagger kb^\dagger \quad (e\mathcal{R}k\mathcal{L}z^\dagger, x^\dagger a^\dagger = a^\dagger) \\ &= kz^\dagger a^\dagger kb^\dagger \quad (E(x^\dagger) = E(z^\dagger),\ e\mathcal{L}x^\dagger) \\ &= kz^\dagger a^\dagger b^\dagger \quad (k\mathcal{L}z^\dagger,\ z^\dagger b^\dagger = b^\dagger,\ a^\dagger kb^\dagger \in E(a^\dagger) = E(b^\dagger)) \\ &= kz^\dagger (ib^\dagger) a^\dagger b^\dagger = kz^\dagger ib^\dagger \quad (a^\dagger \mathcal{R} ib^\dagger, a^\dagger \in E(b^\dagger)) \\ &= kz^\dagger iz^\dagger b^\dagger = kz^\dagger b^\dagger = kb^\dagger. \quad (z^\dagger b^\dagger = b^\dagger, i \in E(z^\dagger)) \end{aligned}$$

By the above identity and its dual, we have $ea^\dagger \mathcal{R} kb^\dagger$. Dually, we can obtain $a^*h\mathcal{L}b^*q$. Hence,

$$(R_e, \delta(x), L_f)(R_g, \delta(y), L_h) = (R_{ea^\dagger}, \delta(a), L_{a^*h})$$

$$= (R_{kb^\dagger}, \delta(b), L_{b^*q}) = (R_k, \delta(z), L_l)(R_p, \delta(w), L_q).$$

(3) Let $m_1 = (R_e, \delta(x), L_f), m_2 = (R_g, \delta(y), L_h), m_3 = (R_s, \delta(z), L_t) \in Q$. Then,

$$(m_1m_2)m_3 = (R_{ea^\dagger}, \delta(a), L_{a^*h})m_3 = (R_{ec^\dagger}, \delta(c), L_{c^*t}),$$

$$m_1(m_2m_3) = m_1(R_{gb^\dagger}, \delta(b), L_{b^*t}) = (R_{ed^\dagger}, \delta(d), L_{d^*t}).$$

By $(\mathcal{QSQE})$, we have

$$c = aP_{a^*h,s}z = aa^*P_{h,s}z = aP_{h,s}z = xP_{f,g}yP_{h,s}z = xP_{f,g}b^\dagger b = xP_{f,gb^\dagger}b = d,$$

which implies that $(m_1m_2)m_3 = m_1(m_2m_3)$. □

Lemma 4.4. *Let $(R_e, \delta(x), L_f) \in Q$. Then $(R_e, \delta(x), L_f) \in E(Q)$ if and only if $xP_{f,e}x = x$.*

Proof. Let $(R_e, \delta(x), L_f) \in Q$, $e\mathcal{L}x^\dagger$ and $f\mathcal{R}x^*$. If $(R_e, \delta(x), L_f) \in E(Q)$, then

$$(R_e, \delta(x), L_f) = (R_{ea^\dagger}, \delta(a), L_{a^*f}),$$

where $a = xP_{f,e}x$. Hence, there exist $i \in E(x^\dagger)$ and $\lambda \in E(x^*)$ such that $xP_{f,e}x = ix\lambda$. Thus,

$$xP_{f,e}x = x^\dagger xP_{f,e}xx^* = x^\dagger ix\lambda x^* = x^\dagger i(x^\dagger xx^*)\lambda x^* = (x^\dagger ix^\dagger)x(x^*\lambda x^*) = x^\dagger xx^* = x.$$

Conversely, if $x = xP_{f,e}x$, then

$$(R_e, \delta(x), L_f)^2 = (R_{ex^\dagger}, \delta(x), L_{x^*f}) = (R_e, \delta(x), L_f) \in E(Q),$$

as required. □

Lemma 4.5. *Let $(R_e, \delta(x), L_f) \in Q$ and $e\mathcal{L}x^\dagger, f\mathcal{R}x^*$ for some $x^\dagger, x^* \in E^\circ$. Then $(R_e, \delta(x^\dagger), L_{x^\dagger}) \in E(Q)$ and $(R_e, \delta(x), L_f)\widetilde{\mathcal{R}}(R_e, \delta(x^\dagger), L_{x^\dagger})$.*

