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On upper and lower generalized Drazin invertible operators

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Abstract. Upper and lower generalized Drazin invertible operators were introduced in [5]. We will characterize these operators and their properties will be studied. In addition, we make connection between the above mentioned operators and operators founded in Fredholm theory. In particular, a bounded operator is Riesz and generalized Drazin invertible if and only if it is Riesz and upper (resp. lower) generalized Drazin invertible.

1. Introduction

Drazin and generalized Drazin invertible operators have been investigated by many authors; see for example [3, 4, 6, 7, 12–15]. Recently, upper and lower generalized Drazin invertible operators were introduced [5, Definition 3.1, Definition 3.3], though under different name: left and right generalized Drazin invertible operators. We find that the term upper (lower) generalized Drazin invertible operators is more appropriate because of the connection of these operators with upper (lower) semi-Browder operators. Namely, according to [11, Theorem 20.10] an operator T is upper (resp. lower) semi-Browder if and only if there exists a decomposition $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$ such that $\dim \mathcal{X}_1 < \infty$, $T\mathcal{X}_i \subset \mathcal{X}_i$ (i=1,2), $T_i = T_{\mathcal{X}_i}$, (i=1,2), $T = T_1 \oplus T_2$, T_1 is nilpotent and T_2 is bounded below (resp. surjective), while from the proof of Proposition 3.2 (resp. 3.4) in [5] it is obvious that an operator T is upper (resp. lower) generalized Drazin invertible if and only if $T = T_1 \oplus T_2$, where T_1 is quasinilpotent and T_2 is bounded below (resp. surjective). One more reason is that these operators are characterized via approximate point (surjective) spectrum [5, Theorem 3.8, Theorem 3.10]. Actually, if an operator T is upper (resp. lower) generalized Drazin invertible, then $0 \notin \operatorname{acc} \sigma_{ap}(T)$ (resp. $0 \notin \operatorname{acc} \sigma_{su}(T)$), where $\sigma_{ap}(T)$ and $\sigma_{su}(T)$ denote approximate point and surjective spectrum of T, respectively. The main objective of this article is to continue studying these operators. Section 2 presents some definitions and preliminary facts. Our results are contained in Section 3. Firstly, we give a new characterization of upper (resp. lower) generalized Drazin invertible operators in a sense that $T \in \mathcal{B}(\mathcal{X})$ is upper (resp. lower) generalized Drazin invertible if and only if there exists a projection $P \in \mathcal{B}(\mathcal{X})$ such that $TP = PT$, $T + P$ is bounded below (resp. surjective) and TP is quasinilpotent. Further, we show that

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a bounded operator is upper semi-Browder if and only if it is upper generalized Drazin invertible and has finite-dimensional quasinilpotent part. Similarly, an operator is lower semi-Browder if and only if it is lower generalized Drazin invertible and its analytical core has finite codimension. We also connect upper (resp. lower) generalized Drazin invertible operators with Riesz operators. In particular, we show that a bounded operator is Riesz and generalized Drazin invertible if and only if it is Riesz and upper (resp. lower) generalized Drazin invertible. What is more, Hilbert space case is also considered. Namely, if $T \in \mathcal{B}(\mathcal{H})$, $\mathcal{H}$ Hilbert space, is self-adjoint and upper generalized Drazin invertible then it is generalized Drazin invertible.

