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Gelfand-Shilov spaces and localization operators

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Abstract. We use Komatsu's approach in the study of Gelfand-Shilov spaces of ultradifferentiable functions in both quasianalytic and non-quasianalytic case. In particular, we prove the kernel theorem for such spaces and study the action of time-frequency representations on Gelfand-Shilov spaces and their dual spaces of ultradistributions. We apply the results to prove the trace class properties of localization operators with ultradistributional symbols. As a bridge between those results we prove and use the description of certain Gelfand-Shilov spaces and their dual spaces as projective and inductive limits of Feichtinger's modulation spaces. For the sake of completeness, we review continuity and compactness properties of localization operators on modulation spaces with polynomial weights, which concerns the space of tempered distributions instead.

1. Introduction

Problems of regularity of solutions to partial differential equations (PDEs) play a central role in the modern theory of PDEs. When solutions of certain PDEs are smooth but not analytic, several intermediate spaces of functions are proposed in order to describe its decay for $|x| \to \infty$ and regularity in $\mathbb{R}^d$. In particular, in the study of properties of solutions of certain parabolic initial-value problems Gelfand and Shilov introduced *the spaces of type S* in [22]. We refer to [23] for the main results on such spaces which are afterwards called Gelfand-Shilov spaces. More recently, Gelfand-Shilov spaces were used in [6, 7] to describe exponential decay and holomorphic extension of solutions to globally elliptic equations, and in [33] in the regularizing properties of the Boltzmann equation. We refer to [36] for a recent overview and for applications in quantum mechanics and traveling waves, and to [56] for the properties of the Bargmann transform on Gelfand-Shilov spaces. The original definition from [23] is afterwards extended to more general decay and regularity conditions by Komatsu's approach to ultradifferentiable functions developed in [31].

In the context of time-frequency analysis, Gelfand-Shilov spaces are connected to modulation spaces [25, 28] and the corresponding pseudodifferential calculus [56, 57]. In particular, Gelfand-Shilov spaces are used in the study of time-frequency localization operators in [14, 15], which gives a new context to the pioneering results of Cordero and Grochenig [11].

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To define localization operators we start with the short-time Fourier transform, a time-frequency representation related to Feichtinger's modulation spaces, cf. [19].

The short-time Fourier transform (STFT in the sequel) of $f \in L^2(\mathbb{R}^d)$ with respect to the window $g \in L^2(\mathbb{R}^d)$ is given by

$$V_g f(x,\omega) = \langle f, M_\omega T_x g\rangle = \int_{\mathbb{R}^d} f(t)\,\overline{g(t-x)}\,e^{-2\pi i\omega t}\,dt, \tag{1}$$

where the translation and modulation operators are defined by

$$T_x f(t) = f(t-x) \quad \text{and} \quad M_\omega f(t) = e^{2\pi i\omega t} f(t) \quad t, x, \omega \in \mathbb{R}^d. \tag{2}$$

This definition can be extended to pairs of dual topological vector spaces whose duality, denoted by $\langle\cdot,\cdot\rangle$ extends the inner product on $L^2(\mathbb{R}^d)$, see Section 3.

The *localization operator* $A_a^{\varphi_1,\varphi_2}$ with *symbol* $a \in L^2(\mathbb{R}^{2d})$ and *windows* $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$ is given by

$$A_a^{\varphi_1,\varphi_2} f(t) = \int_{\mathbb{R}^{2d}} a(x,\omega) V_{\varphi_1} f(x,\omega) M_\omega T_x \varphi_2(t)\,dx d\omega, \quad f \in L^2(\mathbb{R}^d), \tag{3}$$

or, in the weak sense, by

$$\langle A_a^{\varphi_1,\varphi_2} f, g\rangle = \langle a V_{\varphi_1} f, V_{\varphi_2} g\rangle = \langle a, \overline{V_{\varphi_1} f}\, V_{\varphi_2} g\rangle, \quad f, g \in L^2(\mathbb{R}^d). \tag{4}$$

Since, as above, the brackets can be interpreted as duality between different pairs of dual spaces, the definition of a localization operator $A_a^{\varphi_1,\varphi_2}$ extends far beyond $L^2(\mathbb{R}^d)$, see Section 5.

Localization operators of similar type were first introduced by Berezin in the study of general Hamiltonians satisfying the so called Feynman inequality, within a quantization problem in quantum mechanics, [2, 44]. Such operators and their modifications are also called Toeplitz or Berezin-Toeplitz operators, anti-Wick operators and Gabor multipliers, see [21, 55, 56]. We do not intend to discuss the terminology here, and refer to, e.g. [18] for the relation between Toeplitz operators and localization operators.

In signal analysis localization operators are related to localization technique developed by Slepian-Polak-Landau, where time and frequency are considered to be two separate spaces, we refer to [47] for an overview. A different construction is proposed by Daubechies in [17], where time-frequency plane is treated as one geometric whole (phase space). The paper [17], which contains localization in phase space together with basic facts on localization operators with references to applications in optics and signal analysis, initiated farther study of the topic. More precisely, in [17] Daubechies studied localization operators $A_a^{\varphi_1,\varphi_2}$ with Gaussian windows

$$\varphi_1(x) = \varphi_2(x) = \pi^{-d/4}\exp(-x^2/2), \quad x \in \mathbb{R}^d, \quad \text{and with a radial symbol} \quad a \in L^1(\mathbb{R}^{2d}).$$

Such operators are named Daubechies' operators afterwards. The eigenfunctions of Daubechies' operators are $d-$dimensional Hermite functions

$$H_k(x) = H_{k_1}(x_1) H_{k_2}(x_2) \dots H_{k_d}(x_d) = \prod_{j=1}^{d} H_{k_j}(x_j), \quad x \in \mathbb{R}^d, \quad k \in \mathbb{N}_0^d, \tag{5}$$

and

$$H_n(t) = (-1)^n \pi^{-1/4} (2^n n!)^{-1/2} \exp(t^2/2)(\exp(-t^2))^{(n)}, \quad t \in \mathbb{R}, \quad n = 0, 1, \dots,$$

and corresponding eigenvalues can be explicitly calculated. This important issue in applications is discussed in [42]. Moreover, the Hermite functions belong to test function spaces for ultra-distributions, both in non-quasianalytic and in quasianalytic case, and give rise to important representation theorems, [32]. In our analysis this fact is used in Theorem 2.5.

Localization operators of the form $\langle L_{\chi_\Omega} f, g\rangle = \iint_\Omega W(f,g)$, where

$$W(f,g)(x,\omega) = \int f(x+\frac{t}{2})\overline{g(x-\frac{t}{2})}e^{-2\pi i\omega t}\,dt, \quad x,\omega \in \mathbb{R}, \quad f,g \in L^2(\mathbb{R}), \tag{6}$$

is the cross-Wigner distribution or the Wigner transform (see [21]) were studied in [42] in the context of signal analysis. There it is proved that the eigenfunctions of L_{χ_Ω} belong to the Gelfand-Shilov space $\mathcal{S}^{(1)}(\mathbb{R}^d)$ (cf. Section 2 for the definition), if $\Omega \subset [-B,B]\times[-T,T]$ is an open set such that all its cross-sections in both ω and x directions consist of at most M intervals.

Inverse problem for a simply connected localization domain Ω is recently studied in [1]. There it is proved that if one of the eigenfunctions of Daubechies' operator is a Hermite function, then Ω is a disc centered at the origin.

In abstract harmonic analysis, localization operators on a locally compact group G and $L^p(G)$, $1 \le p \le \infty$, were studied in [62] where one can find a product formula and Schatten-von Neumann properties of localization operators, see also [4, 12, 16].

Since the beginning of the XXI century, localization operators in the context of modulation spaces were studied by many authors, cf. [11, 12, 20, 54, 55]. See also the references given there.

For example, different choices of windows and symbols of localization operators give rise to different continuity, compactness and Schatten-von Neumann properties [11, 14, 54, 55], composition formulas and Fredholm property [12, 16], multilinear versions [13], eigenvalue and eigenfunctions estimates [1, 17, 42].

In this paper we follow [14, 15] and study localization operators via Gelfand-Shilov spaces.

For the reader's convenience we briefly describe the content of the paper. We collect notation and definition of some basic spaces in subsections 1.1 and 1.2, respectively. Section 2 contains the definition and basic facts on Gelfand-Shilov spaces. In particular, we recall Theorem 2.3 a classical and important result which shows that the decay at infinity and the regularity of functions in Gelfand-Shilov spaces can be studied separately. In subsection 2.3 we prove the kernel theorem which is used on several occasions afterwards. We follow the proof given in [40] with a slight modification due to the absence of the non-quasianalyticity condition in our result. In Section 3 we study the STFT and the cross-Wigner distribution in Gelfand-Shilov spaces and their dual spaces of ultradistributions, see Theorem 3.2 (compare to, e.g., [56, Section 2]). We define modulation spaces in Section 4 and prove that some Gelfand-Shilov spaces are projective and inductive limits of modulation spaces, Theorem 4.3. Moreover we recall the convolution relations between modulation spaces which will be used in Section 5. Finally, we study localization operators in the context of Gelfand-Shilov spaces in the last section. More precisely, we use the kernel theorem to show that any localization operator is, in fact, a certain Weyl pseudodifferential operator, Theorem 5.2 (this is a well known fact in the context of tempered distributions). This connection is then used in Theorem 5.10 (taken from [15]) to show that some localization operators are trace class operators even if their symbols are certain compactly supported ultradistributions. In addition, we also observe localization operators acting on modulation spaces defined by polynomial type weights, and restate recently published results from [51] which extend some well known results from [11].

1.1. Notation

We define $xy = x\cdot y$, the scalar product on $\mathbb{R}^d$ and denote the Euclidean norm by $\|x\|$. Given a vector $x = (x_1,\ldots,x_d) \in \mathbb{R}^d$, the partial derivative with respect to x_j is denoted by $\partial_j = \frac{\partial}{\partial x_j}$. Given a multi-index $p = (p_1,\ldots,p_d) \ge 0$, i.e., $p \in \mathbb{N}_0^d$ and $p_j \ge 0$, we write $\partial^p = \partial_1^{p_1}\cdots\partial_d^{p_d}$. We write $x^p = (x_1,\ldots,x_d)^{(p_1,\ldots,p_d)} = \prod_{i=1}^d x_i^{p_i}$ and $h\cdot|x|^{1/\alpha} = \sum_{i=1}^d h_i|x_i|^{1/\alpha_i}$. Moreover, for $p \in \mathbb{N}_0^d$ and $\alpha \in \mathbb{R}_+^d$, we set $(p!)^\alpha = (p_1!)^{\alpha_1}\ldots(p_d!)^{\alpha_d}$. In the sequel, a real number $r \in \mathbb{R}_+$ may play the role of the vector with constant components $r_j = r$, so for $\alpha \in \mathbb{R}_+^d$, by writing $\alpha > r$ we mean $\alpha_j > r$ for all $j = 1,\ldots,d$.

For $A = (A_1,\ldots,A_d)$ and $B = (B_1,\ldots,B_d)$, $A > 0$ and $B > 0$ means $A_1,\ldots,A_d,B_1,\ldots,B_d > 0$.

For a multiindex $\alpha \in \mathbb{N}_0^d$ we have $|\alpha| = \alpha_1 + \cdots + \alpha_d$. For given $h > 0$ and multiindex $\alpha \in \mathbb{N}_0^d$ we will (sometimes) use the notation $h^\alpha := h^{|\alpha|}$.

Throughout the paper, we shall use the notation $A \lesssim B$ to indicate $A \leq cB$ for a suitable constant $c > 0$, whereas $A \asymp B$ means that $c^{-1}A \leq B \leq cA$ for some $c \geq 1$. The symbol $X \hookrightarrow Y$ denotes the continuous and dense embedding of the topological vector space X into Y. By $\mathcal{L}_b(X, Y)$ we denote the space of all continuous linear mappings from locally convex topological vector space X into Y equipped with the topology of bounded convergence.

The involution f^* is $f^*(\cdot) = \overline{f(-\cdot)}$, and $\check{f}(\cdot) = f(-\cdot)$. and the convolution of f and g is given by $f * g(x) = \int f(x-y)g(y)dy$, when the integral exists. The Fourier transform is normalized to be $\hat{f}(\omega) = \mathcal{F}f(\omega) = \int f(t)e^{-2\pi i t\omega}dt$.