Proof. Clearly, $(R_e, \delta(x^\dagger), L_{x^\dagger}) \in Q$. In view of Condition $(\mathcal{QSQE})$, we have

$$x^\dagger P_{x^\dagger,e}x^\dagger = x^\dagger(x^\dagger e)x^\dagger = x^\dagger x^\dagger x^\dagger = x^\dagger,$$

whence $(R_e, \delta(x^\dagger), L_{x^\dagger}) \in E(Q)$ by Lemma 4.4. By similar calculations, we can obtain that

$$(R_e, \delta(x^\dagger), L_{x^\dagger})(R_e, \delta(x), L_f) = (R_e, \delta(x), L_f). \tag{1}$$

Now, let $(R_g, \delta(y), L_h) \in E(Q)$ and

$$(R_g, \delta(y), L_h)(R_e, \delta(x), L_f) = (R_e, \delta(x), L_f).$$

Then $yP_{h,g}y = y$ by Lemma 4.4 and $(R_{ga^\dagger}, \delta(a), L_{a^*f}) = (R_e, \delta(x), L_f)$, where $a = yP_{h,e}x$. This implies that

$$yP_{h,g} \in E^\circ, ga^\dagger \mathcal{R} e, a^* f \mathcal{L} f, E(x^\dagger) = E(a^\dagger)$$

by Lemma 2.6. Since $e\mathcal{L}x^\dagger$ and $E(x^\dagger) = E(a^\dagger)$, we have $a^\dagger e = a^\dagger e x^\dagger = a^\dagger x^\dagger$. In view of Condition $(\mathcal{QSQE})$ and the fact $ga^\dagger \mathcal{R} e$, we obtain

$$yP_{h,e}x^\dagger = yP_{h,ga^\dagger e}x^\dagger = yP_{h,ga^\dagger x^\dagger}x^\dagger = (yP_{h,g})a^\dagger x^\dagger \in E^\circ.$$

Since $x\widetilde{\mathcal{R}}x^\dagger$ and $\widetilde{\mathcal{R}}$ is a left congruence, we have $a = yP_{h,e}x\widetilde{\mathcal{R}}yP_{h,e}x^\dagger$. This yields that $a^\dagger \mathcal{R} yP_{h,e}x^\dagger$ and $yP_{h,e}x^\dagger \in E(a^\dagger) = E(x^\dagger)$ since $yP_{h,e}x^\dagger \in E^\circ$. So

$$e\mathcal{R}ga^\dagger \mathcal{R} gyP_{h,e}x^\dagger, \delta(yP_{h,e}x^\dagger) = \delta(x^\dagger), yP_{h,e}x^\dagger \mathcal{L} x^\dagger.$$

In view of Lemma 4.3 (1) and the fact $yP_{h,e}x^\dagger \in E^\circ$, we have

$$(R_g, \delta(y), L_h)(R_e, \delta(x^\dagger), L_{x^\dagger}) = (R_{gyP_{h,e}x^\dagger}, \delta(yP_{h,e}x^\dagger), L_{yP_{h,e}x^\dagger x^\dagger}) = (R_e, \delta(x^\dagger), L_{x^\dagger}). \tag{2}$$

According to items (1) and (2), we have $(R_e, \delta(x^\dagger), L_{x^\dagger})\widetilde{\mathcal{R}}(R_e, \delta(x), L_f)$ by Lemma 2.3. □

Lemma 4.6. *Let $(R_e, \delta(x), L_f)$ and $(R_g, \delta(y), L_h) \in Q$. Then $(R_e, \delta(x), L_f)\tilde{\mathcal{R}}(R_g, \delta(y), L_h)$ if and only if $e\mathcal{R}g$.*

Proof. Now, let $m_1 = (R_e, \delta(x), L_f), n_1 = (R_g, \delta(y), L_h) \in Q$ and

$$e\mathcal{L}x^\dagger, g\mathcal{L}y^\dagger, m_1' = (R_e, \delta(x^\dagger), L_{x^\dagger}), n_1' = (R_g, \delta(y^\dagger), L_{y^\dagger}).$$