2. Preliminaries

Let $\mathcal{X}$ be an infinite dimensional Banach space and $\mathcal{B}(\mathcal{X})$ the Banach algebra of all bounded linear operators acting on $\mathcal{X}$. Given $T \in \mathcal{B}(\mathcal{X})$, we denote by $\mathcal{N}(T), \mathcal{R}(T)$ and $\sigma(T)$ the kernel, the range and the spectrum of T, respectively. Let $\mathbb{N}$ ($\mathbb{N}_0$) denote the set of all positive (non-negative) integers, and let $\mathbb{C}$ denote the set of all complex numbers. Recall that T is said to be *nilpotent* (resp. *quasinilpotent*) when $T^n = 0$ for some $n \in \mathbb{N}$ (resp. $\sigma(T) = \{0\}$). An operator $T \in \mathcal{B}(\mathcal{X})$ is *bounded below* if there exists some $c > 0$ such that $c\|x\| \leq \|Tx\|$ for every $x \in \mathcal{X}$. It is known that T is bounded below if and only if it is injective with closed range. If $\mathcal{R}(T)$ is closed and $\mathcal{N}(T)$ is finite-dimensional, then $T \in \mathcal{B}(\mathcal{X})$ is said to be *upper semi-Fredholm*. An operator $T \in \mathcal{B}(\mathcal{X})$ is *lower semi-Fredholm* if $\mathcal{X}/\mathcal{R}(T)$ is finite-dimensional. The set of upper semi-Fredholm operators (resp. lower semi-Fredholm operators) is denoted by $\Phi_+(\mathcal{X})$ (resp. $\Phi_-(\mathcal{X})$). It is also well known that if $\mathcal{N}(T^n) = \mathcal{N}(T^{n+1})$ for some $n \in \mathbb{N}_0$, then $\mathcal{N}(T^k) = \mathcal{N}(T^n)$ for $k \geq n$. In this case, the *ascent* of T, denoted by $\text{asc}(T)$, is the smallest $n \in \mathbb{N}_0$ such that $\mathcal{N}(T^n) = \mathcal{N}(T^{n+1})$. If such n does not exist, then $\text{asc}(T) = \infty$. Similarly, if $\mathcal{R}(T^{n+1}) = \mathcal{R}(T^n)$ for $n \in \mathbb{N}_0$, then $\mathcal{R}(T^k) = \mathcal{R}(T^n)$ for $k \geq n$. In this case, the *descent* of T, denoted by $\text{dsc}(T)$, is the smallest $n \in \mathbb{N}_0$ such that $\mathcal{R}(T^n) = \mathcal{R}(T^{n+1})$. If such an n does not exist, then $\text{dsc}(T) = \infty$. An operator $T \in \mathcal{B}(\mathcal{X})$ is *upper semi-Browder* if T is upper semi-Fredholm and $\text{asc}(T) < \infty$. If $T \in \mathcal{B}(\mathcal{X})$ is lower semi-Fredholm and $\text{dsc}(T) < \infty$, then T is *lower semi-Browder*. Let $B_+(\mathcal{X})$ (resp. $B_-(\mathcal{X})$) denote the set of all upper (resp. lower) semi-Browder operators. As usual, $\mathcal{K}(\mathcal{X})$ is the set of all *compact* operators on $\mathcal{X}$. Consider $\mathcal{C}(\mathcal{X})$ the Calkin algebra over $\mathcal{X}$, i.e. the quotient algebra $\mathcal{C}(\mathcal{X}) = \mathcal{B}(\mathcal{X})/\mathcal{K}(\mathcal{X})$. Recall that $\mathcal{C}(\mathcal{X})$ is itself a Banach algebra with the quotient norm. Let $\pi : \mathcal{B}(\mathcal{X}) \to \mathcal{C}(\mathcal{X})$ denote the quotient map. An operator $T \in \mathcal{B}(\mathcal{X})$ is *Riesz* if its coset $\pi(T)$ is quasinilpotent in $\mathcal{C}(\mathcal{X})$. An operator $T \in \mathcal{B}(\mathcal{X})$ is *semi-regular* if $\mathcal{R}(T)$ is closed and $\mathcal{N}(T) \subset \mathcal{R}(T^n)$, $n \in \mathbb{N}_0$. It is obvious that bounded below and surjective operators belong to the class of semi-regular operators. If $K \subset \mathbb{C}$, then $\text{acc}\, K$ is the set of limit points of K and $\text{iso}\, K = K \setminus \text{acc}\, K$ is the set of isolated points of K.

The *quasinilpotent part* $H_0(T)$ of an operator $T \in \mathcal{B}(\mathcal{X})$ is defined by

$$H_0(T) = \{x \in \mathcal{X} : \lim_{n \to +\infty} \|T^n x\|^{1/n} = 0\}.$$

The *analytical core* of T, denoted by $K(T)$, is the set of all $x \in \mathcal{X}$ for which there exist $c > 0$ and a sequence $(x_n)_n$ in $\mathcal{X}$ satisfying

$$Tx_1 = x, \ \ Tx_{n+1} = x_n \ \text{ for all } \ n \in \mathbb{N}, \ \ \|x_n\| \leq c^n \|x\| \ \text{ for all } \ n \in \mathbb{N}.$$

The basic properties of $H_0(T)$ and $K(T)$ are summarized in the following two lemmas; see [1, 9, 10].

Lemma 2.1. *Let $T \in \mathcal{B}(\mathcal{X})$. The following statements hold.*
(i) *$H_0(T)$ is a (not necessarily closed) subspace of $\mathcal{X}$;*
(ii) *for each $j \geq 0$, $\mathcal{N}(T^j) \subset H_0(T)$;*
(iii) *$x \in H_0(T)$ if and only if $Tx \in H_0(T)$;*
(iv) *$H_0(T) = \mathcal{X}$ if and only if T is quasinilpotent;*
(v) *for $\lambda \neq 0$, $(T - \lambda I)(H_0(T)) = H_0(T)$;*
(vi) *if T is bounded below, then $H_0(T) = \{0\}$.*

