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Approximation of an Analytic Function Represented by Vector Valued Generalized Dirichlet Series

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Abstract. Andre Boivin and Changzhong Zhu introduced the Dirichlet series with complex exponents and obtained the growth properties of entire functions represented by these series. Later in 2009, Wen Ping Huang, Ju Hong Ning and Jin Tu [3] made independent studies on these series. In our earlier work,we have introduced the concept of growth of analytic functions represented by vector valued Dirichlet series with complex exponents. In these series, we have taken the coefficients from a complex Banach algebra. In the present paper, we have introduced the approximation error of these series with respect to a class of exponential polynomials. We have characterized the order and the type of the analytic function $f(s)$ represented by a vector valued Dirichlet series with complex exponents in terms of the rate of decay of the approximation error introduced. Our results generalize some of the earlier results obtained by A.Nautiyal and D.P.Shukla [4] for classical Dirichlet series.

1. Introduction.

In 1983, B.L.Srivastava [5] introduced a new class of Dirichlet series which is called vector valued Dirichlet series. He modified the classical Dirichlet series $\sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ by considering the sequence $\{a_n\}$ as a member of a complex Banach space. In the present paper, we have considered the vector valued analytic Dirichlet series with complex exponents$\{\lambda_n\}$.

Let us suppose that $\Lambda = \{\lambda_n = |\lambda_n| e^{i\,\alpha_n}; n = 1, 2, 3....\}$ is a sequence of complex numbers in the right half plane satisfying the following conditions:

$$\left.\begin{aligned} \liminf_{n\to\infty} (|\lambda_{n+1}| - |\lambda_n|) &= \delta(A) > 0; \\ \sup_{n\geq 1} \{|\arg \alpha_n|\} &\leqslant \alpha < \frac{\pi}{2}; \\ \limsup_{n\to\infty} \frac{n}{|\lambda_n|} &= D < \infty. \end{aligned}\right\} \tag{1.1}$$

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Let

$$f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n) \ ; \ s = \sigma + it \, ; \tag{1.2}$$

σ and t are real numbers, be a function represented by vector valued Dirichlet series with complex exponents, where $\{a_n\}$ is a sequence from the complex Banach algebra E and the sequence of complex numbers$\{\lambda_n\}$ satisfies the conditions given in (1.1).

B.L.Srivastava [5] has considered the convergence of vector valued Dirichlet series with real exponents and obtained the expression for its abscissa of convergence.In [1] we have obtained the region of convergence of vector valued Dirichlet series (1.2).

Let us suppose that $f(s)$ represents an analytic function in $G_0 = \{\left|\arg s\right| \leqslant \theta_0 < \pi/2 \, ; \, s = \sigma + it \, , \sigma, t \in \mathbb{R}\}$. Let R be defined as

$$R = \lim_{n\to\infty} \inf \frac{\log \|a_n\|^{-1}}{|\lambda_n|}.$$

Suppose that σ_c and σ_a be the abscissa of convergence and abscissa of absolute convergence respectively of the series (1.2). Then it has been shown in [1] that

$$\sigma_c = \sigma_a = -A \quad \text{where} \quad A = R \cos \theta_0 \sec(\alpha - \theta_0).$$

The maximum modulus of the analytic function $f(s)$ can be defined as

$$M(\sigma) = \sup_{-\infty<t<\infty} \{\|f(\sigma + it)\| \, , s = \sigma + it \in G_0\} \, ,$$

and the maximum term of the series (1.2) is defined as

$$m(\sigma) = \max \{\|a_n \exp(s\lambda_n)\| ; s = \sigma + it \in G_0, n \in N\} \, .$$

Let D_A denote the class of all functions $f(s)$ given by (1.2), which are analytic in the half plane $\mathrm{Re}(s) < A$ $(-\infty < A < \infty)$ .If in (1.2) , $a_n = 0$ for $n \geqslant k+1$ and $a_k \neq 0$,then $f(s)$ will be called an exponential polynomial of degree k . The class of all exponential polynomials of degree at most k will be denoted by π_k . For an analytic function $f(s) \in D_A$ represented by vector valued Dirichlet series (1.2) , let us define the order ρ of $f(s)$ by

$$\rho = \lim_{\sigma\to A} \sup \frac{\log\log M(\sigma)}{-\log\left(1 - \exp(\sigma - A)\right)} \, , \, 0 \leqslant \rho \leqslant \infty,$$

and if $0 < \rho < \infty$, then we define the type τ of $f(s)$ by

$$\tau = \lim_{\sigma\to A} \sup \frac{\log M(\sigma)}{(1 - \exp(\sigma - A))^{-\rho}} \ .$$