Then by $(\mathcal{QSQE})$,

$$m_1' n_1' = (R_{eu^\dagger}, \delta(u), L_{u^* y^\dagger}), u = x^\dagger P_{x^\dagger, g} y^\dagger = x^\dagger g \in E^\circ.$$

If $m_1\widetilde{\mathcal{R}}n_1$, then by Lemma 4.5, we have $m_1'\mathcal{R}n_1'$, which is equivalent to $m_1'n_1' = n_1'$ and $n_1'm_1' = m_1'$. But $m_1'n_1' = n_1'$ implies $g\mathcal{R}eu^\dagger$ whence $eg = g$. Dually, $n_1'm_1' = m_1'$ implies $ge = e$. Therefore, $e\mathcal{R}g$. Conversely, if $e\mathcal{R}g$, then by Lemma 2.1, we have

e	g	$eu^\dagger$
$x^\dagger$	$x^\dagger y^\dagger = x^\dagger g = u$	$u^\dagger$
$y^\dagger e$	$y^\dagger$	
	u^*	

. (3)

Hence,

$$m_1' n_1' = (R_{eu^\dagger}, \delta(u), L_{u^* y^\dagger}) = (R_e, \delta(x^\dagger y^\dagger), L_{y^\dagger}) = (R_g, \delta(y^\dagger), L_{y^\dagger}) = n_1'.$$

Dually, we have $n_1'm_1' = m_1'$. Hence, $m_1'\mathcal{R}n_1'$. Again by Lemma 4.5, $m_1\widetilde{\mathcal{R}}n_1$. □

Lemma 4.7. *Q is a semi-abundant semigroup and*

$$Q^\circ = \{(R_{x^\dagger}, \delta(x), L_{x^*}) \in Q | x \in S^\circ\}$$

is a quasi-Ehreshmann $\sim$-subsemigroup of Q isomorphic to S° such that

$$\Gamma_{Q^\circ}((R_e, \delta(x), L_f)) \neq \emptyset$$

for all $(R_e, \delta(x), L_f)$ in Q.

Proof. By Lemma 4.5, each $\widetilde{\mathcal{L}}$-class and each $\widetilde{\mathcal{R}}$-class of Q contains idempotents. Let

$$m_1 = (R_e, \delta(x), L_f), m_2 = (R_g, \delta(y), L_h), m_3 = (R_s, \delta(z), L_t) \in Q$$

and $e\mathcal{L}x^\dagger, g\mathcal{L}y^\dagger, m_1\tilde{\mathcal{R}}n_1$. By Lemma 4.6, we have $e\mathcal{R}g$. In view of the diagram (3), we have $x^\dagger e = x^\dagger$ and $x^\dagger g = x^\dagger y^\dagger$. This implies that $x^\dagger = x^\dagger e\mathcal{R}x^\dagger g = x^\dagger y^\dagger$ whence $zP_{t,e}x^\dagger\mathcal{R}zP_{t,e}x^\dagger y^\dagger$. By ($\mathcal{QSQE}$) and the diagram (3),

$$zP_{t,e}x^\dagger\mathcal{R}zP_{t,e}x^\dagger y^\dagger y^\dagger = zP_{t,ex^\dagger y^\dagger}y^\dagger = zP_{t,g}y^\dagger.$$

On the other hand, since $x\tilde{\mathcal{R}}x^\dagger$, we have $zP_{t,e}x\tilde{\mathcal{R}}zP_{t,e}x^\dagger$. Similarly, we have $zP_{t,g}y\tilde{\mathcal{R}}zP_{t,g}y^\dagger$. Thus, $zP_{t,e}x\tilde{\mathcal{R}}zP_{t,g}y$ and so $(zP_{t,e}x)^\dagger\mathcal{R}(zP_{t,g}y)^\dagger$. This implies that $s(zP_{t,e}x)^\dagger\mathcal{R}s(zP_{t,g}y)^\dagger$. By Lemma 4.6, we have $m_3m_1\tilde{\mathcal{R}}m_3m_2$. We have shown that $\tilde{\mathcal{R}}$ is a left congruence. Dually, $\tilde{\mathcal{L}}$ is a right congruence. Therefore, Q is a semi-abundant semigroup.