Lemma 2.2. *Let $T \in \mathcal{B}(\mathcal{X})$. The following statements hold.*
(i) *$K(T)$ is a (not necessarily closed) subspace of $\mathcal{X}$;*
(ii) *$T(K(T)) = K(T)$;*
(iii) *if $\mathcal{X}_0$ is a closed subspace of $\mathcal{X}$ and $T(\mathcal{X}_0) = \mathcal{X}_0$, then $\mathcal{X}_0 \subset K(T)$;*
(iv) *if T is quasinilpotent, then $K(T) = \{0\}$;*
(v) *if T is surjective, then $K(T) = \mathcal{X}$.*

If M is a subspace of $\mathcal{X}$ such that $T(M) \subset M$, $T \in \mathcal{B}(\mathcal{X})$, it is said that M is *T-invariant*. We define $T_M : M \to M$ as $T_M x = Tx$, $x \in M$. Moreover, if M is closed it is clear that M is itself a Banach space and $T_M \in \mathcal{B}(M)$. If M and N are two closed T-invariant subspaces of $\mathcal{X}$ such that $\mathcal{X} = M \oplus N$ (it means $\mathcal{X} = M + N$ and $M \cap N = \{0\}$), we say that T is completely reduced by the pair (M, N) and it is denoted by $(M, N) \in Red(T)$. We have $\mathcal{N}(T) = \mathcal{N}(T_M) \oplus \mathcal{N}(T_N)$ and $\mathcal{R}(T) = \mathcal{R}(T_M) \oplus \mathcal{R}(T_N)$.

Proposition 2.3. *Let $T \in \mathcal{B}(\mathcal{X})$ and suppose $(M, N) \in Red(T)$. Then, the following statements hold.*
(i) *T is surjective if and only if T_M and T_N are surjective;*
(ii) *T is bounded below if and only if T_M and T_N are bounded below.*

Let $\mathcal{H}$ be a Hilbert space equipped with the scalar product $\langle\ ,\ \rangle$. Two elements $x, y \in \mathcal{H}$ are *orthogonal* if $\langle x, y\rangle = 0$. If $E \subset \mathcal{H}$, then $E^\perp$ denotes the set of all $y \in \mathcal{H}$ that satisfy $\langle y, x\rangle = 0$ for every $x \in E$. It is well known that $E^\perp$ is closed subspace of $\mathcal{H}$. In addition, if E is a closed subspace of $\mathcal{H}$, we also have $E \oplus E^\perp = \mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$ there is a unique operator $T^* \in \mathcal{B}(\mathcal{H})$ such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \quad \text{for every} \quad x, y \in \mathcal{H}.$$

The operator T^* is called *Hilbert-adjoint* operator of T. If $T = T^*$, we say that T is *self-adjoint* operator.

An operator $A \in \mathcal{B}(\mathcal{X})$ is said to be *Drazin invertible*, if there exists $B \in \mathcal{B}(\mathcal{X})$ and some $k \in \mathbb{N}$ such that

$$AB = BA, \quad BAB = B, \quad A^kBA = A^k.$$

For details see [4, 6, 15]. This concept was generalized by Koliha [7]. An operator $A \in \mathcal{B}(\mathcal{X})$ is said to be *generalized Drazin invertible*, if there exists $B \in \mathcal{B}(\mathcal{X})$ such that

$$AB = BA, \quad BAB = B, \quad ABA - A \ \text{ is quasinilpotent.}$$

Necessary and sufficient for $T \in \mathcal{B}(X)$ to be generalized Drazin invertible is that $0 \notin \operatorname{acc}\sigma(T)$, equivalently $K(T)$ and $H_0(T)$ are closed and $\mathcal{X} = K(T) \oplus H_0(T)$; see [7, Theorem 4.2], [9, Théorème 1.6] and [16, Theorem 4]. Also, it is well known that $T \in \mathcal{B}(\mathcal{X})$ is generalized Drazin invertible if and only if there exists a pair $(M, N) \in Red(T)$ such that T_M is invertible and T_N is quasinilpotent.

An operator $T \in \mathcal{B}(\mathcal{X})$ is said to admit a *generalized Kato decomposition*, abbreviated as GKD, if there exists a pair $(M, N) \in Red(T)$ such that T_M is semi-regular and T_N is quasinilpotent. Many classes of operators satisfy GKD property. Some examples are semi-regular operators, upper semi-Fredholm operators, lower semi-Fredholm operators, generalized Drazin invertible operators.

Upper and lower generalized Drazin invertible operators were defined in [5].

Definition 2.4. An operator $T \in \mathcal{B}(\mathcal{X})$ is said to be *upper generalized Drazin invertible* if $H_0(T)$ is closed and complemented with a subspace $M \subset \mathcal{X}$ such that $T(M) \subset M$ and $T(M)$ is closed.