1.2. Basic spaces

In general a weight $w(\cdot)$ on $\mathbb{R}^d$ is a non-negative and continuous function. By $L^p_w(\mathbb{R}^d)$, $p \in [1,\infty]$ we denote the weighted Lebesgue space defined by the norm

$$\|f\|_{L^p_w} = \|fw\|_{L^p} = \left(\int |f(x)|^p w(x)^p dx\right)^{1/p},$$

with the usual modification when $p = \infty$.

Similarly, the weighted mixed-norm space $L^{p,q}_w(\mathbb{R}^{2d})$, $p, q \in [1,\infty]$ consists of (Lebesgue) measurable functions on $\mathbb{R}^{2d}$ such that

$$\|F\|_{L^{p,q}_w} = \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |F(x,\omega)|^p w(x,\omega)^p dx\right)^{q/p} d\omega\right)^{1/q} < \infty.$$

where $w(x,\omega)$ is a weight on $\mathbb{R}^{2d}$.

We denote by $\langle\cdot\rangle^s$ the polynomial weights

$$\langle(x,\omega)\rangle^s = (1 + |x|^2 + |\omega|^2)^{s/2}, \quad (x,\omega) \in \mathbb{R}^{2d}, \quad s \in \mathbb{R},$$

and $\langle x\rangle = \langle 1 + |x|^2\rangle^{1/2}$, when $x \in \mathbb{R}^d$. In particular, when $w(x,\omega) = \langle x\rangle^t\langle\omega\rangle^s$, $s, t \in \mathbb{R}$, we use the notation $L^{p,q}_w(\mathbb{R}^{2d}) = L^{p,q}_{t,s}(\mathbb{R}^{2d})$. and when $w(x) = \langle x\rangle^t$ $t \in \mathbb{R}$, we use the notation $L^p_t(\mathbb{R}^d)$ instead.

We use the brackets $\langle f, g\rangle$ to denote the extension of the inner product $\langle f, g\rangle = \int f(t)\overline{g(t)}dt$ on $L^2(\mathbb{R}^d)$ to any pair of dual spaces. The space of smooth functions with compact support on $\mathbb{R}^d$ is denoted by $\mathcal{D}(\mathbb{R}^d)$. The Schwartz class is denoted by $\mathcal{S}(\mathbb{R}^d)$, the space of tempered distributions by $\mathcal{S}'(\mathbb{R}^d)$. Recall, $\mathcal{S}(\mathbb{R}^d)$ is a Fréchet space, the projective limit of spaces $\mathcal{S}_p(\mathbb{R}^d)$, $p \in \mathbb{N}_0$, defined by the norms:

$$\|\varphi\|_{\mathcal{S}_p} = \sup_{|\alpha|\leq p}(1 + |x|^2)^{p/2}|\partial^\alpha \phi(x)| < \infty, \quad p \in \mathbb{N}_0.$$

Note that $\mathcal{D}(\mathbb{R}^d) \hookrightarrow \mathcal{S}(\mathbb{R}^d)$.

The spaces $\mathcal{S}(\mathbf{R}^d)$ and $\mathcal{S}'(\mathbf{R}^d)$ play an important role in various applications since the Fourier transform is a topological isomorphism between $\mathcal{S}(\mathbf{R}^d)$ and $\mathcal{S}(\mathbf{R}^d)$ which extends to a continuous linear transform from $\mathcal{S}'(\mathbf{R}^d)$ onto itself.

In order to deal with particular problems in applications different generalizations of the Schwartz type spaces were proposed. An example is given by the Gevrey classes given below. Gelfand-Shilov spaces are another important example, see Section 2.

By Ω we denote an open set in $\mathbb{R}^d$, and $K \Subset \Omega$ means that K is compact subset in Ω. For $1 < s < \infty$ we define the Gevrey class $G^s(\Omega)$ by

$$G^s(\Omega) = \{\phi \in C^\infty(\Omega) \mid (\forall K \Subset \Omega)(\exists C > 0)(\exists h > 0) \quad \sup_{x\in K}\left|\partial^\alpha \phi(x)\right| \leq Ch^{|\alpha|}|\alpha|!^s\}.$$

We denote by $G_0^s(\Omega)$ a subspace of $G^s(\Omega)$ which consists of compactly supported functions. We have $\mathcal{A}(\Omega) \hookrightarrow \cap_{s>1} G^s(\Omega)$ and $\cup_{s\geq 1} G^s(\Omega) \hookrightarrow C^\infty(\Omega)$, where $\mathcal{A}(\Omega)$ denotes the space of analytic functions defined by

$$\mathcal{A}(\Omega) = \{\phi \in C^\infty(\Omega) \mid (\forall K \Subset \Omega)(\exists C > 0)(\exists h > 0) \quad \sup_{x\in K} \left|\partial^\alpha \phi(x)\right| \leq C h^{|\alpha|} |\alpha|!\}.$$

We end this section with test function spaces for the spaces of ultradistributions which will be used in Section 5. Let there be given an open set $\Omega \subset \mathbb{R}^d$, and a sequence $(N_q)_{q\in\mathbb{N}_0}$ which satisfies (M.1) and (M.2), see Section 2. The function $\phi \in C^\infty(\Omega)$ is called *ultradifferentiable* function of Beurling class (N_q) (respectively of Roumieu class $\{N_q\}$) if, for any $K \subset\subset \Omega$ and for any $h > 0$ (respectively for some $h > 0$),

$$\|\phi\|_{N_q,K,h} = \sup_{x\in K, \alpha\in\mathbb{N}_0^d} \frac{|\partial^\alpha \phi(x)|}{h^{|\alpha|} N_{|\alpha|}} < \infty.$$

We say that $\phi \in \mathcal{E}^{N_q,K,h}(\Omega)$ if $\|\phi\|_{N_q,K,h} < \infty$ for given K and $h > 0$, and define the following spaces of ultradifferentiable test functions:

$$\mathcal{E}^{(N_q)}(\Omega) := \operatorname{proj} \lim_{K\subset\subset\Omega} \operatorname{proj} \lim_{h\to 0} \mathcal{E}^{N_q,K,h}(\Omega);$$

$$\mathcal{E}^{\{N_q\}}(\Omega) := \operatorname{proj} \lim_{K\subset\subset\Omega} \operatorname{ind} \lim_{h\to\infty} \mathcal{E}^{N_q,K,h}(\Omega).$$

2. Gelfand-Shilov spaces

In this section we introduce Gelfand-Shilov spaces and list important equivalent characterizations. We also prove a kernel theorem which will be used in the study of localization operators.

2.1. Definition

We refer to the original source [23] for the main properties of Gelfand-Shilov spaces. The regularity and decay properties of elements of Gelfand-Shilov spaces are initially measured with respect to sequences of the form $M_p = p^{\alpha p}$, $p \in \mathbb{N}$, $\alpha > 0$ or, equivalently, the Gevrey sequences $M_p = p!^\alpha$, $p \in \mathbb{N}$, $\alpha > 0$.

We follow here Komatsu's approach [31] to spaces of ultra-differentiable functions to extend the original definitions as follows.

Let $(M_p)_{p\in\mathbb{N}_0}$ be a sequence of positive numbers monotonically increasing to infinity which satisfies:
$(M.1)$ $M_p^2 \leq M_{p-1} M_{p+1}, \quad p \in \mathbb{N}$;
$(M.2)$ There exist positive constants A, H such that

$$M_p \leq A H^p \min{}_{0\leq q\leq p} M_{p-q} M_q, \quad p, q \in \mathbb{N}_0,$$

or, equivalently, there exist positive constants A, H such that

$$M_{p+q} \leq A H^{p+q} M_p M_q, \quad p, q \in \mathbb{N}_0;$$

We assume $M_0 = 1$, and that $M_p^{1/p}$ is bounded below by a positive constant.

Remark 2.1. *To give an example, we describe (M.1) and (M.2) as follows. Let $(s_p)_{p\in\mathbb{N}_0}$ be a sequence of positive numbers monotonically increasing to infinity ($s_p \nearrow \infty$) so that for every $p, q \in \mathbb{N}_0$ there exist $A, H > 0$ such that*

$$s_{p+1} \cdots s_{p+q} \leq A H^p s_1 \cdots s_q. \tag{7}$$

Then the sequence $(S_p)_{p\in\mathbb{N}_0}$ given by $S_p = s_1 \cdots s_p$, $S_0 = 1$, satisfy conditions $(M.1)$ and $(M.2)$.

Conversely, if $(S_p)_{p\in\mathbb{N}_0}$ where $S_p = s_1 \cdots s_p$, $S_0 = 1$, satisfies $(M.1)$ then $(s_p)_{p\in\mathbb{N}_0}$ increases to infinity, and if it satisfies $(M.2)$ then (7) holds.

Let $(M_p)_{p\in\mathbb{N}_0}$ and $(N_q)_{q\in\mathbb{N}_0}$ be sequences which satisfy $(M.1)$. We write $M_p \subset N_q$ ($(M_p) \prec (N_q)$, respectively) if there are constants $H, C > 0$ (for any $H > 0$ there is a constant $C > 0$, respectively) such that $M_p \leq CH^p N_p$, $p \in \mathbb{N}_0$. Also, $(M_p)_{p\in\mathbb{N}_0}$ and $(N_q)_{q\in\mathbb{N}_0}$ are said to be equivalent if $M_p \subset N_q$ and $N_q \subset M_p$ hold.

Definition 2.2. *Let there be given sequences of positive numbers $(M_p)_{p\in\mathbb{N}_0}$ and $(N_q)_{q\in\mathbb{N}_0}$ which satisfy $(M.1)$ and $(M.2)$. Let $\mathcal{S}^{N_q,B}_{M_p,A}(\mathbb{R}^d)$ be defined by*

$$\mathcal{S}^{N_q,B}_{M_p,A}(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d) \mid \|x^\alpha \partial^\beta f\|_{L^\infty} \leq CA^\alpha M_{|\alpha|} B^\beta N_{|\beta|}, \;\; \forall \alpha, \beta \in \mathbb{N}^d_0\},$$

for some positive constant C, where $A = (A_1, \dots, A_d)$, $B = (B_1, \dots, B_d)$, $A, B > 0$.

Gelfand-Shilov spaces $\Sigma^{N_q}_{M_p}(\mathbb{R}^d)$ and $\mathcal{S}^{N_q}_{M_p}(\mathbb{R}^d)$ are projective and inductive limits of the spaces $\mathcal{S}^{N_q,B}_{M_p,A}(\mathbb{R}^d)$ with respect to A and B:

$$\Sigma^{N_q}_{M_p}(\mathbb{R}^d) := \text{proj} \lim_{A>0,B>0} \mathcal{S}^{N_q,B}_{M_p,A}(\mathbb{R}^d); \quad \mathcal{S}^{N_q}_{M_p}(\mathbb{R}^d) := \text{ind} \lim_{A>0,B>0} \mathcal{S}^{N_q,B}_{M_p,A}(\mathbb{R}^d).$$

The corresponding dual spaces of $\Sigma^{N_q}_{M_p}(\mathbb{R}^d)$ and $\mathcal{S}^{N_q}_{M_p}(\mathbb{R}^d)$ are the spaces of ultradistributions of Beurling and Roumier type respectively:

$$(\Sigma^{N_q}_{M_p})'(\mathbb{R}^d) := \text{ind} \lim_{A>0,B>0} (\mathcal{S}^{N_q,B}_{M_p,A})'(\mathbb{R}^d); \quad (\mathcal{S}^{N_q}_{M_p})'(\mathbb{R}^d) := \text{proj} \lim_{A>0,B>0} (\mathcal{S}^{N_q,B}_{M_p,A})'(\mathbb{R}^d).$$

Of course, for certain choices of the sequences $(M_p)_{p\in\mathbb{N}_0}$ and $(N_q)_{q\in\mathbb{N}_0}$ the spaces $\Sigma^{N_q}_{M_p}(\mathbb{R}^d)$ and $\mathcal{S}^{N_q}_{M_p}(\mathbb{R}^d)$ are trivial, i.e. they contain only the function $\phi \equiv 0$. Nontrivial Gelfand-Shilov spaces are closed under translation, dilation, multiplication with $x \in \mathbb{R}^d$, and differentiation. Moreover, they are closed under the action of certain differential operators of infinite order (ultradifferentiable operators in the terminology of Komatsu).

By the definition, the spaces $\mathcal{S}^{N_q,B}_{M_p,A}(\mathbb{R}^d)$ are Fréchet spaces. Therefore $\Sigma^{N_q}_{M_p}(\mathbb{R}^d)$ is a Fréchet space, and an (FS)-space as well. The space $\mathcal{S}^{N_q}_{M_p}(\mathbb{R}^d)$ is a (DFS)-space. We refer to [59] for the definition and basic properties of (FS)-spaces and (DFS)-spaces, and to [23] for the proofs of basic properties of Gelfand-Shilov spaces.