Now, define

$$\psi : Q^\circ \to S^\circ, \quad (R_{x^\dagger}, \delta(x), L_{x^*}) \mapsto x.$$

Then, by Lemma 2.6 (3), ψ is bijective. It is also a homomorphism. In fact, by ($\mathcal{QSQE}$),

$$(R_{x^\dagger}, \delta(x), L_{x^*})(R_{y^\dagger}, \delta(y), L_{y^*}) = (, \delta(xP_{x^*,y^\dagger}y),) = (, \delta(xx^*y^\dagger y),) = (, \delta(xy),).$$

Moreover, by Lemma 4.6 and its dual, for each $(R_{x^\dagger}, \delta(x), L_{x^*}) \in Q^\circ$, we have

$$(R_{x^\dagger}, \delta(x^\dagger), L_{x^\dagger})\widetilde{\mathcal{R}}(R_{x^\dagger}, \delta(x), L_{x^*})\widetilde{\mathcal{L}}(R_{x^*}, \delta(x^*), L_{x^*})$$

and

$$(R_{x^\dagger}, \delta(x^\dagger), L_{x^\dagger}), (R_{x^*}, \delta(x^*), L_{x^*}) \in E(Q^\circ).$$

Hence, Q° is a quasi-Ehreshmann ~-subsemigroup of Q.

Let $m = (R_e, \delta(x), L_f) \in Q$ and $e\mathcal{L}x^\dagger, f\mathcal{R}x^*$. Then $\bar{m} = (R_{x^\dagger}, \delta(x), L_{x^*}) \in Q^\circ$. By condition ($\mathcal{QSQE}$), Lemma 4.5, Lemma 4.6 and their dual, we have

$$(R_{x^\dagger}, \delta(x^\dagger), L_{x^\dagger}) = \bar{m}^\dagger\tilde{\mathcal{R}}\bar{m}\tilde{\mathcal{L}}\bar{m}^* = (R_{x^*}, \delta(x^*), L_{x^*})$$

and

$$\bar{m}^\dagger\mathcal{L}e_m = (R_e, \delta(x^\dagger), L_{x^\dagger}) \in E(Q), \ \ \bar{m}^*\mathcal{R}f_m = (R_{x^*}, \delta(x^*), L_f) \in E(Q).$$

It is routine to check that $m = e_m\bar{m}f_m$. This proves that $\Gamma_{Q^\circ}((R_e, \delta(x), L_f)) \neq \emptyset$. □

Lemma 4.8. *The following statements hold:*

(1) $E(Q^\circ) = \{(R_e, \delta(e), L_e) \in Q^\circ | e \in E^\circ\}$*;*

(2) $I_{Q^\circ} = \{(R_g, \delta(h), L_h) \in E(Q) | g\mathcal{L}h \ \& \ h \in E^\circ\}$*;*

(3) $\Lambda_{Q^\circ} = \{(R_g, \delta(g), L_h) \in E(Q) | g\mathcal{R}h \ \& \ g \in E^\circ\}$*.*

Proof. (1) Let $(R_{x^\dagger}, \delta(x), L_{x^*}) \in E(Q^\circ)$. By Lemma 4.6 and condition ($\mathcal{QSQE}$),

$$x = xP_{x^*,x^\dagger}x = xx^*x^\dagger x = xx \in E^\circ.$$

Hence,

$$(R_{x^\dagger}, \delta(x), L_{x^*}) = (R_x, \delta(x), L_x) \in \{(R_e, \delta(e), L_e) \in Q^\circ | e \in E^\circ\}.$$

The reverse inclusion is obvious.