Definition 2.5. An operator $T \in \mathcal{B}(\mathcal{X})$ is said to be *lower generalized Drazin invertible* if $K(T)$ is closed and complemented with a subspace $N \subset \mathcal{X}$ such that $T(N) \subset N$ and $N \subset H_0(T)$.

3. Results

We start with a characterization of upper (resp. lower) generalized Drazin invertible operators.

Theorem 3.1. *An operator $T \in \mathcal{B}(\mathcal{X})$ is upper generalized Drazin invertible if and only if there exists a projection $P \in \mathcal{B}(\mathcal{X})$ such that*

$$TP = PT, \quad T + P \text{ is bounded below}, \quad TP \text{ is quasinilpotent}. \tag{1}$$

If T is upper generalized Drazin invertible, then every projection P from (1) *has uniquely determined range and $\mathcal{R}(P) = H_0(T)$.*

Proof. Let $T \in \mathcal{B}(\mathcal{X})$ be upper generalized Drazin invertible operator. Then $H_0(T)$ is closed and there exists a closed T-invariant subspace M of $\mathcal{X}$ such that $M \oplus H_0(T) = \mathcal{X}$ and $T(M)$ is closed. Using Lemma 2.1(ii) we find $\mathcal{N}(T_M) = M \cap \mathcal{N}(T) \subset M \cap H_0(T) = \{0\}$, so T_M is injective and thus $T_1 = T_M$ is bounded below. It is evident that $T_2 = T_{H_0(T)}$ is quasinilpotent. Let $P \in \mathcal{B}(\mathcal{X})$ be a projection such that $\mathcal{R}(P) = H_0(T)$ and $\mathcal{N}(P) = M$. From $T(H_0(T)) \subset H_0(T)$ and $T(M) \subset M$ we conclude $TP = PT$. Consider arbitrary element $x = x_1 + x_2 \in \mathcal{X}$, $x_1 \in M$, $x_2 \in H_0(T)$ and $n \in \mathbb{N}$. Then

$$\|(TP)^n x\|^{\frac{1}{n}} = \|T^n P^n x\|^{\frac{1}{n}} = \|T^n P x\|^{\frac{1}{n}} = \|T^n x_2\|^{\frac{1}{n}} \to 0 \ (n \to \infty).$$

We see that $H_0(TP) = \mathcal{X}$ and thus TP is quasinilpotent. It is not difficult to see that $(T + P)_M = T_1$ and $(T + P)_{H_0(T)} = T_2 + I_2$, where I_2 is identity on $H_0(T)$. Thus $(T + P)_M$ is bounded below and $(T + P)_{H_0(T)}$ is invertible. Applying Proposition 2.3(ii) we obtain that $T + P$ is bounded below.

Conversely, suppose that there exists a projection $P \in \mathcal{B}(\mathcal{X})$ such that

$$TP = PT, \quad T + P \text{ is bounded below}, \quad TP \text{ is quasinilpotent}.$$

Then $\mathcal{X} = \mathcal{N}(P) \oplus \mathcal{R}(P)$. Also, $\mathcal{N}(P)$ and $\mathcal{R}(P)$ are T-invariant subspaces because T and P commute. The operator $(T + P)_{\mathcal{N}(P)} = T_{\mathcal{N}(P)}$ is bounded below by Proposition 2.3(ii). For $x \in \mathcal{R}(P)$ we have

$$\|T^n_{\mathcal{R}(P)} x\|^{\frac{1}{n}} = \|T^n P^n x\|^{\frac{1}{n}} = \|(TP)^n x\|^{\frac{1}{n}} \to 0 \ (n \to \infty).$$

This means that $H_0(T_{\mathcal{R}(P)}) = \mathcal{R}(P)$ and $T_{\mathcal{R}(P)}$ is quasinilpotent. From Lemma 2.1(vi) it follows that $H_0(T_{\mathcal{N}(P)}) = \{0\}$. The operator T admits a GKD, so $H_0(T) = H_0(T_{\mathcal{N}(P)}) \oplus H_0(T_{\mathcal{R}(P)}) = \mathcal{R}(P)$ by [2, Corollary 1.69] and hence $H_0(T)$ is closed. If we remark that $T(\mathcal{N}(P)) = T_{\mathcal{N}(P)}(\mathcal{N}(P))$ is closed then the result follows. □

We state the similar result for lower generalized Drazin invertible operators.

