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On product of range symmetric matrices in an indefinite inner product space

AR. Meenakshi[a]

[a]*Former Emeritus Professor, Annamalai University,Annamalainagar, 608002, India*

Abstract. Equivalent conditions for the product of range symmetric matrices in an indefinite inner product space in the setting of an indefinite matrix product to be range symmetric is derived. Characterizations of range symmetric block matrix in an inner product space are presented. General conditions under which a range symmetric matrix in an indefinite inner product space can be expressed as a product of range symmetric matrices under an indefinite matrix multiplication are established.

1. Introduction

An indefinite inner product in C^n is a conjugate symmetric **sesquilinear** from $[x, y]$ together with the regularity condition that $[x, y] = 0$ for all $y \in C^n$ only when $x = 0$. Any indefinite inner product is associated with a unique invertible complex matrix J (called a weight) such that $[x, y] =< x, Jy >$ where $<, >$ denotes the Euclidean inner product on C^n. We also make an additional assumption on J, that is, $J^2 = I$, to present the results with much algebraic ease. However, there are two different values for dot product of vectors in indefinite inner product spaces. To overcome these difficulties, a new matrix product, called the indefinite matrix multiplication is introduced and some of its properties are investigated in [7]. A complex matrix A is said to be EP if and only if the range space of A and that of its conjugate transpose A^* are equal. The structure of certain class of EP matrices over the complex field having the same range space has been studied by Baskett and Katz in [1]. Recently, in [5], we have extended the concept of range symmetric matrix to indefinite inner product space and presented some interesting characterizations of range symmetric matrices similar to EP matrices in the setting of an indefinite matrix product. Further, we have established that a range symmetric matrix coincides with $J - EP$ matrix in an inner product space with weight J, studied in [2].

In this manuscript, we have discussed the range symmetry of a product of range symmetric matrices in an indefinite inner product space. General conditions under which a range symmetric matrix can be expressed as a product of range symmetric matrices are also established, which includes as a special case the results found in [3&4]. In section 2, we recall the definitions and preliminary results required in characterizing the structure of complex range symmetric matrices over an indefinite inner product space.

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*Email:*arm_meenakshi@yahoo.co.in (AR. Meenakshi)

Equivalent conditions for the product of range symmetric matrices in $\wp$, an indefinite inner product space in the setting of an indefinite matrix product with weight J, to be range symmetric are presented in Section 3. Factorization of a range symmetric matrix as a product of range symmetric matrices in $\wp$ and as an indefinite product of range symmetric matrices in $\wp$ are presented in Section 4.

2. Preliminaries

We first recall the notion of an indefinite multiplication of matrices.

Definition 2.1: Let $A \in C^{m\times n}, B \in C^{nxk.}$ Let J_n be an arbitrary but fixed $n \times n$ complex matrix such that $J_n = J_n^* = J_n^{-1}$. The indefinite matrix product of A and B (relative to J) is defined as $A \circ B = AJ_nB$.

Definition 2.2: For $A \in C^{m\times n}, A^{[*]} = J_nA^*J_m$ is the adjoint of A relative to J_n and J_m, the weights in the appropriate spaces.

Remark 2.1: When J_n is the identity matrix the product reduces to the usual product of matrices and it can be easily verified that with respect to the indefinite matrix product, $rank(A \circ A^{[*]}) = rank(A^{[*]} \circ A) = rank(A)$, where as this rank property fails under the usual matrix multiplication. Thus the Moore -Penrose inverse of a complex matrix over an indefinite inner product space, with respect to the indefinite matrix product exists and this is one of its main advantages.

Definition 2.3: A matrix $A \in C^{n\times n}$ is said to be J-invertible if there exists $X \in C^{n\times n}$, such that $A \circ X = X \circ A = J_n$. Such an X is denoted as $A^{[-1]} = JA^{-1}J$.

Definition 2.4: $A \in C^{n\times n}$ and $X \in C^{n\times n}$ satisfying $A \circ X \circ A = A$ is called a generalized inverse of A relative to the weight J. $A_J\{1\}$ is the set of all generalized inverses of A relative to the weight J.

Remark 2.2: For the identity matrix J, it reduces to a generalized inverse of A and $A_J\{1\} = A\{1\}$. It can be easily verified that X is a generalized inverse of A under the indefinite matrix product if and only if J_nXJ_m is a generalized inverse of A under the usual product of matrices. Hence $A_J\{1\} = \{X/J_nXJ_m$ is a generalized inverse of $A\}$.