(2) Let $(R_e, \delta(x), L_f) \in I_{Q^\circ}$ and $e\mathcal{L}x^\dagger, f\mathcal{R}x^*$. Then, by Lemma 4.4, Lemma 4.7 and Lemma 2.7, we have

$$xP_{f,e}x = x, (R_e, \delta(x), L_f)\mathcal{L}(R_i, \delta(i), L_i)$$

for some $(R_i, \delta(i), L_i) \in E(Q^\circ)$ where $i \in E^\circ$. Thus, by the dual of Lemma 4.6, $f\mathcal{L}i$ whence $f = fi \in E^\circ$ since $i \in E^\circ$ and $E^\circ\Lambda \subseteq \Lambda$. By ($\mathcal{QSQE}$), $P_{f,e} = fe \in E^\circ I \subseteq E^\circ$. Since $xP_{f,e}x = x$, we have $xP_{f,e}, P_{f,e}x \in E^\circ$. Therefore,

$$x = xP_{f,e}x = (x(fe))((fe)x) \in E^\circ E^\circ \subseteq E^\circ.$$

Moreover, by Lemma 2.1,

x	$x^\dagger$	
x^*	$x^*x^\dagger$	f
	e	$ef = k$

.

Thus,

$$(R_e, \delta(x), L_f) = (R_k, \delta(f), L_f) \in \{(R_g, \delta(h), L_h) \in E(Q) | g\mathcal{L}h \ \&\ h \in E^\circ\}.$$

Conversely, let $(R_g, \delta(h), L_h) \in E(Q)$ and $g\mathcal{L}h \in E^\circ$. Then, by the dual of Lemma 4.6 and Lemma 4.7, we can obtain

$$(R_h, \delta(h), L_h) \in E(Q^\circ), (R_g, \delta(h), L_h)\mathcal{L}(R_h, \delta(h), L_h).$$

By Lemma 2.7, $(R_g, \delta(h), L_h) \in I_{Q^\circ}$.

(3) This is the dual of (2). □

Lemma 4.9. *Q° is a generalized bi-ideal strong quasi-Ehreshmann transversal of Q.*

Proof. By Lemma 4.7, Lemma 3.13 and the definition of strong quasi-Ehreshmann transversals, it suffices to prove that I_{Q° and Λ_{Q° are subbands of Q and $\Lambda_{Q^\circ} I_{Q^\circ} \subseteq Q^\circ$. For the first part, we only prove the case for I_{Q°, the similar argument holds for Λ_{Q°. By Lemma 4.8, let

$$(R_e, \delta(f), L_f), (R_g, \delta(h), L_h) \in I(Q), e\mathcal{L}f \in E^\circ, g\mathcal{L}h \in E^\circ.$$

By ($\mathcal{QSQE}$),

$$a = fP_{f,g}h = P_{ff,gh} = P_{f,g} = fg \in E^\circ.$$

Then by Lemma 4.3 (1) and Lemma 4.8,

$$(R_e, \delta(f), L_f)(R_g, \delta(h), L_h) = (R_{e(fg)}, \delta(fg), L_{(fg)h}) = (R_{eg}, \delta(fg), L_{fg}) \in I_{Q^\circ}.$$

Now, let

$$(R_e, \delta(f), L_f) \in I_{Q^\circ}, (R_g, \delta(g), L_h) \in \Lambda_{Q^\circ}$$

and $e\mathcal{L}f \in E^\circ, h\mathcal{R}g \in E^\circ$ by Lemma 4.8. Then,

$$(R_g, \delta(g), L_h)(R_e, \delta(f), L_f) = (R_{gb^\dagger}, \delta(b), L_{b^*f}).$$

Since $b = gP_{h,e}f$, we have $gb^\dagger = b^\dagger$ and $b^*f = b^*$ by Lemma 2.3 and its dual. Therefore,

$$(R_g, \delta(g), L_h)(R_e, \delta(f), L_f) = (R_{b^\dagger}, \delta(b), L_{b^*}) \in Q^\circ,$$

as required. □

Now, we can give our main result in this section.

Theorem 4.10. *Let (I, Λ, S°, P) be a $\mathcal{QSQE}$-system. Then Q is a semi-abundant semigroup with a generalized bi-ideal strong quasi-Ehreshmann transversal isomorphic to S°; Conversely, every such semigroup can be obtained in this way.*

Proof. The direct part follows from Lemma 4.7 and Lemma 4.9. Conversely, let S be a semi-abundant semigroup with a generalized bi-ideal strong quasi-Ehreshmann transversal S°. Then we define I and Λ as in Section 2 and $P_{f,e} = fe \in S^\circ$ for $e \in I$ and $f \in \Lambda$ by Lemma 3.13. Then, (I, Λ, S°, P) is a $\mathcal{QSQE}$-system by Lemma 2.7 and Lemma 3.11. By the proof of the direct part, we can construct a semi-abundant semigroup Q with a generalized bi-ideal strong quasi-Ehreshmann transversal Q° isomorphic to S°.