In particular, if $(M_p)_{p\in\mathbb{N}_0}$ and $(N_q)_{q\in\mathbb{N}_0}$ are Gevrey sequences: $M_p = p!^r$, $p \in \mathbb{N}_0$ and $N_q = q!^s$, $q \in \mathbb{N}_0$, for some $r, s \geq 0$, then we use the notation $\mathcal{S}^{N_q}_{M_p}(\mathbb{R}^d) = \mathcal{S}^s_r(\mathbb{R}^d)$ and $\Sigma^{N_q}_{M_p}(\mathbb{R}^d) = \Sigma^s_r(\mathbb{R}^d)$.

The choice of Gevrey sequences (which is the most often used choice in the literature) may serve well as an illuminating example in different contexts. In particular, when discussing the nontriviality we have the following:

a) the space $\mathcal{S}^s_r(\mathbb{R}^d)$ is nontrivial if and only if $s + r > 1$, or $s + r = 1$ and $sr > 0$,

b) if $s + r \geq 1$ and $s < 1$, then every $f \in \mathcal{S}^s_r(\mathbb{R}^d)$ can be extended to the complex domain as an entire function,

c) if $s + r \geq 1$ and $s = 1$, then every $f \in \mathcal{S}^s_r(\mathbb{R}^d)$ can be extended to the complex domain as a holomorphic function in a strip.

d) the space $\Sigma^s_r(\mathbb{R}^d)$ is nontrivial if and only if $s + r > 1$, or, if $s + r = 1$ and $sr > 0$ and $(s, r) \neq (1/2, 1/2)$.

We refer to [23] or [36] for the proof in the case of $\mathcal{S}^s_r(\mathbb{R}^d)$, and to [38] for the spaces $\Sigma^s_r(\mathbb{R}^d)$, see also [56].

The discussion here above shows that Gelfand-Shilov classes $\mathcal{S}^s_r(\mathbb{R}^d)$ consist of quasi-analytic functions when $s \in (0, 1)$. This is in sharp contrast with e.g. Gevrey classes $G^s(\mathbf{R}^d)$, $s > 1$, another family of functions commonly used in regularity theory of partial differential equations, whose elements are always non-quasi-analytic. We refer to [43] for microlocal analysis in Gervey classes and note that $G^s_0(\mathbf{R}^d) \hookrightarrow S^s_s(\mathbf{R}^d) \hookrightarrow G^s(\mathbf{R}^d)$, $s > 1$.

When the spaces are nontrivial we have dense and continuous inclusions:

$$\Sigma_r^s(\mathbb{R}^d) \hookrightarrow S_r^s(\mathbb{R}^d) \hookrightarrow \mathcal{S}(\mathbb{R}^d).$$

In Section 5 we will use the following spaces of ultradistributions. We say that $f \in (\Sigma_{M_p}^{N_q})'(\mathbb{R}^d)$ ($f \in (\mathcal{S}_{M_p}^{N_q})'(\mathbb{R}^d)$, respectively) *can be extended* on $\mathcal{E}^{(N_q)}(\Omega)$ (on $\mathcal{E}^{\{N_q\}}(\Omega)$, respectively) if $f \in (\mathcal{E}^{(N_q)})'(\Omega)$ ($f \in (\mathcal{E}^{\{N_q\}})'(\Omega)$, respectively) and

$$_{(\Sigma_{M_p}^{N_q})'(\mathbb{R}^d)}\langle f, \phi\rangle_{\Sigma_{M_p}^{N_q}(\mathbb{R}^d)} = _{(\mathcal{E}^{(N_q)})'(\Omega)}\langle f, \phi\big|_\Omega\rangle_{\mathcal{E}^{(N_q)}(\Omega)}$$

$$\Big(_{(\mathcal{S}_{M_p}^{N_q})'(\mathbb{R}^d)}\langle f, \phi\rangle_{\mathcal{S}_{M_p}^{N_q}(\mathbb{R}^d)} = _{(\mathcal{E}^{\{N_q\}})'(\Omega)}\langle f, \phi\big|_\Omega\rangle_{\mathcal{E}^{\{N_q\}}(\Omega)}, \quad \text{respectively}\Big).$$

Note, if $(\phi_n)_{n\in\mathbb{N}}$ is a sequence in $\Sigma_{M_p}^{N_q}(\mathbb{R}^d)$ (in $\mathcal{S}_{M_p}^{N_q}(\mathbb{R}^d)$, respectively) and if $\phi_n \to \phi$ in $\Sigma_{M_p}^{N_q}(\mathbb{R}^d)$ (in $\mathcal{S}_{M_p}^{N_q}(\mathbb{R}^d)$, respectively) then $\phi_n\big|_\Omega \to \phi\big|_\Omega$ in $\mathcal{E}^{(N_q)}(\Omega)$ (in $\mathcal{E}^{\{N_q\}}(\Omega)$, respectively).

2.2. Equivalent conditions

In this subsection we give a well known equivalent characterization of Gelfand-Shilov spaces.

A starting point is the behavior of Gelfand-Shilov spaces under the action of the Fourier transform. Already in [23] it is shown that the Fourier transform is a topological isomorphism between $S_r^s(\mathbb{R}^d)$ and $S_s^r(\mathbb{R}^d)$ ($\mathcal{F}(S_r^s) = S_s^r$), which extends to a continuous linear transform from $(S_r^s)'(\mathbb{R}^d)$ onto $(S_s^r)'(\mathbb{R}^d)$. In particular, if $s = r$ and $s \geq 1/2$ then $\mathcal{F}(S_s^s)(\mathbb{R}^d) = S_s^s(\mathbb{R}^d)$. Similar assertions hold for $\Sigma_r^s(\mathbb{R}^d)$.

This invariance properties easily follow from the following theorem which also enlightens fundamental properties of Gelfand-Shilov spaces implicitly contained in their definition. Among other things, it states that the decay and regularity estimates of $f \in \mathcal{S}_{M_p}^{N_q}(\mathbb{R}^d)$ can be studied separately.

Theorem 2.3. *Let there be given sequences of positive numbers $(M_p)_{p\in\mathbb{N}_0}$ and $(N_q)_{q\in\mathbb{N}_0}$ which satisfy (M.1) and (M.2) and $p! \subset M_pN_p$ ($p! \prec M_pN_p$, respectively). Then the following conditions are equivalent:*

a) $f \in \mathcal{S}_{M_p}^{N_q}(\mathbb{R}^d)$ ($f \in \Sigma_{M_p}^{N_q}(\mathbb{R}^d)$, *respectively).*

b) There exist constants $A, B \in \mathbb{R}^d$, $A, B > 0$ (for every $A, B \in \mathbb{R}^d$, $A, B > 0$, respectively) and there exist $C > 0$ such that

$$\|x^p f\|_{L^\infty} \leq CA^p M_{|p|} \quad \textit{and} \quad \|\partial^q f\|_{L^\infty} \leq CB^q N_{|q|}, \quad \forall p, q \in \mathbb{N}_0^d.$$

c) There exist constants $A, B \in \mathbb{R}^d$, $A, B > 0$ (for every $A, B \in \mathbb{R}^d$, $A, B > 0$, respectively) and there exist $C > 0$ such that

$$\|x^p f\|_{L^\infty} \leq CA^p M_{|p|} \quad \textit{and} \quad \|\omega^q \hat{f}\|_{L^\infty} \leq CB^q N_{|q|}, \quad \forall p, q \in \mathbb{N}_0^d.$$

d) There exist constants $A, B \in \mathbb{R}^d$, $A, B > 0$ (for every $A, B \in \mathbb{R}^d$, $A, B > 0$, respectively) such that

$$\|f(x)\, exp(M(|Ax|))\|_{L^\infty} < \infty \quad \textit{and} \quad \|\hat{f}(\omega)\, exp(N(|B\omega|))\|_{L^\infty} < \infty,$$

where $M(\cdot)$ and $N(\cdot)$ are the associated functions for the sequences $(M_p)_{p\in\mathbb{N}_0}$ and $(N_q)_{q\in\mathbb{N}_0}$ respectively.

The *associated function* of (M_p) is defined by

$$M(\rho) = \sup_{p\in\mathbb{N}_0} \ln \frac{\rho^p M_0}{M_p}, \quad 0 < \rho < \infty.$$

For example, the associated function for the Gevrey sequence $M_p = p!^r$, $p \in \mathbb{N}_0$ behaves at infinity as $|\cdot|^{1/r}$, cf. [37].

Theorem 2.3 is for the first time proved in [9] and reinvented many times afterwards, see e.g. [15, 28, 30, 36, 39]. The proof is therefore omitted.

By the above characterization $\mathcal{F}\mathcal{S}^{N_q}_{M_p}(\mathbb{R}^d) = \mathcal{S}^{M_p}_{N_q}(\mathbb{R}^d)$. Observe that $\mathcal{S}^{1/2}_{1/2}(\mathbb{R}^d)$ is the smallest non-empty Gelfand-Shilov space invariant under the Fourier transform. Theorem 2.3 implies that $f \in \mathcal{S}^{1/2}_{1/2}(\mathbb{R}^d)$ if and only if $f \in C^\infty(\mathbb{R}^d)$ and there exist constants $h > 0, k > 0$ such that

$$\|f exp(h|\cdot|^2)\|_{L^\infty} < \infty \quad \text{and} \quad \|\hat{f} exp(k|\cdot|^2)\|_{L^\infty} < \infty. \tag{8}$$

(There is a misprint in (2.3) in [15].) Therefore the Hermite functions given by (5) belong to $\mathcal{S}^{1/2}_{1/2}(\mathbb{R}^d)$. This is an important fact when dealing with Gelfand-Shilov spaces, cf. [32, 38].

Note that $\Sigma^{1/2}_{1/2}(\mathbb{R}^d) = \{0\}$ and $\Sigma^s_s(\mathbb{R}^d)$ is dense in the Schwartz space whenever $s > 1/2$. We are also interested in "fine tuning", that is in spaces $\Sigma^{N_q}_{M_p}(\mathbb{R}^d)$ such that

$$\Sigma^{1/2}_{1/2}(\mathbb{R}^d) \hookrightarrow \Sigma^{N_q}_{M_p}(\mathbb{R}^d) \hookrightarrow \mathcal{S}^{N_q}_{M_p}(\mathbb{R}^d) \hookrightarrow \Sigma^s_s(\mathbb{R}^d), \quad s > 1/2.$$

For that reason, we define sequences $(M_p)_{p\in\mathbb{N}_0}$ and $(N_q)_{q\in\mathbb{N}_0}$ by

$$M_p := p!^{\frac{1}{2}} \prod_{k=0}^{p} l_k = p!^{\frac{1}{2}} L_p, \quad p \in \mathbb{N}_0, \quad N_q := q!^{\frac{1}{2}} \prod_{k=0}^{q} r_k = q!^{\frac{1}{2}} R_q, \quad q \in \mathbb{N}_0 \tag{9}$$

where $(r_p)_{p\in\mathbb{N}_0}$ and $(l_p)_{p\in\mathbb{N}_0}$ are sequences of positive numbers monotonically increasing to infinity such that (7) holds with the letter s replaced by r and l respectively and which satisfy: For every $\alpha \in (0, 1]$ and every $k > 1$ so that $kp \in \mathbb{N}, p \in \mathbb{N}$,

$$\max\{(\frac{r_{kp}}{r_p})^2, (\frac{l_{kp}}{l_p})^2\} \leq k^\alpha, \quad p \in \mathbb{N}. \tag{10}$$

Then $p! < M_p N_p$ and the sequences $(R_p)_{p\in\mathbb{N}_0}$ and $(L_p)_{p\in\mathbb{N}_0}$ ($R_p = r_1 \cdots r_p$, $L_p = l_1 \cdots l_p, p \in \mathbb{N}$ $R_0 = 1$, and $L_0 = 1$) satisfy conditions $(M.1)$ and $(M.2)$. Moreover,

$$\max\{R_p, L_p\} \leq p!^{\alpha/2}, p \in \mathbb{N},$$

for every $\alpha \in (0, 1]$. (For $p, q, k \in \mathbb{N}^d_0$ we have $L_{|p|} = \prod_{|k|\leq|p|} l_{|k|}$, and $R_{|q|} = \prod_{|k|\leq|q|} r_{|q|}$.) We will use sequences which satisfy (9) and (10) in Section 5.