In [1] we have proved that if the sequence $\Lambda : \{\lambda_n = |\lambda_n| e^{i\,\alpha_n} ; n = 1, 2, 3....\}$ satisfies the conditions of (1.1) and $D = 0$, then we can express the order ρ of $f(s)$ as

$$\rho = \lim_{\sigma\to A} \sup \frac{\log\log m(\sigma)}{-\log\left(1 - \exp(\sigma - A)\right)} \, ,$$

and if $0 < \rho < \infty$, then the type τ is also given by

$$\tau = \lim_{\sigma\to A} \sup \frac{\log m(\sigma)}{(1 - \exp(\sigma - A))^{-\rho}} \ .$$

Now we introduce the error function.

Let $E_k(f,B)$ denote the error in approximating the function $f(s)$ by exponential polynomials in the half plane $\operatorname{Re} s \leq B, (B < A)$. We define $E_k(f,B)$ as

$$E_k(f,B) = \inf_{b_0,b_1,b_2,\dots\dots,b_k} \left[\max_{-\infty<t<\infty} \|f(B+it) - \sum_{n=1}^{k} b_n \exp((B+it)\lambda_n)\| \right] ;\ k = 1,2,....$$

where $b_n \in E$. Suppose that $\bar{\bar{D}}_B, 0 < B < \infty$, be the class of all functions $f(s)$ given by (1.2) and analytic in $\operatorname{Re}(s) \leqslant B$,i.e., $f(s) \simeq \bar{\bar{D}}_B$ if $f(s) \in D_{A_0}$ for some $\alpha_0 > B$. For $f(s) \in \bar{\bar{D}}_B$, we define $E_n(f,B)$,the error in approximating the function $f(s)$ by exponential polynomials of degree n in the uniform norm as

$$E_n(f,B) = \inf_{p\in\pi_n} \|f - p\|_B\ ,\ n = 0,1,2....$$

where $\|f - p\|_B = \sup_{-\infty<t<\infty} \|f(B+it) - p(B+it)\|$.

In the present paper we have characterized order and type of the analytic function $f(s)$ represented by a vector valued Dirichlet series with complex exponents in terms of the rate of decay of the approximation error $E_n(f,B), B < A, n = 1,2,3....$ In what follows, we shall always assume that $0 < B < A < \infty$.

We now prove

Lemma 1. *Let $f(s)$ be an analytic function represented by Dirichlet series (1.2) and $0 < B < A < \infty$. Then for all $\sigma(\sigma < A)$ sufficiently close to A, we have*

$$E_k(f,B) \leqslant K\, M(\sigma,f)/\exp((\sigma - B)|\lambda_{k+1}| \sec\theta \cos(\alpha+\theta))\ ,\ k = 1,2,.... \tag{1.3}$$

where K is a constant independent of k and σ.

Proof. Let $p_k(s) = \sum_{n=1}^{k} a_n \exp(s\lambda_n)$ be the k^{th} partial sum of the series (1.2). By considering the definition of $E_k(f,B)$ we have

$$E_k(f,B) \leqslant \|f - p_k\|_B \leqslant \sum_{n=k+1}^{\infty} \|a_n \exp(s\lambda_n)\|\,.$$

We have $\|a_n e^{\lambda_n s}\| = \|a_n\| e^{\operatorname{Re}(\lambda_n s)} = \|a_n\| e^{|\lambda_n|(B\cos\alpha_n - t\sin\alpha_n)} = \|a_n\| e^{|\lambda_n| B\sec\theta\cos(\alpha_n+\theta)}$. From the definitions of θ_0 and α ,we have $\sec\theta \leqslant \sec\theta_0$ and $\cos(\alpha_n+\theta) \leqslant \cos(\alpha \sim \theta_0)$. Hence

$$B\sec\theta\cos(\alpha_n+\theta) \leqslant B\sec\theta_0\cos(\alpha\sim\theta_0)\,.$$