Definition 2.5: For $A \in C^{m\times n,}$ a matrix $X \in C^{n\times m}$ is called the Moore-Penrose inverse if it satisfies the following equations:

$A \circ X \circ A = A, X \circ A \circ X = X, (A \circ X)^{[*]} = A \circ X$ and $(X \circ A)^{[*]} = X \circ A$.

Such an X is denoted by $A^{[\dagger]}$ and represented as $A^{[\dagger]} = J_nA^{\dagger}J_m$.

Definition 2.6: The Range space of $A \in C^{m\times n}$ is defined by $R(A) = \{y = A \circ x \in C^m/x \in C^n\}$.

The Null space of A is defined by $Nu(A) = \{x \in C^n/A \circ x = 0\}$. It is clear that $Nu(A^{[*]}) = N(A^*)$.

Property 2.1: Let $A \in C^{n\times n}$. Then

(i) $(A^{[*]})^{[*]} = A$.
(ii) $(A^{[\dagger]})^{[\dagger]} = A$.
(iii) $(AB)^{[*]} = B^{[*]}A^{[*]}$.
(iv) $R(A^{[*]}) = R(A^{[\dagger]})$.
(v) $R(A \circ A^{[*]}) = R(A), R(A^{[*]} \circ A) = R(A^{[*]})$.
(vi) $N(A \circ A^{[*]}) = N(A^{[}*]), N(A^{[*]} \circ A) = N(A)$.

We recall the definition of a range symmetric matrix in $\wp$, an indefinite inner product space with weight J, analogous to that of a range symmetric matrix in the unitary space.

Definition 2.7: A matrix $A \in C^{n\times n}$ is range symmetric in $\wp$ if and only if $R(A) = R(A^{[*]})$.

Remark 2.2: For the identity matrix J it reduces to the definition of range symmetric matrix in unitary space (or) equivalently to an EP matrix [1].

In the sequel, we shall use the following equivalent characterizations of a range symmetric matrix established in our earlier work [5].

Theorem 2.1: For $A \in C^{n\times n}$, the following are equivalent:
(1) A is range symmetric in $\wp$.
(2) AJ is EP.
(3) JA is EP.
(4) $N(A) = N(A^{[*]})$.
(5) $(A \circ A^{[\dagger]}) = (A^{[\dagger]} \circ A)$
(6) $(A^{\dagger}A)^{[*]} = JA^{\dagger}AJ = AA^{\dagger}$.
(7) A is $J - EP$.

3. Product of range symmetric matrices in $\wp$

In this section, we have obtained necessary and sufficient conditions for the indefinite product of two range symmetric matrices of rank r to be range symmetric in . Then we have extended the result to block matrices in $\wp$.

Theorem 3.1: Let A and B be range symmetric matrices of rank r in $\wp$ and $A \circ B$ be of rank r. Then, $A \circ B$ is range symmetric in $\wp$ if and only if $R(A) = R(B)$.

Proof: Since A and B are range symmetric in $\wp$, by Theorem 2.1(2), AJ and BJ are EP_r matrices.
$A \circ B$ is range symmetric in $\wp$ and of rank $r \Leftrightarrow (A \circ B)J$ is EP_r (By Theorem 2.1(7))
$\Leftrightarrow (AJ)(BJ)$ is EP_r (By Definition2.1)
$\Leftrightarrow R(AJ) = R(BJ)$ (By Theorem 2 of [1])
$\Leftrightarrow R(A) = R(B)$.
Hence the Theorem holds.

Henceforth we are concerned with $(m + n) \times (m + n)$ matrices M partitioned in the form

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{3.1}$$

with rank(M)=rank(A), A and D are square matrices of orders m and n respectively. It is well known that (p.21 of [8]) M of the form (3.1) satisfies the following:
$C = YA$ for some $Y \in C^{n\times m}$, $B = AX$ for some $X \in C^{m\times n}$ and $D = CA^{\dagger}B$. (3.2)

Let J, J_m and J_n be the weights associated with the indefinite inner products in C^{m+n}, C^m and C^n respectively. Since $J_m = J_m^* = J_m^{(-1)}$ and $J_n = J_n^* = J_n^{(-1)}$, it can be verified that J is of the form

$$J = \begin{bmatrix} J_m & 0 \\ 0 & J_n \end{bmatrix} \tag{3.3}$$

Theorem 3.2: Let M be of the form (3.1). M is $J - EP$ if and only if A is $J_m - EP$ and there exists an $m \times n$ matrix X such that

$$M = \begin{bmatrix} A & AX \\ X^{[*]}A & X^{[*]}AX \end{bmatrix}.$$

Proof: Since M is of the form (3.1), by using (3.2) M reduces to the form

$$M = \begin{bmatrix} A & AX \\ YA & YAX \end{bmatrix}. \tag{3.4}$$

Since J is of the form (3.3),

$JM = \begin{bmatrix} J_mA & J_mAX \\ J_nYA & J_nYAX \end{bmatrix}$. Then the proof runs as follows:

M is $J-EP \Leftrightarrow JM$ is EP (By Theorem2.1)
$\Leftrightarrow J_mA$ is EP and $J_nYA(J_mA)^\dagger = ((J_mA)^\dagger(J_mAX))^*$ (By Theorem3 of [4])
$\Leftrightarrow A$ is J_m-EP and $J_nYAA^\dagger J_m = (A^\dagger AX)^* = X^*(A^\dagger A)$ (By Theorem 2.1)
$\Leftrightarrow A$ is J_m-EP and $YAA^\dagger = (J_nX^*J_m)(J_mA^\dagger AJ_m)$
$\Leftrightarrow A$ is J_m-EP and $YAA^\dagger = X^([*])AA^\dagger$ (By Definition 2.2 & Theorem2.1)
$\Leftrightarrow A$ is J_m-EP and $YA = X^{[*]}A$.
Then the Theorem follows on substitution for YA in the equation (3.4).
Theorem 3.3: Let

$$M=\begin{bmatrix}A & B\\ C & D\end{bmatrix} \text{ and } L=\begin{bmatrix}F & G\\ H & K\end{bmatrix}$$

be $J-EP$ matrices in $\wp$, both of the form (3.1) and $M\circ L$ be of rank r, then the following are equivalent:
(i) $M\circ L$ is $J-EP$.
(ii) $A\circ F$ is J_m-EP and $CA^\dagger = HF^\dagger$.
(iii) $A\circ F$ is J_m-EP and $A^\dagger B = F^\dagger G$.
Proof: Since both M and L are $J-EP$ matrices of the form (3.1), by Theorem 3.2, there exist $m\times n$ matrices X and Y such that $M=\begin{bmatrix}A & AX\\ X^{[*]}A & X^{[*]}AX\end{bmatrix}$ and $L=\begin{bmatrix}F & FY\\ Y^{[*]}F & Y^{[*]}FY\end{bmatrix}$.

$$\text{Now, } M\circ L = \begin{bmatrix}A & AX\\ X^{[*]}A & X^{[*]}AX\end{bmatrix}\begin{bmatrix}J_m & 0\\ 0 & J_n\end{bmatrix}\begin{bmatrix}F & FY\\ Y^{[*]}F & Y^{[*]}FY\end{bmatrix}.$$
$$= \begin{bmatrix}A(J_m+XY^*J_m)F & A(J_m+XY^*J_m)FY\\ X^{[*]}A(J_m+XY^*J_m)F & X^{[*]}A(J_m+XY^*J_m)FY\end{bmatrix}.$$
$$= \begin{bmatrix}AZF & AZFY\\ X^{[*]}AZF & X^{[*]}AZFY\end{bmatrix} \text{ where } Z=(J_m+XY^*J_m).$$