Theorem 3.2. *An operator $T \in \mathcal{B}(\mathcal{X})$ is lower generalized Drazin invertible if and only if there exists a projection $P \in \mathcal{B}(\mathcal{X})$ such that*

$$TP = PT, \quad T + P \text{ is surjective}, \quad TP \text{ is quasinilpotent}. \tag{2}$$

If T is lower generalized Drazin invertible, then every projection P from (2) *has uniquely determined kernel and $\mathcal{N}(P) = K(T)$.*

Proof. Apply the analysis similar to that in the proof of Theorem 3.1 using in particular [2, Theorem 1.41] instead [2, Corollary 1.69]. □

Proposition 3.3. *Let $\mathcal{X}$ be a Banach space and $T \in \mathcal{B}(\mathcal{X})$. The following statements hold.*
(i) *T is upper generalized Drazin invertible and* $\dim H_0(T) < \infty$ *if and only if $T \in B_+(\mathcal{X})$;*
(ii) *T is lower generalized Drazin invertible and* $\dim \mathcal{X}/K(T) < \infty$ *if and only if $T \in B_-(\mathcal{X})$.*

Proof. (i). Let T be upper generalized Drazin invertible operator such that $\dim H_0(T) < \infty$. From the proof of Theorem 3.1 we know that there exist two closed T-invariant subspaces of $\mathcal{X}$, say $\mathcal{X}_1$ and $\mathcal{X}_2$, such that $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$, $\mathcal{X}_1 = H_0(T)$, $T_{\mathcal{X}_1}$ is quasinilpotent and $T_{\mathcal{X}_2}$ is bounded below. We see that $\dim \mathcal{X}_1 < \infty$ and $T_{\mathcal{X}_1}$ is nilpotent, since every quasinilpotent operator on a finite-dimensional space is nilpotent. Applying [11, Theorem 20.10] completes the proof.

If T is upper semi-Browder then [11, Theorem 20.10] ensures that there exists a decomposition $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$, $\dim \mathcal{X}_1 < \infty$, where $\mathcal{X}_1$ and $\mathcal{X}_2$ are closed T-invariant subspaces of $\mathcal{X}$, such that $T_{\mathcal{X}_1}$ is nilpotent and $T_{\mathcal{X}_2}$ is bounded below. The operator T admits a GKD, so $H_0(T) = \mathcal{X}_1 \oplus H_0(T_{\mathcal{X}_2})$ by [2, Corollary 1.69]. Since $H_0(T_{\mathcal{X}_2}) = \{0\}$ it follows that $\mathcal{X}_1 = H_0(T)$, so $H_0(T)$ is finite-dimensional. In addition, $T(\mathcal{X}_2) = T_{\mathcal{X}_2}(\mathcal{X}_2)$ is closed and hence T is upper generalized Drazin invertible.
(ii). Follows similarly as part (i). □

Example 3.4. The condition $\dim H_0(T) < \infty$ can not be omitted from Proposition 3.3(i). Indeed, if $Q \in \mathcal{B}(\mathcal{X})$ is quasinilpotent operator, then Q is upper generalized Drazin invertible and $\dim H_0(T) = \infty$, but $Q \notin B_+(\mathcal{X})$. Similarly, Q is also lower generalized Drazin invertible operator and $\dim \mathcal{X}/K(T) = \infty$, but it is not lower semi-Browder.

Proposition 3.3 and Example 3.4 prove that the set of upper (resp. lower) semi-Browder operators is a proper subset of the set of upper (resp. lower) generalized Drazin invertible operators.

For two upper (resp. lower) generalized Drazin invertible operators A and B it is said to have *equal projections* if there exists some projection $P \in \mathcal{B}(\mathcal{X})$ such that pairs (A, P) and (B, P) satisfy equation (1) (resp. (2)).

Proposition 3.5. (i) *Let $A_1, A_2 \in \mathcal{B}(\mathcal{X})$ be commuting upper (resp. lower) generalized Drazin invertible operators with equal projections. Then A_1A_2 is upper (resp. lower) generalized Drazin invertible and has equal projection with A_1 and A_2;*
(ii) *If $T \in \mathcal{B}(\mathcal{X})$ is upper (resp. lower) generalized Drazin invertible then T^n is also upper (resp. lower) generalized Drazin invertible for all $n \in \mathbb{N}$.*

Proof. We give proof for upper generalized Drazin invertible operators. The second case follows analogously.
(i). According to Theorem 3.1 and assumption there exists a projection $P \in \mathcal{B}(\mathcal{X})$ such that

$$A_iP = PA_i, \quad A_i + P \text{ is bounded below}, \quad A_iP \text{ is quasinilpotent}, \quad i = 1, 2.$$