This gives

$$\begin{aligned} E_k(f,B) &\leqslant \sum_{n=k+1}^{\infty} \|a_n\| \exp(B\sec\theta_0\cos(\alpha-\theta_0)|\lambda_n|) \\ &\leqslant M(\sigma) \sum_{n=k+1}^{\infty} \exp((B-\sigma)\sec\theta_0\cos(\alpha-\theta_0)|\lambda_n|)\,. \end{aligned}$$

We have $\liminf_{n\to\infty} (|\lambda_{n+1}| - |\lambda_n|) = \delta(A) > 0$. Hence we can choose $0 < \delta' < \delta$ such that $(|\lambda_{n+1}| - |\lambda_n|) \geqslant$

$\delta', n \geqslant 0$. Therefore for $\sigma \geqslant (A+B)/2$, we have

$$\begin{aligned}E_k(f,B) &\leqslant M(\sigma)\ e^{|\lambda_{n+1}|(B-\sigma)\sec\theta_0\cos(\alpha-\theta_0)} \sum_{n=k+1}^{\infty} e^{(|\lambda_n|-|\lambda_{n+1}|)(B-\sigma)\sec\theta_0\cos(\alpha-\theta_0)} \\ &\leqslant M(\sigma)\ e^{|\lambda_{n+1}|(B-\sigma)\sec\theta_0\cos(\alpha-\theta_0)} \sum_{n=k+1}^{\infty} e^{[\delta'(B-A)n\sec\theta_0\cos(\alpha-\theta_0)]/2} \\ &= \frac{M(\sigma)\ e^{|\lambda_{n+1}|(B-\sigma)\sec\theta_0\cos(\alpha-\theta_0)}}{1-e^{[\delta_0(B-A)\sec\theta_0\cos(\alpha-\theta_0)]/2}}.\end{aligned}$$

Denoting by $K = 1/[1-e^{[\delta'(B-A)\sec\theta_0\cos(\alpha-\theta_0)]/2}]$, we get (1.3).

□

Lemma 2. *Let $f(s) \in \bar{\bar{D}}_B, 0 < B < \infty$, be given by (1.2). Then for $n \geqslant 1$, we have*

$$\|a_n\| \exp(B|\lambda_n|\sec\theta_0\cos(\alpha-\theta_0)) \leqslant 2\ E_{n-1}(f,B). \tag{1.4}$$

Proof. Let us consider

$$\begin{aligned}\frac{1}{t_*}\int_{t_0}^{t_*} f(B+it)\ \exp(-it\lambda_n)\ dt &= \frac{1}{t_*}\sum_{k=1}^{\infty}\int_{t_0}^{t_*} a_k \exp((B+it)\lambda_k)\exp(-it\lambda_n)\ dt \\ &= \frac{1}{t_*}\sum_{k=1}^{\infty} a_k\exp(B\lambda_k)\int_{t_0}^{t_*}\exp(it(\lambda_k-\lambda_n))\,dt \\ &= \frac{1}{t_*}\sum_{k=1}^{\infty} a_k\exp(B\lambda_k)\int_{t_0}^{t_*}\exp(it|\lambda_k-\lambda_n|\,e^{i\phi})\,dt\end{aligned}$$

where $0 < \phi < \pi/2$ .Therefore for $k \neq n$,we get

$$\frac{1}{t_*}\int_{t_0}^{t_*}\exp(it|\lambda_k-\lambda_n|\,e^{i\phi})\,dt = \left[\frac{\exp(it_*|\lambda_k-\lambda_n|\,e^{i\phi}) - \exp(it_0|\lambda_k-\lambda_n|\,e^{i\phi})}{t_*}\right]\frac{1}{i|\lambda_k-\lambda_n|\,e^{i\phi}}$$

Further,since $\sin\phi > 0$,we have

$$\left|\frac{\exp(t_*|\lambda_k-\lambda_n|\,(i\cos\phi-\sin\phi)}{t_*}\right| \leqslant \frac{1}{t_*}.$$

Now for $k = n$, $\frac{1}{t_*}\int_{t_0}^{t_*}\exp(it|\lambda_k-\lambda_n|\,e^{i\phi})\,dt = \frac{t_*-t_0}{t_*}$.

Hence $\lim_{t_*\to\infty}\frac{1}{t_*}\int_{t_0}^{t_*} f(B+it)\ \exp(-it\lambda_n)\ dt = a_n\exp(B\lambda_n)$.