Let

$$\varphi : Q \to S,\ (R_e, \delta(x), L_f) \mapsto exf.$$

By Lemma 3.12, φ is well-defined and injective. Let $m \in S$. Then, there exist $e, f \in E(S)$ and $\bar{m} \in S^\circ$ such that $(e, \bar{m}, f) \in \Omega_m$. Hence, $(R_e, \delta(\bar{m}), L_f) \in Q$ and

$$\varphi(R_e, \delta(\bar{m}), L_f) = e\bar{m}f = m.$$

That is, φ is surjective. Let $(R_e, \delta(x), L_f), (R_g, \delta(y), L_h) \in Q$. Then,

$$\begin{aligned}
\varphi((R_e, \delta(x), L_f)(R_g, \delta(y), L_h)) &= \varphi((R_{e(xP_{f,g}y)^\dagger}, \delta(xP_{f,g}y), L_{(xP_{f,g}y)^*h})) \\
&= \varphi((R_{e(xfgy)^\dagger}, \delta(xfgy), L_{(xfgy)^*h})) \\
&= e(xfgy)^\dagger \cdot xfgy \cdot (xfgy)^*h \\
&= exfgyh \\
&= \varphi(R_e, \delta(x), L_f) \cdot \varphi(R_g, \delta(y), L_h).
\end{aligned}$$

This implies that φ is indeed an isomorphism from Q onto S. □

Now, we apply our Theorem 4.10 to the class of regular semigroups with a generalized bi-ideal orthodox transversal. The following theorem gives a structure theorem for regular semigroups with generalized bi-ideal orthodox transversals, which substantively is the Theorem 3.4 in Chen [4].

Corollary 4.11. *Let (I, Λ, S°, P) be a $\mathcal{QSQE}$-system such that S° is an orthodox semigroup. Then Q is a regular semigroup with a generalized bi-ideal orthodox transversal isomorphic to S°. Conversely, every such semigroup can be obtained in this way.*

Proof. It follows from Theorem 3.2 and Theorem 4.10. □

5. Some Remarks

In this section, we give some remarks on the results obtained in this paper. Let S be a semigroup and $x, y \in S$. The Green's *-relations can be defined as follows. That $x\mathcal{R}^*y$ means that $ax = bx$ if and only if $ay = by$ for all $a, b \in S^1$. The relation $\mathcal{L}^*$ can be defined dually. Denote $\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*$. Clearly, $\mathcal{L}^*$ is a right congruence and $\mathcal{R}^*$ is a left congruence. A semigroup is called *abundant* if each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class contains idempotents. An abundant semigroup S is *quasi-adequate* if its idempotents form a subsemigroup of S. An abundant subsemigroup U of an abundant semigroup S is called a $*$*-subsemigroup* of S if

$$\mathcal{L}^*(U) = \mathcal{L}^*(S) \cap (U \times U), \mathcal{R}^*(U) = \mathcal{R}^*(S) \cap (U \times U).$$

It is well known (and easy to prove) that abundant semigroups are always semi-abundant semigroups and quasi-adequate semigroups are always quasi-Ehresmann semigroups. Moreover, in an abundant semigroup S, we have $\mathcal{L}^* = \widetilde{\mathcal{L}}, \mathcal{R}^* = \widetilde{\mathcal{R}}$ and $\mathcal{H}^* = \widetilde{\mathcal{H}}$ and so $*$-subsemigroups of S and $\sim$-subsemigroups of S are equal. Thus, we have the following remark.