It is easy to see that A_1A_2 and P commute. Since A_1 and A_2P commute and A_2P is quasinilpotent, then A_1A_2P is quasinilpotent. Further, we have

$$(A_1 + P)(A_2 + P) = (A_1A_2 + P) + (A_1P + PA_2).$$

The operator $(A_1 + P)(A_2 + P)$ is bounded below and commutes with quasinilpotent operator $A_1P + PA_2$, so $A_1A_2 + P$ is also bounded below. The assertion follows from Theorem 3.1.
(ii). Apply part (i) and use induction. □

Lemma 3.6. *Let $A \in \mathcal{B}(\mathcal{X})$ and let $B \in \mathcal{B}(\mathcal{X})$ be invertible operator such that $AB = BA$. Then $H_0(AB) = H_0(A)$ and $K(AB) = K(A)$.*

Proof. Fix $x \in H_0(A)$. Then $\|(AB)^n x\|^{\frac{1}{n}} = \|B^nA^nx\|^{\frac{1}{n}} \leq \|B^n\|^{\frac{1}{n}}\|A^nx\|^{\frac{1}{n}} \leq \|B\|\|A^nx\|^{\frac{1}{n}}$. Since $\|A^nx\|^{\frac{1}{n}} \to 0$ $(n \to \infty)$, then $\|(AB)^nx\|^{\frac{1}{n}} \to 0$ $(n \to \infty)$, and thus $x \in H_0(AB)$. If we apply the inclusion that has just been demonstrated to operators AB and B^{-1} we obtain the desired result.

For $x \in K(A)$ there exist a sequence $(x_n)_n \subset \mathcal{X}$ and constant $c > 0$ such that

$$Ax_1 = x, \quad Ax_{n+1} = x_n \text{ for all } n \in \mathbb{N} \text{ and } \|x_n\| \leq c^n\|x\| \text{ for all } n \in \mathbb{N}.$$

An easy computation shows that sequence $y_n = (B^{-1})^n x_n,\ n \in \mathbb{N}$, and constant $\delta = \|B^{-1}\|c > 0$ satisfy

$$ABy_1 = x,\ \ ABy_{n+1} = y_n \ \text{ for all } n \in \mathbb{N} \ \text{ and } \ \|y_n\| \le \delta^n \|x\| \ \text{ for all } n \in \mathbb{N},$$

so $x \in K(AB)$. We now apply this argument again, with A replaced by AB and B replaced by B^{-1}, to obtain $K(AB) \subset K(A)$. □

Proposition 3.7. (i) *Let $A \in \mathcal{B}(\mathcal{X})$ be upper generalized Drazin invertible operator and let M be the subspace from Definition 2.4. If $B \in \mathcal{B}(\mathcal{X})$ is invertible operator commuting with A such that $B(M) \subset M$, then AB is also upper generalized Drazin invertible.*
(ii) *Let $A \in \mathcal{B}(\mathcal{X})$ be lower generalized Drazin invertible operator and let N be the subspace from Definition 2.5. If $B \in \mathcal{B}(\mathcal{X})$ is invertible operator commuting with A such that $B(N) \subset N$, then AB is also lower generalized Drazin invertible.*

Proof. (i). According to Lemma 3.6 we have $H_0(AB) = H_0(A)$. It follows that $H_0(AB)$ is closed and complemented with M. It is obvious that $AB(M) \subset M$. The set $AB(M) = B(A(M))$ is closed because $A(M)$ is closed and B is invertible and the proof is complete.
(ii). Apply Lemma 3.6. □

Corollary 3.8. *If $A \in \mathcal{B}(\mathcal{X})$ is upper (resp. lower) generalized Drazin invertible and $\lambda \in \mathbb{C}$, then λA is also upper (resp. lower) generalized Drazin invertible.*

Proof. Case $\lambda = 0$ is evident. For $\lambda \neq 0$ the result follows from Proposition 3.7 if we put $B = \lambda I$. □

Example 3.9. Let $A \in \mathcal{B}(\mathcal{X})$ be upper generalized Drazin invertible and let M be the subspace as in Definition 2.4. Suppose that $H_0(A)$ is non-trivial subspace, i.e. $0 \subsetneq H_0(A) \subsetneq \mathcal{X}$. If projection $P \in \mathcal{B}(\mathcal{X})$ is such that $\mathcal{R}(P) = H_0(A)$ and $\mathcal{N}(P) = M$, then the operator $B = \lambda P + \mu(I - P)$, where λ and μ are nonzero complex numbers, satisfies the conditions of Proposition 3.7(i). If $\lambda = \mu$ then it is evident. Consider the case $\lambda \neq \mu$. The expression for B can be rewritten as $B = (\lambda - \mu)(P - \frac{\mu}{\mu-\lambda} I)$. From $\frac{\mu}{\mu-\lambda} \notin \{0, 1\} = \sigma(P)$, we see that B is invertible. It is clear that $AB = BA$ and $B(M) \subset M$. We can find an operator that satisfies the conditions of Proposition 3.7(ii) in a similar way.