From the above equation, we get

$$a_n\exp(B\lambda_n) = \lim_{t_*\to\infty}\frac{1}{t_*}\int_{t_0}^{t_*}(f(B+it)-p(B+it))\ \exp(-it\lambda_n)\ dt$$

for any $p(s) \in \pi_{n-1}$.For any $p(s) \in \pi_{n-1}$,the above relation easily gives

$$\|a_n\| \exp(B\,|\lambda_n| \sec\theta_0 \cos(\alpha-\theta_0)) \leqslant \|f-p\|_B. \tag{1.5}$$

By the definition of error term $E_n(f,B)$ there exists $\widetilde{p}(s) \in \pi_{n-1}$ such that

$$\|f-\widetilde{p}\|_B \leqslant 2E_{n-1}(f,B). \tag{1.6}$$

Taking in particular $p(s) = \widetilde{p}(s)$ in (1.5) and using (1.6) we get (1.4) and Lemma 2 follows. □

The coefficient characterizations for the order and type of classical analytic functions represented by Dirichlet series are well known. The same kind of formulae do not hold for vector valued Dirichlet series having complex exponents as shown in [1].

We introduce some new characterizations of order and type for the vector valued function defined by (1.2).

Let $f(s) \in D_A$ be defined by (1.2).We put the following expressions .

$$\limsup_{n\to\infty} \frac{\log^+\{\log E_n(f,B) + ((A-B)\sec\theta_0\cos(\alpha-\theta_0))\sec\alpha\,\mathrm{Re}\,\lambda_{n+1}\}}{\log \mathrm{Re}\,\lambda_{n+1}} = \frac{\rho_r}{1+\rho r}\ ; 0 \leqslant \rho_r \leqslant \infty,$$

and

$$\limsup_{n\to\infty} \frac{\log^+\{\log E_n(f,B) + (A-B)\sec\theta_0\cos(\alpha-\theta_0)|\lambda_{n+1}|\}}{\log|\lambda_{n+1}|} = \frac{\rho_m}{1+\rho_m}\ ; 0 \leqslant \rho_m \leqslant \infty.$$

For $0 < \rho < \infty$, we define the corresponding formulae for type τ_r and τ_m.Hence we put

$$\tau_r = \limsup_{n\to\infty} \left(\frac{\rho}{\mathrm{Re}\,\lambda_{n+1}}\right)^{\rho} \left[\frac{\{\log E_n(f,B) + ((A-B)\sec\theta_0\cos(\alpha-\theta_0))\sec\alpha\,\mathrm{Re}\,\lambda_{n+1}\}}{(1+\rho)}\right]^{1+\rho},$$

and

$$\tau_m = \limsup_{n\to\infty} \left(\frac{\rho}{|\lambda_{n+1}|}\right)^{\rho} \left[\frac{\{\log E_n(f,B) + (A-B)\sec\theta_0\cos(\alpha-\theta_0)|\lambda_{n+1}|\}}{(1+\rho)}\right]^{1+\rho}.$$

2. Main Results.

In this section, we obtain the characterizations of growth parameters of $f(s)$ in terms of the approximation error. We first obtain the characterization of abscissa of convergence. We prove

Theorem 1. *Let $f(s) \in \overline{D_B}, 0 < B < \infty$ .Then $f(s) \in D_A, B < A < \infty$, if and only if*

$$\limsup_{n\to\infty} \frac{\log E_n(f,B)}{|\lambda_{n+1}|} = (B-A)\sec\theta_0\cos(\alpha-\theta_0) \tag{2.1}$$

Proof. From Lemma 1, we have

$$E_n(f,B) \leqslant K\,M(\sigma,f)\exp((B-\sigma)|\lambda_{n+1}|\sec\theta_0\cos(\alpha-\theta_0))\ ,\ \sigma < A;\ n = 0,1,2,....$$

which gives $\log E_n(f,B) \leqslant \log K + \log M(\sigma,f) + (B-\sigma)|\lambda_{n+1}|\sec\theta_0\cos(\alpha-\theta_0)$. Since $\log M(\sigma,f)$ is finite for a given σ ,we get

$$\limsup_{n\to\infty} \frac{\log E_n(f,B)}{|\lambda_{n+1}|} \leqslant (B-\sigma)\sec\theta_0\cos(\alpha-\theta_0)$$

i.e.,

$$\lim_{n\to\infty}\sup \frac{\log E_n(f,B)}{|\lambda_{n+1}|} \leqslant (B-A)\sec\theta_0\cos(\alpha-\theta_0) \tag{2.2}$$