Remark 5.1. *Quasi-Ehresmann transversals of abundant semigroups are generalizations of orthodox transversals of regular semigroups in the range of abundant semigroups.*

On the other hand, Ni [18] introduced *quasi-adequate transversals* of abundant semigroups (with the notations in this paper) as follows: A quasi-adequate $*$-subsemigroup S° of an abundant semigroup S is called a *quasi-adequate transversal* of S if

(i) $\Gamma_x \neq \emptyset$ for all $x \in S$.

(ii) $\Gamma_e \Gamma_s \subseteq \Gamma_{se}$ and $\Gamma_s \Gamma_e \subseteq \Gamma_{es}$ for all $e \in E(S)$ and $s \in E(S^\circ)$.

From Ni [18], a multiplicative orthodox transversal of a regular semigroup S is always a multiplicative quasi-adequate transversal of S. In the following, we give an example to show that, in general, an orthodox transversal S° of a regular semigroup S may not be a quasi-adequate transversal of S even if S° is also a generalized bi-ideal of S.

Example 5.2. *Let S be an inverse monoid with identity 1 which is not a Clifford semigroup. Then there exist $\alpha \in S$ and $i \in E(S)$ such that $\alpha i \neq i\alpha$. Suppose that $M \equiv \mathcal{M}(S, 2, 2, P)$ is the Rees matrix semigroup over S, where the entries of its sandwich matrix $P = (p_{uv})_{2\times 2}$ are*

$$p_{11} = p_{12} = p_{21} = 1, p_{22} = \alpha^{-1}.$$

Denote $M^\circ = \{(1, x, 1) | x \in S\}$. Then M° is an inverse subsemigroup and a generalized bi-ideal of M, and $V_{M^\circ}((u, x, v)) = \{(1, x^{-1}, 1)\}$ for all $(u, x, v) \in M$ where x^{-1} is the unique inverse of x in S. For $(u, x, v) \in M$, we denote $(u, x, v)^\circ = (1, x^{-1}, 1)$. Now, let $(u_1, x_1, v_1), (u_2, x_2, v_2) \in M$ and

$$\{(u_1, x_1, v_1), (u_2, x_2, v_2)\} \cap M^\circ \neq \emptyset.$$

It is easy to check that

$$V_{M^\circ}((u_1, x_1, v_1)(u_2, x_2, v_2)) = \{((u_1, x_1, v_1)(u_2, x_2, v_2))^\circ\}$$
$$= \{(u_2, x_2, v_2)^\circ (u_1, x_1, v_1)^\circ\} = V_{M^\circ}((u_2, x_2, v_2)) V_{M^\circ}((u_1, x_1, v_1)).$$

This implies that M° is an orthodox transversal of M.

On the other hand, since M is regular and M° is an inverse subsemigroup of M, M is abundant and M° is a quasi-adequate $*$-subsemigroup of M certainly. Let $(u, x, v) \in M$. Then $(1, x, 1) \in M^\circ$ and

$$(1, x, 1)^\dagger = (1, xx^{-1}, 1), (1, x, 1)^* = (1, x^{-1}x, 1).$$

It is easy to see that

$$((u, x, v)(u, x, v)^\circ, (1, x, 1), (u, x, v)^\circ (u, x, v)) \in \Omega_{(u,x,v)}$$

and so $(1, x, 1) \in \Gamma_{(u,x,v)}$. If $(1, y, 1) \in \Gamma_{(u,x,v)}$, then there exist

$$(u_1, z_1, v_1), (u_2, z_2, v_2) \in E(M)$$

such that

$$(u, x, v) = (u_1, z_1, v_1)(1, y, 1)(u_2, z_2, v_2)$$

and

$$(u_1, z_1, v_1) \mathcal{L} (1, y, 1)^\dagger = (1, yy^{-1}, 1), (u_2, z_2, v_2) \mathcal{R} (1, y, 1)^* = (1, y^{-1}y, 1).$$