2.3. Kernel theorem

Next we prove a kernel theorem which will be used in the sequel. It is an extension of the famous Schwartz kernel theorem (see [48, 59]) to the spaces of ultradistributions. We follow the proof given in [40] in the case of non-quasianalytic Gelfand-Shilov spaces. The only difference is that the density of $\mathcal{D}(\mathbb{R}^d)$ can not be used here. Instead we use arguments based on Hermite expansions in Gelfand-Shilov spaces, see [32, 35].

We introduce additional conditions for a sequence of positive numbers $(M_p)_{p\in\mathbb{N}_0}$:

$\{N.1\}$ There exist positive constants A, H such that

$$p!^{1/2} \leq AH^p M_p, \quad p \in \mathbb{N}_0,$$

and

$(N.1)$ For every $H > 0$ there exists $A > 0$ such that

$$p!^{1/2} \leq AH^p M_p, \quad p \in \mathbb{N}_0.$$

The conditions $\{N.1\}$ and $(N.1)$ are taken from [34] where they are called *nontriviality conditions* for the spaces $\mathcal{S}^{M_p}_{M_p}(\mathbb{R}^d)$ and $\Sigma^{M_p}_{M_p}(\mathbb{R}^d)$ respectively. In fact, the following lemma is proved in [32].

Lemma 2.4. *Let there be given a sequence of positive numbers $(M_p)_{p\in\mathbb{N}_0}$ which satisfies $(M.1)$ and*
$(M.2)'$ *There exist positive constants A, H such that $M_{p+1} \le AH^p M_p$, $p \in \mathbb{N}_0$.*
Then the following are equivalent:

a) The Hermite functions are contained in $\mathcal{S}^{N_q}_{M_p}(\mathbb{R}^d)$ (in $\Sigma^{N_q}_{M_p}(\mathbb{R}^d)$, respectively).

b) $(M_p)_{p\in\mathbb{N}_0}$ satisfies $\{N.1\}$ ($(M_p)_{p\in\mathbb{N}_0}$ satisfies $(N.1)$, respectively).

c) There are positive constants A, B and H such that

$$p!^{1/2} M_q \le AB^{p+q} H^p M_{p+q}, \quad p, q \in \mathbb{N}_0.$$

(There is $B > 0$ such that for every $H > 0$ there exists $A > 0$ such that

$$p!^{1/2} M_q \le AB^{p+q} H^p M_{p+q}, \quad p, q \in \mathbb{N}_0.$$

We note that the condition $(M.2)'$ is weaker than the condition $(M.2)$, and refer to [32, Remark 3.3] for the proof of Lemma 2.4.

Theorem 2.5. *Let there be given a sequence of positive numbers $(M_p)_{p\in\mathbb{N}_0}$ which satisfies $(M.1)$, $(M.2)$ and $\{N.1\}$. Then the following isomorphisms hold:*

a) $\mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_1}) \hat{\otimes} \mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_2}) \cong \mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_1+d_2}) \cong \mathcal{L}_b((\mathcal{S}^{M_p}_{M_p})'(\mathbb{R}^{d_1}), \mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_2}))$,

b) $(\mathcal{S}^{M_p}_{M_p})'(\mathbb{R}^{d_1}) \hat{\otimes} (\mathcal{S}^{M_p}_{M_p})'(\mathbb{R}^{d_2}) \cong (\mathcal{S}^{M_p}_{M_p})'(\mathbb{R}^{d_1+d_2}) \cong \mathcal{L}_b(\mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_1}), (\mathcal{S}^{M_p}_{M_p})'(\mathbb{R}^{d_2}))$.

If the sequence $(M_p)_{p\in\mathbb{N}_0}$ satisfies $(M.1)$, $(M.2)$ and $(N.1)$ instead, then the following isomorphisms hold:

c) $\Sigma^{M_p}_{M_p}(\mathbb{R}^{d_1}) \hat{\otimes} \Sigma^{M_p}_{M_p}(\mathbb{R}^{d_2}) \cong \Sigma^{M_p}_{M_p}(\mathbb{R}^{d_1+d_2}) \cong \mathcal{L}_b((\Sigma^{M_p}_{M_p})'(\mathbb{R}^{d_1}), \Sigma^{M_p}_{M_p}(\mathbb{R}^{d_2}))$,

d) $(\Sigma^{M_p}_{M_p})'(\mathbb{R}^{d_1}) \hat{\otimes} (\Sigma^{M_p}_{M_p})'(\mathbb{R}^{d_2}) \cong (\Sigma^{M_p}_{M_p})'(\mathbb{R}^{d_1+d_2}) \cong \mathcal{L}_b(\Sigma^{M_p}_{M_p}(\mathbb{R}^{d_1}), (\Sigma^{M_p}_{M_p})'(\mathbb{R}^{d_2}))$,

Proof. Let $(M_p)_{p\in\mathbb{N}_0}$ satisfy $(M.1)$, $(M.2)$ and $\{N.1\}$.

By [32, Remark 3.3] it follows that $\{N.1\}$ is equivalent to $H_k(x) \in \mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_1})$, $x \in \mathbb{R}^{d_1}$, $k \in \mathbb{N}^{d_1}_0$, and $H_l(y) \in \mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_2})$, $y \in \mathbb{R}^{d_2}$, $l \in \mathbb{N}^{d_2}_0$, where $H_k(x)$ and $H_l(y)$ are the Hermite functions given by (5). Now, by representation theorems from [32] and [34] and the fact that $H_{(k,l)}(x, y) \in \mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_1+d_2})$, $(x, y) \in \mathbb{R}^{d_1+d_2}$, $(k, l) \in \mathbb{N}^{d_1+d_2}_0$, it follows that $\mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_1}) \otimes \mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_2})$ is dense in $\mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_1+d_2})$.

By the nuclearity of $\mathcal{S}^{M_p}_{M_p}(\mathbb{R}^d)$ it follows that the topologies $\mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_1}) \otimes_\pi \mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_2})$ and $\mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_1}) \otimes_\epsilon \mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_2})$ coincide. We refer to [59, Chapter 43] for the definition and basic facts on the π and ϵ topologies.

For the isomorphism $\mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_1}) \hat{\otimes} \mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_2}) \cong \mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_1+d_2})$ to hold, it therefore remains to prove that $\mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_1+d_2})$ induces the $\pi = \epsilon$ topology on $\mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_1}) \hat{\otimes} \mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_2})$.

Consider now the mapping $B : \mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_1}) \times \mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_2}) \to \mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_1+d_2})$ given by $B : (\phi, \varphi) \mapsto \phi \otimes \varphi$. This is a separately continuous bilinear mapping. Now, since $\mathcal{S}^{M_p}_{M_p}(\mathbb{R}^d)$ is a (DFS)-space (and therefore a barreled (DF)-space) it follows that B is (jointly) continuous. This implies the continuity of the inclusion $\mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_1}) \otimes_\pi \mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_2}) \to \mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_1+d_2})$, wherefrom it follows that the topology π is stronger than the induced one.

Next, for a given equicontinuous subsets $A' \subset \mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_1})$ and $B' \subset \mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_2})$ we estimate $|\langle F_x \otimes \tilde{F}_y, \Phi(x, y)\rangle|$, $F_x \in A'$ and $\tilde{F}_y \in B'$. Here it is convenient to use a particular family of norms which defines a topology equivalent to the one given by Definition 2.2. In fact, it can be shown (see [8]) that $\phi \in \mathcal{S}^{M_p}_{M_p}(\mathbb{R}^d)$ if and only if

$$\|\phi\|_{(k_p),(\tilde{k}_p)} := \sup_{x\in\mathbb{R}^d,\ \alpha\in\mathbb{N}_0^d} \frac{|\partial^\alpha \phi(x)|\ exp(N_{\tilde{k}_p}(|x|))}{M_{|\alpha|}\prod_{p=1}^{|\alpha|} k_p} < \infty.$$

where (k_p), and $(\tilde{k}_p)$ are sequences of positive numbers monotonically increasing to infinity, and, by a slight abuse of notation, $N_{\tilde{k}_p}(|x|)$ denotes the associated function of the sequence $(M_p \prod_{j=1}^{p} \tilde{k}_p)$, $p \in \mathbb{N}_0$, i.e.

$$N_{\tilde{k}_p}(\rho) = \sup_{p\in\mathbb{N}_0} \ln \frac{\rho^p M_0}{M_p \prod_{j=1}^{p} \tilde{k}_p}, \quad 0 < \rho < \infty.$$

In fact, it is known that several other families of norms define on $\mathcal{S}^{M_p}_{M_p}(\mathbb{R}^d)$ topologies equivalent to the one given above. See, for example [8, Chapter 2] for details and the proof. Moreover, by [41, Lemma 2.3] we may assume that

$$\prod_{j=1}^{p+q} k_j \le 2^{p+q} \prod_{j=1}^{p} k_j \prod_{j=1}^{q} k_j, \quad \text{and} \quad \prod_{j=1}^{p+q} \tilde{k}_j \le 2^{p+q} \prod_{j=1}^{p} \tilde{k}_j \prod_{j=1}^{q} \tilde{k}_j, \quad p, q \in \mathbb{N}_0. \tag{11}$$

Therefore we know that there exist sequences (k_p), and $(\tilde{k}_p)$ of positive numbers monotonically increasing to infinity, satisfying (11) such that

$$\sup_{F\in A'} |\langle F, \phi\rangle| \le C\|\phi\|_{(k_p),(\tilde{k}_p)}, \quad \phi \in \mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_1}) \quad \text{and} \quad \sup_{\tilde{F}\in B'} |\langle \tilde{F}, \phi\rangle| \le C\|\varphi\|_{(k_p),(\tilde{k}_p)}, \quad \varphi \in \mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_2}).$$

Now we have

$$\begin{aligned} |\langle F_x \otimes \tilde{F}_y, \Phi(x,y)\rangle| &= |\langle F_x, \langle \tilde{F}_y, \Phi(x,y)\rangle\rangle| \le C \sup_{y,\beta} \frac{|\langle F_x, \partial_y^\beta \Phi(x,y)\rangle| exp(N_{\tilde{k}_p}(|y|))}{M_{|\beta|}\prod_{j=1}^{|\beta|} k_j} \\ &\le C_1 \sup_{x,y,\alpha,\beta} \frac{|\partial_x^\alpha \partial_y^\beta \Phi(x,y)\rangle| exp(N_{\tilde{k}_p}(|x|)) exp(N_{\tilde{k}_p}(|y|))}{M_{|\alpha|}M_{|\beta|}\prod_{j=1}^{|\alpha|} k_j \prod_{j=1}^{|\beta|} k_j} \\ &\le C_2 \sup_{x,y,\alpha,\beta} \frac{|\partial_x^\alpha \partial_y^\beta \Phi(x,y)\rangle| exp(N_{\tilde{r}_p}(|(x,y)|))}{M_{|\alpha|+|\beta|}\prod_{j=1}^{|\alpha|+|\beta|} r_j} = C_3\|\Phi\|_{(k_p),(\tilde{k}_p)}, \quad \Phi \in \mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_1+d_2}), \end{aligned}$$

where C, C_1, C_2 and C_3 are positive constants independent on x, y, α, β, $r_p = k_p/(2H)$, $\tilde{r}_p = \tilde{k}_p/(2H)$, $p \in \mathbb{N}_0$, with H determined by $(M.2)$, and we have used [31, Proposition 2.6] or [41, Lemma 2.4] which implies that $N_{\tilde{k}_p}(|x|)) + N_{\tilde{k}_p}(|y|) \le CN_{\tilde{r}_p}(|(x,y)|)$ for some $C > 0$.

Therefore the ϵ topology on $\mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_1}) \otimes \mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_2})$ is weaker that the induced one from $\mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_1+d_2})$. This gives $\mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_1}) \hat{\otimes} \mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_2}) \cong \mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_1+d_2})$.