Now by Lemma 2,

$$\|a_{n+1}\|\exp(B|\lambda_{n+1}|\sec\theta_0\cos(\alpha-\theta_0)) \leqslant 2\,E_n(f,B)$$

which gives

$$\lim_{n\to\infty}\sup \frac{\log\|a_{n+1}\|}{|\lambda_{n+1}|} \leqslant \lim_{n\to\infty}\sup \frac{\log E_n(f,B)}{|\lambda_{n+1}|} - B\sec\theta_0\cos(\alpha-\theta_0)$$

Since

$$\lim_{n\to\infty}\sup \frac{\log\|a_n\|}{|\lambda_n|} = -A\sec\theta_0\cos(\alpha-\theta_0),$$

we get

$$\lim_{n\to\infty}\sup \frac{\log E_n(f,B)}{|\lambda_{n+1}|} \geqslant (B-A)\sec\theta_0\cos(\alpha-\theta_0) \tag{2.3}$$

On combining (2.2) and (2.3) we obtain (2.1) and Theorem 1 follows. □

Now we compare the above growth parameters introduced in section 1 with the order ρ and type τ respectively, of the analytic function $f(s)$.

Theorem 2. *For the analytic function $f(s)$ we have*

$$\rho_r \leqslant \rho \leqslant \rho_m \sec\theta_0\cos(\alpha-\theta_0). \tag{2.4}$$

Proof. Using the definition of order ρ of $f(s)$, we have for arbitrarily small $\varepsilon > 0$

$$\frac{\log\log M(\sigma)}{-\log(1-\exp(\sigma-A))} < \rho+\varepsilon,\quad \sigma_0(\varepsilon) < \sigma < A$$

or, $\log M(\sigma) < (1-\exp(\sigma-A))^{-(\rho+\varepsilon)} \leqslant (A-\sigma)^{-(\rho+\varepsilon)}$
since $(1-\exp(\sigma-A)) \simeq A-\sigma$ as $\sigma\to A$. Using Lemma 1, we have

$$\log E_n(f,B) \leqslant \log K + \log M(\sigma,f) + (B-\sigma)\sec\theta_0\cos(\alpha-\theta_0)|\lambda_{n+1}|.$$

Therefore $\log E_n(f,B) \leqslant \log K + (B-\sigma)\sec\theta_0\cos(\alpha-\theta_0)|\lambda_{n+1}| + (A-\sigma)^{-(\rho+\varepsilon)}$.
We put $\phi(\sigma) = (B-\sigma)\sec\theta_0\cos(\alpha-\theta_0)|\lambda_{n+1}| + (A-\sigma)^{-(\rho+\varepsilon)}$. Then the maximum value of $\phi(\sigma)$ is obtained as

$$\phi(\sigma) = (|\lambda_{n+1}|\sec\theta_0\cos(\alpha-\theta_0))^{\frac{\rho+\varepsilon}{1+\rho+\varepsilon}}(\rho+\varepsilon)^{-\frac{\rho+\varepsilon}{1+\rho+\varepsilon}}(1+\rho+\varepsilon) + (B-A)|\lambda_{n+1}|\sec\theta_0\cos(\alpha-\theta_0)$$

at the point

$$\sigma = A - \left[\frac{\rho+\varepsilon}{|\lambda_{n+1}|\sec\theta_0\cos(\alpha-\theta_0)}\right]^{1/(1+\rho+\varepsilon)}.$$

This gives us

$$\log E_n(f,B) + ((A-B)\sec\theta_0\cos(\alpha-\theta_0))\sec\alpha\,\mathrm{Re}\,\lambda_{n+1}$$
$$\leqslant \log K + (\sec\alpha\sec\theta_0\cos(\alpha-\theta_0)\,\mathrm{Re}\,\lambda_{n+1})^{\frac{\rho+\varepsilon}{1+\rho+\varepsilon}}Y$$

where $Y = (\rho+\varepsilon)^{-\frac{\rho+\varepsilon}{1+\rho+\varepsilon}}(1+\rho+\varepsilon)$ and $(|\lambda_n|/\mathrm{Re}\,\lambda_n) = \sec\alpha_n \leqslant \sec\alpha$. Therefore