This implies that

$$u_1 = u, v_2 = v, v_1 = u_2 = 1$$

and

$$z_1, z_2 \in E(S), z_1 \mathcal{L} yy^{-1}, z_2 \mathcal{L} y^{-1}y$$

in S whence $z_1 = yy^{-1}$ and $z_2 = y^{-1}y$ since S is inverse. Thus, we have

$$(u, x, v) = (u_1, z_1, v_1)(1, y, 1)(u_2, z_2, v_2) = (u, yy^{-1}, 1)(1, y, 1)(1, y^{-1}y, v) = (u, y, v)$$

and so $(1, y, 1) = (1, x, 1)$. We have shown that $\Gamma_{(u,x,v)} = \{(1, x, 1)\}$ for all $(u, x, v) \in M$. For $(2, \alpha, 2) \in E(M)$ and $(1, i, 1) \in E(M^\circ)$, we have

$$\Gamma_{(2,\alpha,2)} = \{(1, \alpha, 1)\}, \Gamma_{(1,i,1)} = \{(1, i, 1)\},$$

$$\Gamma_{((1,i,1)(2,\alpha,2))} = \Gamma_{(1,i\alpha,1)} = \{(1, i\alpha, 1)\}.$$

and

$$\Gamma_{(2,\alpha,2)}\Gamma_{(1,i,1)} = \{(1, \alpha, 1)(1, i, 1)\} = \{(1, \alpha i, 1)\}.$$

Since $\alpha i \neq i\alpha$, it follows that $\Gamma_{(2,\alpha,2)}\Gamma_{(1,i,1)}$ is not contained in $\Gamma_{(1,i,1)(2,\alpha,2)}$. This implies that M° is not a quasi-adequate transversal of M.

The above Example 5.2 implies the following remark.

Remark 5.3. *Quasi-adequate transversals of abundant semigroups are not generalizations of orthodox transversals of regular semigroups in the range of abundant semigroups.*

To explore some relations between quasi-adequate transversals and quasi-Ehresmann transversals of abundant semigroups, we need the following proposition.

Proposition 5.4. *Let S be an abundant semigroup and S° a generalized bi-ideal quasi-adequate transversal of S. Then*

$$E(S^\circ)I \subseteq E(S^\circ), IE(S^\circ) \subseteq I, E(S^\circ)\Lambda \subseteq \Lambda, \Lambda E(S^\circ) \subseteq E(S^\circ),$$

where I and Λ are defined in the statements before Lemma 2.7.

Proof. In fact, let $e \in I$ and $f \in E(S^\circ)$. By Lemma 2.7, there exists $e^\circ \in I$ such that $e\mathcal{L}e^\circ$, and so $e^\circ \in \Gamma_e$ and $e^\circ f \in E(S^\circ)$. Since S° is a generalized bi-ideal of S, we have $fe = fee^\circ \in S^\circ$. Obviously, $f \in \Gamma_f$. By the definition of quasi-adequate transversals, $e^\circ f \in \Gamma_e\Gamma_f \subseteq \Gamma_{fe}$. By Lemma 2.10 and $e^\circ f \in E(S^\circ) \subseteq RegS^\circ$, it follows that $e^\circ f \in V_{S^\circ}(V_{S^\circ}(fe))$. Noticing that $e^\circ f \in E(S^\circ), fe \in S^\circ$ and $RegS^\circ$ is orthodox, we obtain $fe \in E(S^\circ)$. On the other hand, by the above discussions, we can see that $e^\circ f$ and ef are in the same $\mathcal{D}$-class of $E(S^\circ)$. In view of the fact $e\mathcal{L}e^\circ$, we have $ef\mathcal{L}e^\circ f \in E(S^\circ)$ and

$$(ef)^2 = efef = ee^\circ fee^\circ f = e(e^\circ ffee^\circ f) = ee^\circ f = ef.$$

Again by Lemma 2.7, we have $ef \in I$. Dually, we can prove that $E(S^\circ)\Lambda \subseteq \Lambda$ and $\Lambda E(S^\circ) \subseteq E(S^\circ)$. □

In view of Definition 3.1, Theorem 3.7 and Proposition 5.4, we have the remark below.

Remark 5.5. *A generalized bi-ideal quasi-adequate transversal of an abundant semigroup S is always a generalized bi-ideal strong quasi-Ehresmann transversal of S. The converse is not true by the Example 5.2.*

However, up to now we do not know whether a quasi-adequate transversal of an abundant semigroup is a quasi-Ehresmann transversal in general. This would be an interesting problem to be considered in the future research works.

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