Clearly, $N(AZF)\subseteq N(X^{[*]}AZF), N(AZF)^{[*]}\subseteq N(AZFY)^{[*]}$ and the Schur complement of AZF in $M\circ L$ is zero. Indeed,
$M\circ L/AZF = X^{[*]}AZFY - X^{[*]}(AZF)(AZF)^\dagger(AZF)Y = X^{[*]}AZFY - X^{[*]}AZFY = O$.
Hence, rank(AZF)=rank$(M\circ L)$=r. Thus $M\circ L$ is also of the form (3.1) . Since M and L are range symmetric in $\wp$, by Theorem 2.1 M and L are $J-EP$. Then, by Theorem 3.2 A and F are J_m-EP. Clearly, $R(AZF)\subseteq R(A)$ and by property 2.1(iii) $R(AZF)^{[*]}\subseteq R(F)^{[*]}$. Since, rank$(AZF)$=rank$(A)$=rank$(F)$=$r$, we have $R(AZF)=R(A)$ and $R((AZF)^{[*]} = R(F)^([*])$. Therefore,
$(AZF)(AZF)^\dagger = AA^\dagger$ and $(AZF)^\dagger(AZF) = F^\dagger F$. (3.5)
Since A and F are J_m-EP, by Theorem 2.1(6), we have
$(A^\dagger A)^{[*]} = (J_mA^\dagger AJ_m) = AA^\dagger$ and $(F^\dagger F)^{[*]} = (J_mF^\dagger FJ_m) = FF^\dagger$. (3.6)
Further, $A\circ F$ is$J_m-EP \Leftrightarrow R(A)=R(F)$ (By Theorem3.1).
$\Leftrightarrow R(AZF)=R(A)=R(F)=R(F)^{[*]}=R((AZF)^{[*]}$.
$\Leftrightarrow AZF$ is J_m-EP.
Thus for J_m-EP matrices A and F, with $R(A)=R(F)$, we have
$A\circ F$ is $J_m-EP \Leftrightarrow AZF$ is J_m-EP.
Now the proof runs as follows:
$M\circ L is J-EP_r \Leftrightarrow AZF is J_m-EP_r$ and $X^{[*]}(AZF)(AZF)^\dagger = ((AZF)^\dagger(AZF)Y)^{[*]}$.
$\Leftrightarrow AZF$ is J_m-EP_r and $X^{[*]}AA^\dagger = (F^\dagger FY)^{[*]}$ (By using 3.5)
$\Leftrightarrow A\circ F is J_m-EP_r$ and $X^{[*]}AA^\dagger = Y^{[*]}(FF)^{[*]}$ (By (3.7) & Property 2.1(iii))
$\Leftrightarrow A\circ F is J_m-EP_r$ and $X^{[*]}AA^\dagger = Y^{[*]}FF^\dagger$ (By using 3.6)
$\Leftrightarrow A\circ F is J_m-EP_r$ and $CA^\dagger = HF^\dagger$ (By using $C=X^{[*]}A$ and $H=Y^{[*]}F$)
$\Leftrightarrow A\circ F is J_m-EP_r$ and $A^\dagger B = F^\dagger G$ (By Theorem 3 of [4]).
Hence the Theorem holds.

4. Factorization

In this section, a set of conditions under which a complex matrix can be expressed as a product of range symmetric matrices in $\wp$ and as the indefinite matrix product of range symmetric matrices in are derived. Since, A is $J-EP$ and A is range symmetric in $\wp$ are equivalent by Theorem 2.1, hence forth we use, A is $J-EP_r$ if A is range symmetric in $\wp$ and of rank r.

Definition 4.1: A matrix $P \in C^{n\times n}$ is said to be J-unitary if $P \circ P^{[*]} = P^{[*]} \circ P = J_n$.

Lemma 4.1: Let $A \in C_r^{n\times n}$. Then A is $J-EP_r$ if and only if $A = P\begin{bmatrix} D & O \\ O & O \end{bmatrix}P^{[*]}$, where P is J-unitary and $D \in C_r^{r\times r}$.

Proof: A is $J-EP_r \Leftrightarrow AJ$ is EP_r (By Theorem 2.1(7))

$$\Leftrightarrow AJ = U\begin{bmatrix} D_1 & O \\ O & O \end{bmatrix}U^*, \text{ where } U \text{ is unitary and } D_1 \in C_r^{r\times r} \text{ (By Theorem 1 of [6]).}$$
$$\Leftrightarrow A = (UJ)\begin{bmatrix} D & O \\ O & O \end{bmatrix}U^*J, \text{ where } D \in C_r^{r\times r}.$$
$$\Leftrightarrow A = P\begin{bmatrix} D & O \\ O & O \end{bmatrix}P^{[*]}, \text{ where } P = UJ \text{ is } J\text{- unitary and } D \in C_r^{r\times r}.$$

Hence the Lemma holds.

Lemma 4.2: Let A, B be $J-EP_r$ matrices. Then $Nu(A) = Nu(B) \Leftrightarrow Nu(PAP^{[*]}) = Nu(PBP^{[*]})$, where P is J-unitary.
Proof: $(\Rightarrow).x \in Nu(PAP^{[*]}) \Leftrightarrow PAP^{[*]} \circ x = 0.$

$$\Leftrightarrow PAJP^*x = 0.$$
$$\Leftrightarrow A \circ y = 0, \text{ where } y = P^*x \text{ (By Definition 4.1 \& Definition 2.1)}$$
$$\Leftrightarrow B \circ y = 0, \text{ (By using } Nu(A) = Nu(B)).$$
$$\Leftrightarrow PBJP^*x = 0.$$
$$\Leftrightarrow ((PBP^{[*]}) \circ x = 0$$
$$\Leftrightarrow x \in Nu(PBP^{[*]}).$$

Reverse implication can be proved in the same manner and hence omitted.
Theorem 4.1: Let M of the form (3.1) be range symmetric in $\wp$. Then M can be expressed as a product of range symmetric matrices in $\wp$. Also M can be written as an indefinite matrix product of range symmetric matrices in $\wp$.