Now, since $(\mathcal{S}^{M_p}_{M_p})'(\mathbb{R}^{d_1})$ and $\mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_2})$ are complete, and since $(\mathcal{S}^{M_p}_{M_p})'(\mathbb{R}^{d_1})$ is barreled and $\mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_1})$ is nuclear and complete, then by [59, Proposition 50.5] it follows that $\mathcal{L}_b((\mathcal{S}^{M_p}_{M_p})'(\mathbb{R}^{d_1}), \mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_2}))$ is complete and

$$\mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_1}) \hat{\otimes} \mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_2}) \cong \mathcal{L}_b((\mathcal{S}^{M_p}_{M_p})'(\mathbb{R}^{d_1}), \mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_2})).$$

This proves a) and we leave the other claims to the reader, see also [40]. □

The isomorphisms in Theorem 2.5 b) tells us that for a given kernel-distribution $k(x,y)$ on $\mathbb{R}^{d_1+d_2}$ we may associate a continuous linear mapping k of $\mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_2})$ into $(\mathcal{S}^{M_p}_{M_p})'(\mathbb{R}^{d_1})$ as follows:

$$\langle k_\varphi, \phi\rangle = \langle k(x,y), \phi(x)\varphi(y)\rangle, \quad \phi \in \mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{d_1}),$$

which is commonly written as $k_\varphi(\cdot) = \int k(\cdot, y)\varphi(y)dy$. By Theorem 2.5 b) it follows that the correspondence between $k(x,y)$ and k is an isomorphism. Note also that the transpose ${}^tk$ of the mapping k is given by ${}^tk_\phi(\cdot) = \displaystyle\int k(x,\cdot)\phi(x)dx$.

By the above isomorphisms we conclude that for any continuous and linear mapping between $\mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{2d})$ to $(\mathcal{S}^{M_p}_{M_p})'(\mathbb{R}^{2d})$ one can assign a uniquely determined kernel with the above mentioned properties. We will use this fact in the proof of Theorem 5.2, and refer to [59, Chapter 52] for applications of kernel theorems in linear partial differential equations.

Remark 2.6. *The choice of the Fourier transform invariant spaces of the form $\mathcal{S}^{M_p}_{M_p}(\mathbb{R}^d)$ in Theorem 2.5 is not accidental. We refer to [24] where it is proved that if the Hermite expansion $\sum_{k\in\mathbb{N}^d} a_k H_k(x)$ converges to f (a_k are the Hermite coefficients of f) in the sense of $\mathcal{S}^s_r(\mathbb{R}^d)$ ($\Sigma^s_r(\mathbb{R}^d)$, respectively), $r < s$, then it belongs to $\mathcal{S}^r_r(\mathbb{R}^d)$ ($\Sigma^r_r(\mathbb{R}^d)$, respectively).*

3. Time-frequency analysis and Gelfand-Shilov spaces

In this section we extend the action of the short-time Fourier transform to dual spaces of Gelfand-Shilov spaces. To that end we observe the following version of Definition 2.2.

Definition 3.1. *Let there be given sequences of positive numbers $(M_p)_{p\in\mathbb{N}_0}$, $(N_q)_{q\in\mathbb{N}_0}$, $(\tilde M_p)_{p\in\mathbb{N}_0}$, $(\tilde N_q)_{q\in\mathbb{N}_0}$ which satisfy (M.1) and (M.2). We define $\mathcal{S}^{N_q,\tilde N_q,B}_{M_p,\tilde M_p,A}(\mathbb{R}^{2d})$ by*

$$\mathcal{S}^{N_q,\tilde N_q,B}_{M_p,\tilde M_p,A}(\mathbb{R}^{2d}) = \{f \in C^\infty(\mathbb{R}^{2d}) \mid \|x^{\alpha_1}\omega^{\alpha_2}\partial_x^{\beta_1}\partial_\omega^{\beta_2} f\|_{L^\infty} \le CA^{|\alpha_1+\alpha_2|} M_{|\alpha_1|}\tilde M_{|\alpha_2|} B^{|\beta_1+\beta_2|} N_{|\beta_1|}\tilde N_{|\beta_2|}, \quad \forall \alpha_1,\alpha_2,\beta_1,\beta_2 \in \mathbb{N}_0^d\},$$

for some $A,B,C > 0$. Gelfand-Shilov spaces *are projective and inductive limits of $\mathcal{S}^{N_q,\tilde N_q,B}_{M_p,\tilde M_p,A}(\mathbb{R}^{2d})$:*

$$\Sigma^{N_q,\tilde N_q}_{M_p,\tilde M_p}(\mathbb{R}^{2d}) := \text{proj} \lim_{A>0,B>0} \mathcal{S}^{N_q,\tilde N_q,B}_{M_p,\tilde M_p,A}(\mathbb{R}^{2d}); \quad \mathcal{S}^{N_q,\tilde N_q}_{M_p,\tilde M_p}(\mathbb{R}^{2d}) := \text{ind} \lim_{A>0,B>0} \mathcal{S}^{N_q,\tilde N_q,B}_{M_p,\tilde M_p,A}(\mathbb{R}^{2d}).$$

Clearly, the corresponding dual spaces are given by

$$(\Sigma^{N_q,\tilde N_q}_{M_p,\tilde M_p})'(\mathbb{R}^{2d}) := \text{ind} \lim_{A>0,B>0} (\mathcal{S}^{N_q,\tilde N_q,B}_{M_p,\tilde M_p,A})'(\mathbb{R}^{2d}); \quad (\mathcal{S}^{N_q,\tilde N_q}_{M_p,\tilde M_p})'(\mathbb{R}^{2d}) := \text{proj} \lim_{A>0,B>0} (\mathcal{S}^{N_q,\tilde N_q,B}_{M_p,\tilde M_p,A})'(\mathbb{R}^{2d}).$$

We note that, by Theorem 2.3, the Fourier transform is a homeomorphism from $\Sigma^{N_q,\tilde N_q}_{M_p,\tilde M_p}(\mathbb{R}^{2d})$ to $\Sigma^{M_p,\tilde M_p}_{N_q,\tilde N_q}(\mathbb{R}^{2d})$ and, if $\mathcal{F}_1 f$ denotes the partial Fourier transform of $f(x,\omega)$ with respect to the x variable, and if $\mathcal{F}_2 f$ denotes the partial Fourier transform of $f(x,\omega)$ with respect to the ω variable, then $\mathcal{F}_1$ and $\mathcal{F}_2$ are homeomorphisms from $\Sigma^{N_q,\tilde N_q}_{M_p,\tilde M_p}(\mathbb{R}^{2d})$ to $\Sigma^{N_q,\tilde M_p}_{M_p,\tilde N_q}(\mathbb{R}^{2d})$ and $\Sigma^{N_q,\tilde M_p}_{M_p,\tilde N_q}(\mathbb{R}^{2d})$, respectively. Similar facts hold when $\Sigma^{N_q,\tilde N_q}_{M_p,\tilde M_p}(\mathbb{R}^{2d})$ is replaced by $\mathcal{S}^{N_q,\tilde N_q}_{M_p,\tilde M_p}(\mathbb{R}^{2d})$, $(\Sigma^{N_q,\tilde N_q}_{M_p,\tilde M_p})'(\mathbb{R}^{2d})$ or $(\mathcal{S}^{N_q,\tilde N_q}_{M_p,\tilde M_p})'(\mathbb{R}^{2d})$.

When $M_p = \tilde M_p$ and $N_q = \tilde N_q$ we use usual abbreviated notation: $\mathcal{S}^{N_q}_{M_p}(\mathbb{R}^{2d}) = \mathcal{S}^{N_q,\tilde N_q}_{M_p,\tilde M_p}(\mathbb{R}^{2d})$ and similarly for other spaces.

Let $(M_p)_{p\in\mathbb{N}_0}$ satisfy $(M.1)$, $(M.2)$ and $\{N.1\}$ ($(N.1)$, respectively). For a fixed non-zero $g \in \mathcal{S}^{M_p}_{M_p}(\mathbb{R}^d)$ ($g \in \Sigma^{M_p}_{M_p}(\mathbb{R}^d)$, respectively) the short-time Fourier transform (STFT) of $f \in \mathcal{S}^{M_p}_{M_p}(\mathbb{R}^d)$ ($f \in \Sigma^{M_p}_{M_p}(\mathbb{R}^d)$, respectively) with respect to the window g is given by (1), i.e.

$$V_g f(x,\omega) = \langle f, M_\omega T_x g\rangle = \int_{\mathbb{R}^d} f(t)\,\overline{g(t-x)}\,e^{-2\pi i\omega t}\,dt, \quad x,\omega \in \mathbb{R}^d,$$

and the definition can be extended to $f \in (\mathcal{S}^{M_p}_{M_p})'(\mathbb{R}^d)$ ($f \in (\Sigma^{M_p}_{M_p})'(\mathbb{R}^d)$, respectively) by duality.

Similarly, another time-frequency representation, the *cross-Wigner distribution* $W(f,g)$, defined by (6) can be extended to $f \in (\mathcal{S}^{M_p}_{M_p})'(\mathbb{R}^d)$ ($f \in (\Sigma^{M_p}_{M_p})'(\mathbb{R}^d)$, respectively) when $g \in \mathcal{S}^{M_p}_{M_p}(\mathbb{R}^d)$ ($g \in \Sigma^{M_p}_{M_p}(\mathbb{R}^d)$, respectively). In fact, a straightforward calculation gives

$$W(f,g)(x,\omega) = 2^d e^{4\pi i x\omega} V_{\check{g}} f(2x, 2\omega), \quad x,\omega \in \mathbb{R}^d,$$

see e.g. [25, Lemma 4.3.1]. Since Gelfand-Shilov spaces $\mathcal{S}^{M_p}_{M_p}(\mathbb{R}^{2d})$ are closed under dilations and modulations, it is enough to prove the following theorem for one of the time-frequency representations.

Theorem 3.2. *Let there be given sequences $(M_p)_{p\in\mathbb{N}_0}$ and $(N_q)_{q\in\mathbb{N}_0}$ which satisfy (M.1), (M.2) and $\{N.1\}$.*

a) Let $f, g \in \mathcal{S}^{N_q}_{M_p}(\mathbb{R}^d)$. Then $W(f,g)(x,\xi) \in \mathcal{S}^{N_q,M_p}_{M_p,N_q}(\mathbb{R}^{2d})$ and extends uniquely to a continuous map from $(\mathcal{S}^{N_q}_{M_p})'(\mathbb{R}^d) \times (\mathcal{S}^{M_p}_{N_q})'(\mathbb{R}^d)$ into $(\mathcal{S}^{N_q,M_p}_{M_p,N_q})'(\mathbb{R}^{2d})$. The same is true for the short-time Fourier transform.

b) Conversely, if $W(f,g) \in \mathcal{S}^{N_q,M_p}_{M_p,N_q}(\mathbb{R}^{2d})$ (if $V_g f \in \mathcal{S}^{N_q,M_p}_{M_p,N_q}(\mathbb{R}^{2d})$, respectively) then $f, g \in \mathcal{S}^{N_q}_{M_p}(\mathbb{R}^d)$.

Let the sequences $(M_p)_{p\in\mathbb{N}_0}$ and $(N_q)_{q\in\mathbb{N}_0}$ satisfy (M.1), (M.2) and $(N.1)$ instead.

c) Let $f, g \in \Sigma^{N_q}_{M_p}(\mathbb{R}^d)$. Then $W(f,g)(x,\xi) \in \Sigma^{N_q,M_p}_{M_p,N_q}(\mathbb{R}^{2d})$ and extends uniquely to a continuous map from $(\Sigma^{N_q}_{M_p})'(\mathbb{R}^d) \times (\Sigma^{M_p}_{N_q})'(\mathbb{R}^d)$ into $(\Sigma^{N_q,M_p}_{M_p,N_q})'(\mathbb{R}^{2d})$. The same is true for the short-time Fourier transform.

d) Conversely, if $W(f,g) \in \Sigma^{N_q,M_p}_{M_p,N_q}(\mathbb{R}^{2d})$ (if $V_g f \in \Sigma^{N_q,M_p}_{M_p,N_q}(\mathbb{R}^{2d})$, respectively) then $f, g \in \Sigma^{N_q}_{M_p}(\mathbb{R}^d)$.

Proof. We prove only a) and b) since c) and d) can be proved similarly.

a) We first consider a slightly more general situation by assuming that $g \in \mathcal{S}^{\tilde{N}_q}_{\tilde{M}_p}(\mathbb{R}^d)$ where $(\tilde{M}_p)_{p\in\mathbb{N}_0}$ and $(\tilde{N}_q)_{q\in\mathbb{N}_0}$ satisfy (M.1), (M.2), $\tilde{M}_p \subset M_p$ and $\tilde{N}_q \subset N_q$. Obviously, $f(x) \otimes g(t) \in \mathcal{S}^{N_q,\tilde{N}_q}_{M_p,\tilde{M}_p}(\mathbb{R}^d \times \mathbb{R}^d)$.