Further, Hilbert space case will be considered.

Remark 3.10. Let $\mathcal{H}$ be a Hilbert space and let $T \in \mathcal{B}(\mathcal{H})$ be self-adjoint. If a subspace $M \subset \mathcal{H}$ is invariant for T, then $M^\perp$ is also invariant for T. Indeed, let $y \in M^\perp$. Then $\langle y, Tx \rangle = 0$ for every $x \in M$. Hence $\langle Ty, x \rangle = \langle y, T^* x \rangle = \langle y, Tx \rangle = 0$ and $Ty \in M^\perp$.

Proposition 3.11. *Let $\mathcal{H}$ be a Hilbert space. If $T \in \mathcal{B}(\mathcal{H})$ is self-adjoint and upper generalized Drazin invertible, then it is generalized Drazin invertible. Moreover, there is only one projection which satisfies* (1).

Proof. According to Theorem 3.1 there exists a projection $P \in \mathcal{B}(\mathcal{H})$ such that

$$TP = PT, \quad T + P \text{ is bounded below}, \quad TP \text{ is quasinilpotent.}$$

The subspace $\mathcal{R}(T + P)$ is closed and T-invariant. The subspace $\mathcal{R}(T + P)^\perp$ is also closed and $T(\mathcal{R}(T + P)^\perp) \subset \mathcal{R}(T + P)^\perp$ by Remark 3.10. From the bounded inverse theorem it follows that we can find an operator $W \in \mathcal{B}(\mathcal{R}(T + P), \mathcal{H})$ such that $W(T + P) = I$ (identity on $\mathcal{H}$). It will be shown that the operator $B = WQ(I - P) \in \mathcal{B}(\mathcal{H})$, where $Q \in \mathcal{B}(\mathcal{H})$ is the orthogonal projection on $\mathcal{R}(T + P)$, satisfies

$$TB = BT, \ \ BTB = B, \ \ T - TBT \ \text{ is quasinilpotent.}$$

Firstly, we show that operators T and $WQ \in \mathcal{B}(\mathcal{H})$ commute. Choose $x \in \mathcal{H}$. Then $x = x_1 + x_2$, $x_1 \in \mathcal{R}(T + P)$, $x_2 \in \mathcal{R}(T + P)^\perp$. Also $x_1 = (T + P)z$ for some $z \in \mathcal{H}$. We have

$$TWQx = TWx_1 = TW(T + P)z = Tz,$$
$$WQTx = WQ(Tx_1 + Tx_2) = WTx_1 = WT(T + P)z = W(T + P)Tz = Tz.$$

Now, $BT = WQ(I-P)T = TWQ(I-P) = TB$. Let remark that $WQ(T+P) = I$ (identity on $\mathcal{H}$). The operator

$$T - TBT = T - TWQ(I-P)T = T - TWQ(I-P)(T+P) = T - T(I-P) = TP$$

is quasinilpotent. Finally,

$$BTB = BTWQ(I-P) = BWQ(T+P)(I-P) = B(I-P) = B.$$

We have just shown that T is generalized Drazin invertible. According to [7, Theorem 3.1 and Theorem 5.4] there exists only one projection $P_1 \in \mathcal{B}(\mathcal{X})$ such that

$$TP_1 = P_1T, \quad T + P_1 \text{ is invertible}, \quad TP_1 \text{ is quasinilpotent},$$

and it is given by $P_1 = I - BT$. We take expression for B and obtain

$$P_1 = I - WQ(I-P)T = I - WQ(T+P)(I-P) = P,$$

so there is no other projections that satisfy (1) except $P = P_1$. □

Proposition 3.11 motivates us to put the following question.

Question. Suppose that $T \in \mathcal{B}(\mathcal{X})$ is upper generalized Drazin invertible operator that is also generalized Drazin invertible. There exists only one projection $P \in \mathcal{B}(\mathcal{X})$ that commutes with T such that $T + P$ is invertible and TP is quasinilpotent. Obviously, P satisfies (1). Does it exist a projection $Q \in \mathcal{B}(\mathcal{X})$ which commutes with T such that $T + Q$ is bounded below but not invertible and TQ is quasinilpotent? Some answers are possible in the context of a Hilbert space. From Proposition 3.11 we see that if T is self-adjoint and upper generalized Drazin invertible (and hence generalized Drazin invertible) the answer is negative. Similar question can be asked for lower generalized Drazin invertible operators that are generalized Drazin invertible.

Finally, we are going to characterize operators that are Riesz and upper (resp. lower) generalized Drazin invertible at the same time. In order to do that, some preparation is needed. We recall some properties of Riesz operators.