Proof: Since M is range symmetric in $\wp$ and of the form (3.1), by Theorem 3.2, A is $J_m - EP$ and there exists an $m \times n$ matrix X such that $M = \begin{bmatrix} A & AX \\ X^{[*]}A & X^{[*]}AX \end{bmatrix}$. Then, by using $J = \begin{bmatrix} J_m & 0 \\ 0 & J_n \end{bmatrix}$, we have

$$JM = \begin{bmatrix} J_mA & J_mAX \\ J_nX^{[*]}A & J_nX^{[*]}AX \end{bmatrix}. \tag{4.1}$$

Now, let us consider the matrices

$$P = \begin{bmatrix} J_mAA^{\dagger}J_m & J_mAA^{\dagger}J_mX \\ X^*J_mAA^{\dagger}J_m & X^*J_mAA^{\dagger}J_mX \end{bmatrix}, L = \begin{bmatrix} J_mA & O \\ O & O \end{bmatrix} \text{ and } Q = \begin{bmatrix} A^{\dagger}A & A^{\dagger}AX \\ X^{[*]}A^{\dagger}A & X^{[*]}A^{\dagger}AX \end{bmatrix}.$$

Clearly, P, L and Q satisfy (3.2), hence, rank(P)=rank(Q)=rank(L)=rank(A)=r. Since A is $J_m - EP$, by Theorem 2.1(7) J_mA is EP. Therefore L is EP. Again by Theorem 2.1(6), $J_mAA^{\dagger}J_m = A^{\dagger}A$. Hence, $P^* = P = Q = Q^*$. P and Q are EP matrices being symmetric. Now, by using (4.1), we have,

$$PLQ = \begin{bmatrix} J_mA & O \\ X^*J_mA & O \end{bmatrix}\begin{bmatrix} A^{\dagger}A & A^{\dagger}AX \\ X^{[*]}A^{\dagger}A & X^{[*]}A^{\dagger}AX \end{bmatrix} = \begin{bmatrix} J_mA & J_mAX \\ X^{[*]}J_mA & X^{[*]}J_mAX \end{bmatrix} = \begin{bmatrix} J_mA & J_mAX \\ J_nX^{[*]}A & J_nX^{[*]}AX \end{bmatrix} = JM.$$

Thus JM is a product of EP_r matrices. Also, M can be written as $M = (JP)(LJ)(JQ)$ and we conclude that M is expressed as a product of $J-EP$ matrices. Again M can be written as $M = (JP)J(JL)J(JQ) = (JP) \circ (JL) \circ (JQ)$ and we conclude that M is expressed as the indefinite matrix product of $J - EP$ matrices. Hence the Theorem holds.

5. Conclusion

Here we have determined conditions for the indefinite product of range symmetric matrices in an inner product space to be range symmetric and discussed some decomposition of a matrix into product of range symmetric matrices as a generalization of our earlier works established in [3&4].

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References

[1] T. S. Baskett, I. J. Katz, Theorem on product of EPr matrices, Lin. Alg. and its Appln. 2 (1969) 87–103.
[2] S. Jayaraman, EP matrices in indefinite inner product spaces, Funct. Anal. Approx. Comp. 4 (2) (2012) 23–31.
[3] AR. Meenakshi, Products of EP Partitioned Matrices, C. R. Math. Rep. Acad. Sci. Canada III (1981) No. 6 317–322.
[4] AR. Meenakshi, On Schur Complements in an EP matrix, Periodica Math. Hungarica 16 (3)(1985) 193–200.
[5] AR. Meenakshi, Range symmetric matrices in indefinite inner product space, Int. J. fuzzy Math. Archive 5 (2) (2015) 49–56.
[6] M. H. Pearl, On generalized inverses of matrices, Proc. Camb. Phil. Soc. 62 (1966) 673–677.
[7] K. Ramanathan, K. Kamaraj, K. C. Sivakumar, Indefinite product of matrices and applications to indefinite inner product spaces, J. Anal. 12 (2004) 135–142.
[8] C. R. Rao and S. K. Mitra, Generalized inverses of matrices and its applications, Wiley, New York, 1971.