Observe now $\varphi(x,t) := f(x+\frac{t}{2})g(x-\frac{t}{2})$. If we show

$$\sup_{x,t\in\mathbf{R}^d} |x^\alpha t^\beta \varphi(x,t)| \le C h^{|\alpha|+|\beta|} M_{|\alpha|} M_{|\beta|}, \tag{12}$$

and

$$\sup_{x,t\in\mathbf{R}^d} |\partial_x^\alpha \partial_t^\beta \varphi(x,t)| \le C k^{|\alpha|+|\beta|} N_{|\alpha|} N_{|\beta|} \tag{13}$$

for some $h, k > 0$, then by Theorem 2.3 it follows that $\varphi \in \mathcal{S}^{N_q,N_q}_{M_p,M_p}(\mathbb{R}^{2d})$.

The first inequality easily follows from assumptions on f and g and a change of variables:

$$\begin{aligned}
\sup_{x,t\in\mathbf{R}^d} |x^\alpha t^\beta f(x+\frac{t}{2})g(x-\frac{t}{2})| &= \sup_{y,t\in\mathbf{R}^d} |(y-t/2)^\alpha t^\beta f(y)g(y-t)| \\
&\leq 2^{-|\alpha|} \sup_{y,t\in\mathbf{R}^d} |(y-(t-y))^\alpha (-1)^{|\beta|}((y-t)-y)^\beta f(y)g(y-t)| \\
&= 2^{-|\alpha|} \sup_{y,z\in\mathbf{R}^d} |(y-z)^\alpha (z-y)^\beta f(y)g(z)| \\
&\leq C_{\alpha,\beta} \sup_{y,z\in\mathbf{R}^d} |(z-y)^{\alpha+\beta} f(y)g(z)|.
\end{aligned}$$

Now (12) follows from the assumptions on f, g and $\tilde{M}_p \subset M_p$. In order to prove (13), we use the Leibniz formula which gives

$$\begin{aligned}
|\partial_x^\alpha \partial_t^\beta \varphi(x,t)| &= |\sum_{\delta\leq\alpha}\sum_{\gamma\leq\beta}\binom{\alpha}{\delta}\binom{\beta}{\gamma}\frac{1}{2^{|\alpha|+|\beta|}}\partial_x^\delta\partial_t^\gamma f(x+t/2)\partial_x^{\alpha-\delta}\partial_t^{\beta-\gamma} g(x-t/2)| \\
&\leq C_{\alpha,\beta} \sup_{x,t\in\mathbf{R}^d} |\partial_x^\delta\partial_t^\gamma f(x+t/2)\partial_x^{\alpha-\delta}\partial_t^{\beta-\gamma} g(x-t/2)|.
\end{aligned}$$

Next we use $\tilde{N}_q \subset N_q$ and conditions (M.1) and (M.2) applied to the sequence (N_q) to obtain (13). Therefore, $\varphi \in \mathcal{S}^{N_q,N_q}_{M_p,M_p}(\mathbb{R}^{2d})$.

Now, the partial Fourier transform of φ with respect to the second variable is continuous bijection between $\mathcal{S}^{N_q,N_q}_{M_p,M_p}(\mathbb{R}^{2d})$ and $\mathcal{S}^{N_q,M_p}_{M_p,N_q}(\mathbb{R}^{2d})$, that is

$$\Phi(x,\omega) = \int e^{-2\pi it\omega}\varphi(x,t)dt \in \mathcal{S}^{N_q,M_p}_{M_pN_q}(\mathbb{R}^{2d}) \quad \text{if and only if} \quad \varphi \in \mathcal{S}^{N_q,N_q}_{M_p,M_p}(\mathbb{R}^{2d}).$$

Moreover, $\Phi(x,\omega)$ can be extended to a map from $(\mathcal{S}^{N_q}_{M_p})'(\mathbb{R}^d) \times (\mathcal{S}^{M_p}_{N_q})'(\mathbb{R}^d)$ into $(\mathcal{S}^{N_q,M_p}_{M_p,N_q})'(\mathbb{R}^{2d})$ by duality.

Thus a) is proved in a general case for transforms of the type $\int e^{-2\pi it\omega}\varphi(x,t)dt$ with $\varphi \in \mathcal{S}^{N_q,N_q}_{M_p,M_p}(\mathbb{R}^{2d})$. In particular, the assertion holds for the cross Wigner ditribution and the short-time Fourier transform.

b) Let τ and θ denote the continuous bijections $\tau(x,t) = (x+\frac{t}{2}, x-\frac{t}{2})$ and $\theta(x,t) = (t, t-x)$ on $\mathcal{S}^{N_q,N_q}_{M_p,M_p}(\mathbb{R}^{2d})$. Then $W(f,g) = (\mathcal{F}_2 \circ \tau^*)(f \otimes \overline{g})$ and, similarly, $V_g(f) = (\mathcal{F}_2 \circ \theta^*)(f \otimes \overline{g})$, where $\mathcal{F}_2$ denotes the partial Fourier transform with respect to the second variable and the pullback operators τ^* and θ^* are continuous bijections on $\mathcal{S}^{N_q,N_q}_{M_p,M_p}(\mathbb{R}^{2d})$. Since $\mathcal{F}_2$ is a continuous bijection between $\mathcal{S}^{N_q,N_q}_{M_p,M_p}(\mathbb{R}^{2d})$ and $\mathcal{S}^{N_q,M_p}_{M_p,N_q}(\mathbb{R}^{2d})$, we obtain

$$V_g(f) \in \mathcal{S}^{N_q,M_p}_{M_p,N_q}(\mathbb{R}^{2d}) \Leftrightarrow W(f,g) \in \mathcal{S}^{N_q,M_p}_{M_p,N_q}(\mathbb{R}^{2d}) \Leftrightarrow f \otimes \overline{g} \in \mathcal{S}^{N_q,N_q}_{M_p,M_p}(\mathbb{R}^{2d}) \Leftrightarrow f, g \in \mathcal{S}^{N_q}_{M_p}(\mathbb{R}^d).$$

□

Remark 3.3. *We refer to [56] for related results, see also [49, Theorems 3.8-9].*

4. Modulation spaces

In this section we introduce Feichtinger's modulation spaces, see [19, 25]. Since such spaces are defined by imposing certain decay conditions to the STFT, by the previous section it follows that Gelfand-Shilov spaces can be described via modulation spaces. This is indeed the case, see Theorem 4.3. In such way some applications of modulation spaces to the study of different types of operators might be transferred into the context of Gelfand-Shilov spaces thus leading to a different type of results, see Section 5.

The modulation space norm $M^{p,q}_m(\mathbb{R}^d)$ of a function f on $\mathbb{R}^d$ is given by the $L^{p,q}_m(\mathbb{R}^{2d})$ norm of its STFT $V_g f$, defined on the time-frequency space $\mathbb{R}^{2d}$, with respect to a *suitable* window function g on $\mathbb{R}^d$. Depending on the growth of the weight function m, different Gelfand-Shilov classes may be chosen as fitting test function spaces for modulation spaces, see [14, 49]. The widest class of weights allowing to define modulation spaces is the weight class $\mathcal{N}$ consisting of continuous and positive functions m such that

$$m(z) = o(e^{cz^2}), \quad \text{for } |z| \to \infty, \quad \forall c > 0, \tag{14}$$

with $z \in \mathbb{R}^{2d}$. For instance, every function $m(z) = e^{s|z|^b}$, with $s > 0$ and $0 \le b < 2$, is in $\mathcal{N}$. Thus, the weight m may grow faster than exponentially at infinity. We notice that there is a limit in enlarging the weight class for modulation spaces, imposed by Hardy's theorem: if $m(z) \ge Ce^{cz^2}$, for some $c > \pi/2$, then the corresponding modulation spaces are trivial cf. [28].

Definition 4.1. *Let $m \in \mathcal{N}$, and g a non-zero* window *function in $\mathcal{S}^{1/2}_{1/2}(\mathbb{R}^d)$. For $1 \le p, q \le \infty$ and $f \in (\mathcal{S}^{1/2}_{1/2})'(\mathbb{R}^d)$ we define the modulation space norm (on $\mathcal{S}^{1/2}_{1/2}(\mathbb{R}^d)$) by*

$$\|f\|_{M^{p,q}_m} = \|V_g f\|_{L^{p,q}_m} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x,\omega)|^p m(x,\omega)^p \, dx \right)^{q/p} d\omega \right)^{1/q}$$

(with obvious changes if either $p = \infty$ or $q = \infty$). If $p, q < \infty$, the modulation space $M^{p,q}_m$ is the norm completion of $\mathcal{S}^{1/2}_{1/2}$ in the $M^{p,q}_m$-norm. If $p = \infty$ or $q = \infty$, then $M^{p,q}_m$ is the completion of $\mathcal{S}^{1/2}_{1/2}$ in the weak topology. If $p = q$, $M^p_m := M^{p,p}_m$, and, if $m \equiv 1$, then $M^{p,q}$ and M^p stand for $M^{p,q}_m$ and $M^{p,p}_m$, respectively.*

By the definition, $M^{p,q}_m$ are Banach spaces. Of course, the definition of $M^{p,q}_m$ may depend on the choice of the window function g. So, we choose the Gaussian window $\varphi(x) = e^{-\pi x^2} \in \mathcal{S}^{1/2}_{1/2}(\mathbb{R}^d)$ once and for all to define modulation spaces and we shall work always with it in the sequel.

Remark 4.2. *If $f, g \in \mathcal{S}^{1/2}_{1/2}(\mathbb{R}^d)$ then, by* (14), (8) *and Theorem 3.2 it follows that $\|f\|_{M^{p,q}_m} < \infty$. In fact, if $m \in \mathcal{N}$ we choose $c = h - \epsilon > 0$ in* (14)*, for a suitable $\epsilon > 0$, where $h > 0$ is chosen so that $\|V_g f e^{h|\cdot|^2}\|_{L^\infty} < \infty$. Hence*

$$\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x,\omega)|^p m(x,\omega)^p \, dx \right)^{q/p} d\omega$$
$$\le C \, \|(V_g f) e^{h|\cdot|^2}\|_{L^\infty} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |m(x,\omega)|^p e^{-hp|(x,\omega)|^2} \, dx \right)^{q/p} d\omega < \infty.$$

Moreover, the Gelfand-Shilov class $\mathcal{S}^{1/2}_{1/2}$ is densely embedded in M^1_m, with $m \in \mathcal{N}$, cf. [10].

When the involved weights are of polynomial growth (or decay) at infinity, we introduce a special notation as follows. Let $m(x,\omega) = \langle x \rangle^t \langle \omega \rangle^s$, $s, t \in \mathbb{R}$, $p, q \in [1, \infty]$ and $\varphi(x) = e^{-\pi x^2} \in \mathcal{S}^{1/2}_{1/2}(\mathbb{R}^d)$. Then the modulation space $M^{p,q}_{s,t}(\mathbb{R}^d)$ consists of distributions whose STFT belong to $L^{p,q}_{t,s}(\mathbb{R}^{2d})$: $f \in \mathcal{S}'(\mathbb{R}^d)$ belongs to $M^{p,q}_{s,t}(\mathbb{R}^d)$ if

$$\|f\|_{M^{p,q}_{s,t}} \equiv \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_\phi f(x,\omega) \langle x \rangle^t \langle \omega \rangle^s|^p \, dx \right)^{q/p} d\omega \right)^{1/q} < \infty \tag{15}$$

(with obvious interpretation of the integrals when $p = \infty$ or $q = \infty$). We will also use the usual abbreviated notation: $M^{p,p}_{0,0} = M^p$, $M^{p,p}_{t,t} = M^p_t$, etc.

Gelfand-Shilov type spaces can be characterized by modulation spaces in the following way.