Remark 3.12. Let $T \in \mathcal{B}(\mathcal{X})$. The following statements are equivalent.
(i) T is Riesz;
(ii) $\{\lambda \in \mathbb{C} : T - \lambda \in \Phi_+(\mathcal{X})\} = \mathbb{C} \setminus \{0\}$;
(iii) $\{\lambda \in \mathbb{C} : T - \lambda \in \Phi_-(\mathcal{X})\} = \mathbb{C} \setminus \{0\}$.

If $T \in \mathcal{B}(\mathcal{X})$ is Riesz, then the following hold.
(iv) $0 \in \sigma(T)$;
(v) $\sigma(T)$ is finite or a sequence which converges to 0;
(vi) If M is a closed T-invariant subspace of $\mathcal{X}$, then T_M is Riesz.

If M is a closed subspace of $\mathcal{X}$, then $Q_M : \mathcal{X} \to \mathcal{X}/M$ is the natural epimorphism. The following two classes of operators were introduced by Aiena. For more details we refer the reader to [1, 2].

$$\Omega_+(\mathcal{X}) = \left\{ T \in \mathcal{B}(\mathcal{X}) : \begin{array}{l} T_M \text{ is an (into) isomorphism for no infinite-} \\ \text{dimensional, } T\text{-invariant subspace } M \text{ of } \mathcal{X} \end{array} \right\},$$

$$\Omega_-(\mathcal{X}) = \left\{ T \in \mathcal{B}(\mathcal{X}) : \begin{array}{l} Q_M T \text{ is surjective for no infinite-} \\ \text{codimensional, } T\text{-invariant subspace } M \text{ of } \mathcal{X} \end{array} \right\}.$$

We are able to establish the aforementioned characterization.

Theorem 3.13. *Let $T \in \mathcal{B}(\mathcal{X})$. The following statements are equivalent.*
(i) $T \in \Omega_+(\mathcal{X})$ *and* $\sigma(T)$ *is finite;*
(ii) $T \in \Omega_-(\mathcal{X})$ *and* $\sigma(T)$ *is finite;*
(iii) *T is Riesz and generalized Drazin invertible;*
(iv) *T is Riesz and upper generalized Drazin invertible;*
(v) *T is Riesz and lower generalized Drazin invertible.*

Proof. (i) $\Longrightarrow$ (iii). Suppose $T \in \Omega_+(\mathcal{X})$ and $\sigma(T)$ is finite. T is Riesz by [1, Theorem 3.21]. The spectrum of T is finite and $0 \in \sigma(T)$, so $0 \in \operatorname{iso}\sigma(T)$. It proves that T is generalized Drazin invertible.
(iii) $\Longrightarrow$ (i). Let T be Riesz and generalized Drazin invertible. From [1, Theorem 3.21] it follows that $T \in \Omega_+(\mathcal{X})$ and that the spectrum $\sigma(T)$ is either finite or a sequence which converges to 0. The spectrum must be finite, because in the second case $0 \in \operatorname{acc}\sigma(T)$, which contradicts the fact that T is generalized Drazin invertible.
(ii) $\Longleftrightarrow$ (iii) follows similarly as (i) $\Longleftrightarrow$ (iii). Implications (iii) $\Longrightarrow$ (iv) and (iii) $\Longrightarrow$ (v) are clear.
(iv) $\Longrightarrow$ (iii). Suppose that T is Riesz and upper generalized Drazin invertible. There exists a closed T-invariant subspace M of $\mathcal{X}$ such that $H_0(T) \oplus M = \mathcal{X}$ and $T(M)$ is closed. From the proof of Theorem 3.1 we know that $T_{H_0(T)}$ is quasinilpotent and T_M is bounded below. Bounded below operators are upper semi-Fredholm, then $T_M \in \Phi_+(M)$. On the other hand, T_M is Riesz. From the point of view of Remark 3.12 this is impossible, so M is finite-dimensional. We conclude that T_M is invertible and we obtain the desired result.
(v) $\Longrightarrow$ (iii). Suppose that T is Riesz and lower generalized Drazin invertible. Then $K(T)$ is closed and there exists a closed T-invariant subspace N of $\mathcal{X}$ such that $K(T) \oplus N = \mathcal{X}$ and $N \subset H_0(T)$. That $T_{K(T)}$ is surjective follows from $T(K(T)) = K(T)$. The assumption $N \subset H_0(T)$ implies that T_N is quasinilpotent. Obviously, $T_{K(T)}$ is Riesz and lower semi-Fredholm. Remark 3.12 gives that $K(T)$ is finite-dimensional and thus $T_{K(T)}$ is invertible, so T is generalized Drazin invertible. □

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