$$\frac{\log^+\{\log E_n(f,B)\ +((A-B)\sec\theta_0\cos(\alpha-\theta_0))\sec\alpha\,\mathrm{Re}\,\lambda_{n+1}\}}{\log \mathrm{Re}\,\lambda_{n+1}}$$

$$\leqslant \frac{\log^+\left[\log K+(\sec\alpha\sec\theta_0\cos(\alpha-\theta_0)\,\mathrm{Re}\,\lambda_{n+1})^{\frac{\rho+\varepsilon}{1+\rho+\varepsilon}}Y\right]}{\log \mathrm{Re}\,\lambda_{n+1}}$$

$$\leqslant \frac{\log^+\left[(\mathrm{Re}\,\lambda_{n+1})^{\frac{\rho+\varepsilon}{1+\rho+\varepsilon}}\left((\mathrm{Re}\,\lambda_{n+1})^{-\frac{\rho+\varepsilon}{1+\rho+\varepsilon}}\log K+(\sec\alpha\sec\theta_0\cos(\alpha-\theta_0))^{\frac{\rho+\varepsilon}{1+\rho+\varepsilon}}Y\right)\right]}{\log \mathrm{Re}\,\lambda_{n+1}}$$

$$\leqslant \frac{\rho+\varepsilon}{1+\rho+\varepsilon}+\frac{\log[(\sec\alpha\sec\theta_0\cos(\alpha-\theta_0))^{\frac{\rho+\varepsilon}{1+\rho+\varepsilon}}Y]}{\log \mathrm{Re}\,\lambda_n}+o(1).$$

Hence on proceeding to limits as $n\to\infty$, we have

$$\limsup_{n\to\infty}\frac{\log^+\{\log E_n(f,B)\ +((A-B)\sec\theta_0\cos(\alpha-\theta_0))\sec\alpha\,\mathrm{Re}\,\lambda_{n+1}\}}{\log \mathrm{Re}\,\lambda_{n+1}}\leqslant\frac{\rho}{1+\rho}$$

i.e. $\frac{\rho_r}{1+\rho_r}\leqslant\frac{\rho}{1+\rho}$,

which implies that $\rho_r\leqslant\rho$. This proves the left hand inequality of (2.4).

Now to prove the other half of the inequality, we consider the other characterization of order of $f(s)$ in terms of the approximation error as

$$\limsup_{n\to\infty}\frac{\log^+\{\log E_n(f,B)+(A-B)\sec\theta_0\cos(\alpha-\theta_0)|\lambda_{n+1}|\}}{\log|\lambda_{n+1}|}=\frac{\rho_m}{1+\rho_m}\ ;0\leqslant\rho_m\leqslant\infty.$$

Then for a given $\varepsilon>0$ and all sufficiently large values of n, we have

$$\log E_n(f,B)+(A-B)\sec\theta_0\cos(\alpha-\theta_0)|\lambda_{n+1}|\ \leqslant\ |\lambda_{n+1}|^{\frac{\rho_m}{1+\rho_m}+\varepsilon} \tag{2.5}$$

By using Lemma 2 we have

$$\begin{aligned}M(\sigma,f)&\leqslant\sum_{n=1}^{\infty}\|a_n\|\exp(\mathrm{Re}(s\lambda_n))\\&\leqslant\sum_{n=1}^{\infty}\|a_n\|\exp(\sigma\sec\theta_0\cos(\alpha-\theta_0)|\lambda_n|)\\&\leqslant 2\sum_{n=0}^{\infty}E_{n-1}(f,B)\ \exp[(\sigma-B)\sec\theta_0\cos(\alpha-\theta_0)|\lambda_n|]\\&\leqslant 2M(\sigma,f_B)\end{aligned}$$

where $M(\sigma,f_B)$ denotes the maximum modulus of the analytic function $f_B(s)$ defined as:

$$f_B(s)=\sum_{n=1}^{\infty}E_n(f,B)\ \exp(-B\sec\theta_0\cos(\alpha-\theta_0)|\lambda_{n+1}|)\exp(\mathrm{s}\lambda_{n+1}).$$