Theorem 4.3. *Let there be given sequence $(N_q)_{q \in \mathbb{N}_0}$ such that (M.1) and (M.2) holds, and let $1 \le p, q \le \infty$, $p' = \frac{p}{p-1}$, $q' = \frac{q}{q-1}$. Then we have:*

$$\mathcal{S}(\mathbb{R}^d) = \operatorname{proj} \lim_{s \to \infty} M^{p,q}_s(\mathbb{R}^d),$$

$$\Sigma_{N_q}^{N_q}(\mathbb{R}^d) = \operatorname{proj}\lim_{s\to\infty} M^{p,q}_{e^{N(s|\cdot|)}}(\mathbb{R}^d), \quad \mathcal{S}_{N_q}^{N_q}(\mathbb{R}^d) = \operatorname{ind}\lim_{s\to\infty} M^{p,q}_{e^{N(s|\cdot|)}}(\mathbb{R}^d)$$

and, by duality,

$$\mathcal{S}'(\mathbb{R}^d) = \operatorname{ind}\lim_{s\to\infty} M^{p',q'}_{-s}(\mathbb{R}^d),$$

$$(\Sigma_{N_q}^{N_q})'(\mathbb{R}^d) = \operatorname{ind}\lim_{s\to\infty} M^{p',q'}_{e^{-N(s|\cdot|)}}(\mathbb{R}^d), \quad (\mathcal{S}_{N_q}^{N_q})'(\mathbb{R}^d) = \operatorname{proj}\lim_{s\to\infty} M^{p',q'}_{e^{-N(s|\cdot|)}}(\mathbb{R}^d),$$

in the set theoretical sense. By $N(\cdot)$ we denote the associated function of the sequence $(N_q)_{q\in\mathbb{N}_0}$.

Proof. We refer to [25, Proposition 11.3.1 d)] for the characterization of the Schwartz class and the space of tempered distributions.

The proof for Gelfand-Shilov spaces and their dual spaces in the non-quasianalytic case is given in [50, Theorem 5.1]. Here we repeat the proof given in [15] and show $\Sigma_{N_q}^{N_q}(\mathbb{R}^d) = \operatorname{proj}\lim_{s\to\infty} M^{p,q}_{e^{N(s|\cdot|)}}(\mathbb{R}^d)$. We leave the other claims to the reader.

We first show that $\operatorname{proj}\lim_{s\to\infty} M^{\infty}_{e^{N(s|\cdot|)}} = \operatorname{proj}\lim_{s\to\infty} M^{p,q}_{e^{N(s|\cdot|)}}$, $1 \le p,q \le \infty$. For fixed $g \in \Sigma_{N_q}^{N_q}$ and any weight $m \in \mathcal{N}$ we have

$$\|f\|_{M^{p,q}_m} = \|V_g f\|_{M^{p,q}_m} \le \|V_g f e^{N(s|\cdot|)}\|_{L^\infty} \|e^{-N(s|\cdot|)}\|_{L^{p,q}_m}$$

and therefore $\operatorname{proj}\lim_{s\to\infty} M^{\infty}_{e^{N(s|\cdot|)}} \subset \operatorname{proj}\lim_{s\to\infty} M^{p,q}_{e^{N(s|\cdot|)}}$. For the opposite inclusion we use

$$V_g f(x,\xi) e^{N(s|(x,\omega)|)} = \int_{-\infty}^{x}\int_{-\infty}^{\xi} \frac{\partial^2}{\partial t \partial \eta}(V_g f(t,\eta) e^{N(s|(t,\eta)|)}) dt d\eta$$

which implies

$$|V_g f(x,\xi) e^{N(s|(x,\omega)|)}| \le \|\frac{\partial^2}{\partial t\partial\eta} V_g f(t,\eta)\cdot e^{N(s|(t,\eta)|)}\|_{L^1} + \|V_g f \cdot \frac{\partial^2}{\partial t\partial\eta} e^{N(s|(t,\eta)|)}\|_{L^1}.$$

Now, the estimates similar to the ones given in the proof of [50, Theorem 5.1] give

$$|V_g f(x,\xi) e^{N(s|(x,\omega)|)}| \le C(\|f\|_{M^{p,q}_{e^{N(s_1|\cdot|)}}} + \|f\|_{M^{p,q}_{e^{N(s_2|\cdot|)}}} + \|f\|_{M^{p,q}_{e^{N(s_3|\cdot|)}}})$$

for certain $s_1, s_2, s_3 > s$.

Let $f, g \in \Sigma_{N_q}^{N_q}(\mathbb{R}^d)$. By Theorem 3.2 it follows that

$$\|V_g f(x,\xi) e^{N(s|(x,\xi)|)})\|_{L^\infty} < \infty$$

for every $s \ge 0$ and therefore $f \in \operatorname{proj}\lim_{s\to\infty} M^{\infty}_{e^{N(s|\cdot|)}}(\mathbb{R}^d)$.

Conversely, $f \in \operatorname{proj}\lim_{s\to\infty} M^{\infty}_{e^{N(s|\cdot|)}}(\mathbb{R}^d)$ means that $\sup_{x,\xi\in\mathbb{R}^d} |V_g f(x,\xi) e^{N(s|(x,\xi)|)}| < \infty$, for every $s \ge 0$ and for any given $g \in \Sigma_{N_q}^{N_q}(\mathbb{R}^d)$.

Assume that $h \in \Sigma_{N_q}^{N_q}(\mathbb{R}^d)$ another function such that $\langle h, g\rangle \neq 0$. Then, for every $s \ge 0$ the following inversion formula holds in $M^{\infty}_{e^{N(s|\cdot|)}}(\mathbb{R}^d)$:

$$f(x) = \frac{1}{\langle h, g\rangle} \iint V_g f(x,\xi)(M_\xi T_x h)(t) dx d\xi, \quad f \in \operatorname{proj}\lim_{s\to\infty} M^{\infty}_{e^{N(s|\cdot|)}}(\mathbb{R}^d).$$

In fact, observe that the STFT can be written as $V_g f(x,\xi) = (\widehat{f\overline{T_x g}})(\xi)$ so that

$$\iint V_g f(x,\xi) M_\xi T_x h(t) dx d\xi = \iint (\widehat{f\overline{T_x g}})(\xi) e^{2\pi i \xi t} d\xi h(t-x) dx = \int f(t)\overline{g(t-x)} h(t-x) dx = \langle h, g\rangle f(t).$$

Now, since $h \in \Sigma_{N_q}^{N_q}(\mathbb{R}^d)$ it follows that for a given $k > 0$ we have

$$\sup_{t \in \mathbf{R}^d} e^{N(k|t|)} |(M_\xi T_x h)(t)| \le e^{N(k|x|)} \sup_{t \in \mathbf{R}^d} |e^{2\pi t \xi}| |T_x(e^{N(k|t|)} h(t))| \le C e^{N(k|x|)}.$$

Therefore, by choosing $s > k$ we obtain

$$\sup_{t \in \mathbf{R}^d} e^{N(k|t|)} |f(t)| = \frac{1}{|\langle h, g \rangle|} \sup_{t \in \mathbf{R}^d} e^{N(k|t|)} |\iint V_g f(x,\xi)(M_\xi T_x h)(t) dx d\xi| \le C \iint |V_g f(x,\xi)| e^{N(k|x|)} dx d\xi < \infty.$$

To show that $\sup_{\eta \in \mathbf{R}^d} e^{N(k|\eta|)} |\hat{f}(\eta)| < \infty$ we use similar arguments, together with

$$\mathcal{F}(M_\xi T_x h)(\eta) = e^{2\pi i x \xi} M_{-x} T_\xi \mathcal{F} h(\eta),$$

and $\hat{h} \in \Sigma_{N_q}^{N_q}(\mathbb{R}^d)$.

Therefore it follows that $f \in \Sigma_{N_q}^{N_q}(\mathbb{R}^d)$.

We note that the results for dual spaces follow directly from the duality relations for modulation spaces: $(M^{p,q}_{e^{N(s|\cdot|)}})' = M^{p',q'}_{e^{-N(s|\cdot|)}}$, see [19, Theorem 7.6.1] or [25, Theorem 11.3.6]). □

4.1. Convolution relations

In this subsection we consider convolution relations for modulation spaces. We recall theorems on polynomial and (sub)exponential type weights from [11, 15], and begin with a recently published sharp convolution estimates, cf. [58].

We introduce the *Young functional*:

$$\mathsf{R}(\mathrm{p}) \equiv 2 - \frac{1}{p_0} - \frac{1}{p_1} - \frac{1}{p_2}, \qquad p = (p_0, p_1, p_2) \in [1,\infty]^3. \tag{16}$$

When $\mathsf{R}(\mathrm{p}) = 0$, the Young inequality for convolution reads as

$$\|f_1 * f_2\|_{L^{p_0'}} \le \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}, \quad f_j \in L^{p_j}(\mathbb{R}^d), \quad j = 1,2.$$

We give an extension of this inequality to weighted Lebesgue spaces and modulation spaces when the Young functional satisfies $0 \le \mathsf{R}(\mathrm{p}) \le 1/2$.

Proposition 4.4. *Let $s_j, t_j \in \mathbb{R}$, $p_j, q_j \in [1,\infty]$, $j = 0,1,2$. Assume that $0 \le \mathsf{R}(\mathrm{p}) \le 1/2$, $\mathsf{R}(\mathrm{q}) \le 1$,*

$$0 \le t_j + t_k, \qquad j,k = 0,1,2, \quad j \ne k, \tag{17}$$

$$0 \le t_0 + t_1 + t_2 - d \cdot \mathsf{R}(\mathrm{p}), \quad \text{and} \tag{18}$$

$$0 \le s_0 + s_1 + s_2, \tag{19}$$

with strict inequality in (18) *when $\mathsf{R}(\mathrm{p}) > 0$ and $t_j = d \cdot \mathsf{R}(\mathrm{p})$ for some $j = 0,1,2$.*

Then $(f_1, f_2) \mapsto f_1 * f_2$ on $C_0^\infty(\mathbb{R}^d)$ extends uniquely to a continuous map from

(1) $L^{p_1}_{t_1}(\mathbb{R}^d) \times L^{p_2}_{t_2}(\mathbb{R}^d)$ to $L^{p_0'}_{-t_0}(\mathbb{R}^d)$;

(2) $M^{p_1,q_1}_{s_1,t_1}(\mathbb{R}^d) \times M^{p_2,q_2}_{s_2,t_2}(\mathbb{R}^d)$ to $M^{p_0',q_0'}_{-s_0,-t_0}(\mathbb{R}^d)$.

For the proof, additional remarks and applications we refer to [58]. It is based on a detailed study of an auxiliary three-linear map over carefully chosen regions in $\mathbb{R}^d$, see Subsections 3.1 and 3.2 in [58]. Moreover, the result is sharp in the following sense.

Proposition 4.5. *Let $p_j, q_j \in [1,\infty]$ and $s_j, t_j \in \mathbb{R}$, $j = 0,1,2$. Assume that at least one of the following statements hold true:*

(1) *the map* $(f_1, f_2) \mapsto f_1 * f_2$ *on* $C_0^\infty(\mathbb{R}^d)$ *is continuously extendable to a map from* $L^{p_1}_{t_1}(\mathbb{R}^d) \times L^{p_2}_{t_2}(\mathbb{R}^d)$ *to* $L^{p_0'}_{-t_0}(\mathbb{R}^d)$;

(2) *the map* $(f_1, f_2) \mapsto f_1 * f_2$ *on* $C_0^\infty(\mathbb{R}^d)$ *is continuously extendable to a map from* $M^{p_1,q_1}_{s_1,t_1}(\mathbb{R}^d) \times M^{p_2,q_2}_{s_2,t_2}(\mathbb{R}^d)$ *to* $M^{p_0',q_0'}_{-s_0,-t_0}(\mathbb{R}^d)$;

Then (17) *and* (18) *hold true.*

Again, we refer to [58] for the proof. In the case of more general weights we use the following result which can be proved by a slight modification of the proof of [11, Proposition 2.4].

Proposition 4.6. *Let* $\nu \in \mathcal{N}(\mathbb{R}^d)$ *be a weight function only in the frequency variables* $\nu(x, \omega) = \nu(\omega)$ *and* $1 \leq p, q, r, s, t \leq \infty$. *If*

$$\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}, \qquad and \qquad \frac{1}{t} + \frac{1}{t'} = 1\,,$$

then

$$M^{p,st}_{1\otimes\nu}(\mathbb{R}^d) * M^{q,st'}_{1\otimes\nu^{-1}}(\mathbb{R}^d) \hookrightarrow M^{r,s}(\mathbb{R}^d) \tag{20}$$

with norm inequality $\|f * h\|_{M^{r,s}} \lesssim \|f\|_{M^{p,st}_{1\otimes\nu}} \|h\|_{M^{q,st'}_{1\otimes\nu^{-1}}}$.

We finish this section with the modulation space norm estimate of the cross-Wigner transform which will be used in the sequel.