In view of Theorem 1,the region of convergence of above series is same as that of series (1.2). Thus, if ρ_B denotes the order of $f_B(s)$ then the above inequality, $\rho\leqslant\rho_B$. Now let $m_B(\sigma)$ denote the maximum term of $f_B(s)$. Then we have

$$\begin{aligned}\log m_B(\sigma)&=\log\ E_n(f,B)\ +(\sigma-B)\sec\theta_0\cos(\alpha-\theta_0)|\lambda_{n+1}|\\&\leqslant\max\{|\lambda_{n+1}|^{\frac{\rho_m}{1+\rho_m}+\varepsilon}+(\sigma-A)\sec\theta_0\cos(\alpha-\theta_0)|\lambda_{n+1}|\},\text{ by using (2.5).}\end{aligned}$$

Let us assume that $\phi(t) = t^a + bt, t > 0,$ where $t = |\lambda_{n+1}|, a = (\rho_m/1+\rho_m) + \varepsilon$ and $b = (\sigma - A)\sec\theta\cos(\alpha-\theta)$. Then $\phi(t)$ attains its maximum value $\phi(t) = -b^{a/a-1}\left[a^{-a/a-1} + a^{-1/a-1}\right]$ at the point $t = \left(\frac{-b}{a}\right)^{1/a-1}$. Since $\varepsilon > 0$ is arbitrary, we have

$$\rho_B = \limsup_{\sigma\to A} \frac{\log\log m_B(\sigma)}{-\log(1-\exp(\sigma-A))} \leqslant \limsup_{\sigma\to A} \frac{\log(A-\sigma)\sec\theta_0\cos(\alpha-\theta_0)}{-\log(A-\sigma)} . \frac{\varepsilon(\rho_m+1)+\rho_m}{\varepsilon(\rho_m+1)-1},$$

i.e.,$\rho \leqslant \rho_m \sec\theta_0\cos(\alpha-\theta_0)$.
Combining the two estimates of ρ , we get (2.4). This proves Theorem 2. □

Corollary 1: If $\{\lambda_n\}$ is a sequence of positive real numbers then $\alpha = 0$. Further let D =0. Then we have from (2.3), $\rho_r = \rho = \rho_m$. This gives the coefficient characterization of the order of analytic function represented by the classical Dirichlet series in terms of the approximation error [4,Theorem 2].

Next we obtain the characterization of the type. We prove

Theorem 3. *For the analytic function $f(s)$ defined by (1.2) we have*

$$\tau_r\,(\sec\alpha)^{-\rho} \leqslant \tau \leqslant \tau_m[(\cos\theta_0)\sec(\alpha-\theta_0)]^{\rho}. \tag{2.6}$$

Proof. Since the type τ of $f(s)$ is given by

$$\tau = \limsup_{\sigma\to A} \frac{\log M(\sigma)}{(1-\exp(\sigma-A))^{-\rho}} ,$$

therefore for a given $\varepsilon > 0$, we have

$$\log M(\sigma) < (\tau+\varepsilon)(1-\exp(\sigma-A))^{-\rho}.$$

Combining this with (1.4), we have

$$\log E_n(f,B) \leqslant \log K + (B-\sigma)\sec\theta_0\cos(\alpha-\theta_0)|\lambda_{n+1}| + (\tau+\varepsilon)(A-\sigma)^{-\rho}.$$

Now we put

$$\psi(\sigma) = (\tau+\varepsilon)(A-\sigma)^{-\rho} + (B-\sigma)\sec\theta_0\cos(\alpha-\theta_0)|\lambda_{n+1}|.$$

The function $\psi(\sigma)$ attains its maximum value given by

$$\left[\rho^{-\rho/1+\rho} + \rho^{1/1+\rho}\right](\tau+\varepsilon)^{1/1+\rho}(|\lambda_{n+1}|)^{\rho/1+\rho} + (B-A)\sec\theta_0\cos(\alpha-\theta_0)|\lambda_{n+1}|$$

at the point $\sigma = A - \left[\frac{(\tau+\varepsilon)\rho}{\sec\theta_0\cos(\alpha-\theta_0)|\lambda_{n+1}|}\right]^{1/1+\rho}$.Thus we have

$$\left[\frac{\{\log E_n(f,B) + ((A-B)\sec\theta_0\cos(\alpha-\theta_0))\sec\alpha\operatorname{Re}\lambda_{n+1}\}}{(1+\rho)}\right]^{1+\rho}$$
$$\leqslant \left[\frac{\log K + (\sec\alpha\operatorname{Re}\lambda_{n+1})^{\rho/1+\rho}(\tau+\varepsilon)^{1/1+\rho}X}{1+\rho}\right]^{1+\rho},$$

where $X = \left[\rho^{-\rho/1+\rho} + \rho^{1/1+\rho}\right]$ and $(|\lambda_n|/\operatorname{Re}\lambda_n) = \sec\alpha_n \leqslant \sec\alpha$.
Thus

$$\left(\frac{\rho}{\operatorname{Re}\lambda_{n+1}}\right)^{\rho}\left[\frac{\{\log E_n(f,B) + ((A-B)\sec\theta_0\cos(\alpha-\theta_0))\sec\alpha\operatorname{Re}\lambda_{n+1}\}}{(1+\rho)}\right]^{1+\rho}$$

$$\leqslant \left(\frac{\rho}{\operatorname{Re}\lambda_{n+1}}\right)^{\rho}\left[\frac{\log K + (\sec\alpha \operatorname{Re}\lambda_{n+1})^{\rho/1+\rho}(\tau+\varepsilon)^{1/1+\rho}X}{1+\rho}\right]^{1+\rho}$$

$$\leqslant (\rho)^{\rho}(\tau+\varepsilon)\left[\left(\frac{X}{1+\rho}\right)^{1+\rho}(\sec\alpha)^{\rho} + o(1)\right].$$

On proceeding to limits as $n \to \infty$, we have

$$\limsup_{n\to\infty}\left(\frac{\rho}{\operatorname{Re}\lambda_{n+1}}\right)^{\rho}\left[\frac{\{\log E_n(f,B) + ((A-B)\sec\theta_0\cos(\alpha-\theta_0))\sec\alpha \operatorname{Re}\lambda_{n+1}\}}{(1+\rho)}\right]^{1+\rho}$$

$$\leqslant \tau\,(\sec\alpha)^{\rho}\left\{(\rho)^{\rho}\left(\frac{\rho^{-\rho/1+\rho}+\rho^{1/1+\rho}}{1+\rho}\right)^{1+\rho}\right\}$$

$$\leqslant \tau\,(\sec\alpha)^{\rho}.$$

Hence we get $\tau_r \leqslant \tau\,(\sec\alpha)^{\rho}$.
To prove the remaining part of (2.6) ,we consider the expression for τ_m. Then for a given $\varepsilon > 0$ and all sufficiently large n we have

$$\log E_n(f,B) + (A-B)\sec\theta_0\cos(\alpha-\theta_0)|\lambda_{n+1}| \leqslant (\tau_m+\varepsilon)^{1/1+\rho}\left(\frac{|\lambda_{n+1}|}{\rho}\right)^{\rho/1+\rho}(1+\rho)\,.$$

Using the definition of the maximum term $m_B(\sigma)$, we obtain

$$\log m_B(\sigma) \leqslant \max\left[(\tau_m+\varepsilon)^{1/1+\rho}\left(\frac{|\lambda_{n+1}|}{\rho}\right)^{\rho/1+\rho}(1+\rho) + (\sigma-A)\sec\theta_0\cos(\alpha-\theta_0)|\lambda_{n+1}|\right],$$

$$\leqslant \max\left[(\tau_m+\varepsilon)^{1/1+\rho}\left(\frac{t}{\rho}\right)^{\rho/1+\rho}(1+\rho) + (\sigma-A)\sec\theta_0\cos(\alpha-\theta_0)\,(t)\right]; t = |\lambda_n|,$$

$$\leqslant \frac{(\tau_m+\varepsilon)}{[(A-\sigma)\sec\theta_0\cos(\alpha-\theta_0)\,]^{\rho}}.$$

From the definition of f_B , we have $\log m(\sigma) \leqslant \log m_B(\sigma)$. On proceeding to limits as $\sigma \to A$, we have

$$\tau \leqslant \tau_m(\cos\theta_0)^{\rho}\sec(\alpha-\theta_0))^{\rho}.$$

Combining the two inequalities obtained, we get (2.6). □

Corollary 2: If $\{\lambda_n\}$ is a sequence of positive real numbers then $\alpha = 0$ and if D = 0 then we get $\tau_r = \tau = \tau_m$. This easily leads to the type formula for analytic functions [4,Theorem 3].

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