Proposition 4.7. *Let* $t_j \in \mathbb{R}$, $p_j \in [1, \infty]$, $j = 0, 1, 2$, $0 \leq \mathsf{R}(\mathrm{p}) \leq 1/2$, $0 \leq t_j + t_k$, $j, k = 0, 1, 2$, $j \neq k$, *and* $0 \leq t_0 + t_1 + t_2 - d \cdot \mathsf{R}(\mathrm{p})$, *with strict inequality when* $\mathsf{R}(\mathrm{p}) > 0$ *and* $t_j = d \cdot \mathsf{R}(\mathrm{p})$ *for some* $j = 0, 1, 2$.

If $\varphi_j \in M^{p_j}_{t_j}(\mathbb{R}^d)$, $j = 1, 2$, *then the map* $(\varphi_1, \varphi_2) \mapsto W(\varphi_2, \varphi_1)$ *where* W *is the cross-Wigner distribution given by* (6) *is sesquilinear continuous map from* $M^{p_2}_{t_2}(\mathbb{R}^d) \times M^{p_1}_{t_1}(\mathbb{R}^d)$ *to* $M^{1,p_0'}_{-t_0,0}(\mathbb{R}^{2d})$.

We refer to [51] for the proof and remark that Proposition 4.7 extends some known results. For example, when $s = -t_0 = t_1 = t_2 \geq 0$, $p = p_0' = p_2 \in [1, \infty]$ and $p_1 = 1$, we have

$$\|W(\varphi_2, \varphi_1)\|_{M^{1,p}_{s,0}} \lesssim \|\varphi_1\|_{M^1_s} \|\varphi_2\|_{M^p_s}, \tag{21}$$

so we recover [11, Proposition 2.5] (with a slightly different notation).

5. Localization operators

We start with a formal definition of the time-frequency localization operator $A_a^{\varphi_1,\varphi_2}$ with symbol a and windows φ_1, φ_2:

Definition 5.1. *Let* $f \in \Sigma^{N_q}_{N_q}(\mathbb{R}^d)$ *where the sequence* (N_q) *satisfies* $(M.1)$, $(M.2)$ *and* $(N.1)$. *The* localization operator $A_a^{\varphi_1,\varphi_2}$ *with* symbol $a \in (\Sigma^{N_q}_{N_q})'(\mathbb{R}^{2d})$ *and* windows $\varphi_1, \varphi_2 \in \Sigma^{N_q}_{N_q}(\mathbb{R}^d)$ *is given by*

$$A_a^{\varphi_1,\varphi_2} f(t) = \int_{\mathbb{R}^{2d}} a(x, \omega) V_{\varphi_1} f(x, \omega) M_\omega T_x \varphi_2(t)\, dx d\omega. \tag{22}$$

In the weak sense,

$$\langle A_a^{\varphi_1,\varphi_2} f, g\rangle = \langle a V_{\varphi_1} f, V_{\varphi_2} g\rangle = \langle a, \overline{V_{\varphi_1} f}\, V_{\varphi_2} g\rangle, \tag{23}$$

where the brackets express a suitable duality between a pair of dual spaces. If (N_q) satisfies $(M.1)$, $(M.2)$ and $\{N.1\}$ instead, $a \in (\mathcal{S}^{N_q}_{N_q})'(\mathbb{R}^{2d})$ and $\varphi_1, \varphi_2 \in \mathcal{S}^{N_q}_{N_q}(\mathbb{R}^d)$, then the weak definition (23) shows that $A_a^{\varphi_1,\varphi_2}$ is a well-defined continuous operator from $\mathcal{S}^{N_q}_{N_q}$ to $(\mathcal{S}^{N_q}_{N_q})'$.

In the study of compactness properties of localization operators in terms of modulation spaces Cordero and Grochenig used a combination of properties of modulation spaces and a representation of localization operators as pesudodifferential operators, see [11].

Such pseudodifferential operators are introduced by Weyl (see [60]) where operators of the form

$$L_\sigma f(x) = (2\pi)^{-d} \iint e^{i(x-y)\cdot\xi} \sigma(\frac{x+y}{2}, \xi) f(y) dy d\xi, \quad f \in L^2((\mathbb{R}^d) \tag{24}$$

are proposed as a good model for the quantization of the classical mechanical observable $\sigma(x, \xi) \in L^2((\mathbb{R}^{2d})$. Then L_σ is called the Weyl pseudodifferential operator (or the Weyl transform) with the symbol σ and (24) is called the Weyl correspondence between the operator and its symbol.

In fact, it can be shown that the Weyl pseudodifferential operator L_σ can be weakly defined by the means of the cross-Wigner distribution (6) as follows:

$$\langle L_\sigma f, g\rangle = \langle \sigma, W(g, f)\rangle, \qquad \sigma, f, g \in L^2(\mathbb{R}^d). \tag{25}$$

This definition can be extended to ultradistributional symbols by duality.

The formula (25) can be found in many places. We refer to [61] where it plays a major role in the study of L_σ.

The main result which will be used in the sequel is the operator equality $A_a^{\varphi_1,\varphi_2} = L_\sigma$, where

$$\sigma = a * W(\varphi_2, \varphi_1)\,. \tag{26}$$

This fact is proved in the case of tempered distributions in [3, 21, 45]. For the sake of completeness, we give here a proof similar to the one given in [51]. However, here we use the kernel Theorem 2.5.

Theorem 5.2. *Let there be given a sequence (N_q) which satisfies $(M.1)$, $(M.2)$ and $(N.1)$. If $a \in (\Sigma_{N_q}^{N_q})'(\mathbb{R}^{2d})$ and $\varphi_1, \varphi_2 \in \Sigma_{N_q}^{N_q}(\mathbb{R}^d)$, then the localization operator $A_a^{\varphi_1,\varphi_2}$ is a Weyl pseudodifferential operator with the Weyl symbol $\sigma = a * W(\varphi_2, \varphi_1)$, in other words,*

$$A_a^{\varphi_1,\varphi_2} = L_{a*W(\varphi_2,\varphi_1)}. \tag{27}$$

Similarly, (27) holds if (N_q) satisfies $(M.1)$, $(M.2)$ and $\{N.1\}$ instead, If $a \in (\mathcal{S}_{N_q}^{N_q})'(\mathbb{R}^{2d})$ and $\varphi_1, \varphi_2 \in \mathcal{S}_{N_q}^{N_q}(\mathbb{R}^d)$.

Proof. We prove only the projective limit case, and the same arguments hold in the inductive limit case.

In the calculations below, we use the fact that the integrals below are absolutely convergent, so that the change of order of integration is allowed. Moreover, certain oscillatory integrals are meaningful when interpreted in the sense of $(\Sigma_{N_q}^{N_q})'(\mathbb{R}^d)$. In particular, if δ denotes the Dirac distribution, then the Fourier inversion formula in the sense of distributions gives $\int e^{2\pi i x\omega} d\omega = \delta(x)$. and $\int \phi(y)\delta(x-y)dy = \phi(x)$, when $\phi \in \Sigma_{N_q}^{N_q}(\mathbb{R}^d)$.

By the kernel Theorem 2.5 and its consequence for any linear and continuous operator $T : \Sigma_{N_q}^{N_q}(\mathbb{R}^{2d}) \to (\Sigma_{N_q}^{N_q})'(\mathbb{R}^{2d})$, there exists a uniquely determined $k \in (\Sigma_{N_q}^{N_q})'(\mathbb{R}^{2d})$ such that

$$\langle Tf, g\rangle = \langle k, g \otimes \overline{f}\rangle, \quad f, g \in \Sigma_{N_q}^{N_q}(\mathbb{R}^{2d}).$$

Therefore it is enough to show that the kernels of $A_a^{\varphi_1,\varphi_2}$ and L_σ coincide when $\sigma = a * W(\varphi_2, \varphi_1)$.

From (23) it is immediate to see that

$$\begin{aligned}
&\langle A_a^{\varphi_1,\varphi_2} f, g\rangle \\
&= \iint_{\mathbb{R}^{2d}} a(x,\omega) \left(\int_{\mathbb{R}^d} f(y) \overline{M_\omega T_x \varphi_1}(y) dy \right) \left(\int_{\mathbb{R}^d} \overline{g}(t) M_\omega T_x \varphi_2(t) dt \right) dx d\omega \\
&\qquad = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\iint_{\mathbb{R}^{2d}} a(x,\omega) \overline{M_\omega T_x \varphi_1}(y) M_\omega T_x \varphi_2(t) dx d\omega \right) f(y) \overline{g}(t) dt dy = \langle k, g \otimes \overline{f}\rangle,
\end{aligned}$$

where

$$k(t,y) = \iint_{\mathbb{R}^{2d}} a(x,\omega)\overline{M_\omega T_x \varphi_1}(y) M_\omega T_x \varphi_2(t) dx d\omega. \tag{28}$$

In order to show that $k(t,y)$ is at the same time the kernel of $L_{a*W(\varphi_2,\varphi_1)}$, we first calculate the convolution $a * W(\varphi_2,\varphi_1)(p,q)$. By $W(f,g) = \overline{W(g,f)}$ and the covariance property of the Wigner transform

$$W(T_x M_\omega f, T_x M_\omega g)(p,q) = W(f,g)(p-x, q-\omega),$$

see [25], we have:

$$\begin{aligned}
& a * W(\varphi_2,\varphi_1)(p,q) \\
&= \iint_{\mathbb{R}^{2d}} a(x,\omega) W(\varphi_2,\varphi_1)(p-x,q-\omega) dx d\omega \\
&= \iint_{\mathbb{R}^{2d}} a(x,\omega) W(T_x M_\omega \varphi_2, T_x M_\omega \varphi_1)(p,q) dx d\omega \\
&= \iint_{\mathbb{R}^{2d}} a(x,\omega) \left(\int_{\mathbb{R}^d} T_x M_\omega \varphi_2 (p + \frac{s}{2}) \overline{T_x M_\omega \varphi_1}(p - \frac{s}{2}) e^{-2\pi i q s} ds \right) dx d\omega \\
&= \iint_{\mathbb{R}^{2d}} a(x,\omega) \left(\int_{\mathbb{R}^d} M_\omega T_x \varphi_2 (p + \frac{s}{2}) \overline{M_\omega T_x \varphi_1}(p - \frac{s}{2}) e^{-2\pi i q s} ds \right) dx d\omega,
\end{aligned}$$

where we have used the commutation relation $T_x M_\omega = e^{-2\pi i x\cdot\omega} M_\omega T_x$.

Now, by the change of variables $p + \frac{s}{2} = t$ and $p - \frac{s}{2} = y$ it follows that

$$\begin{aligned}
\langle k, g \otimes \overline{f} \rangle &= \iint_{\mathbb{R}^{2d}} \iint_{\mathbb{R}^{2d}} a(x,\omega) M_\omega T_x \varphi_2(t) \overline{M_\omega T_x \varphi_1}(y) dx d\omega f(y) \overline{g}(t) dt dy \\
&= \iint_{\mathbb{R}^{2d}} a(x,\omega) \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} M_\omega T_x \varphi_2 (p + \frac{s}{2}) \overline{M_\omega T_x \varphi_1}(p - \frac{s}{2}) \overline{g}(p + \frac{s}{2}) f(p - \frac{s}{2}) ds \Big) dp dx d\omega.
\end{aligned}$$

Now, by the discussion from the beginning of the proof and with suitable interpretations it follows that we may put

$$\overline{g}(p + \frac{s}{2}) f(p - \frac{s}{2}) = \int_{\mathbb{R}^d} \overline{g}(p + \frac{r}{2}) f(p - \frac{r}{2}) \delta(s-r) dr = \iint_{\mathbb{R}^d} e^{-2\pi i q(s-r)} \overline{g}(p + \frac{r}{2}) f(p - \frac{r}{2}) dq dr,$$

which gives

$$\begin{aligned}
& \langle k, g \otimes \overline{f} \rangle = \\
&= \iint_{\mathbb{R}^{2d}} a(x,\omega) \iint_{\mathbb{R}^{2d}} \Big(\iint_{\mathbb{R}^{2d}} M_\omega T_x \varphi_2 (p + \frac{s}{2}) \overline{M_\omega T_x \varphi_1}(p - \frac{s}{2}) e^{-2\pi i q(s-r)} \overline{g}(p + \frac{r}{2}) f(p - \frac{r}{2}) ds dr \Big) dp dq dx d\omega.
\end{aligned}$$

Therefore

$$\langle k, g \otimes \overline{f} \rangle = \langle a * W(\varphi_2,\varphi_1), W(g,f) \rangle = \langle L_{a*W(\varphi_2,\varphi_1)} f, g \rangle$$

(see (25)), and the proof is finished. □

5.1. Continuity properties

We shall focus now on trace-class results for localization operators. First, let us recall that the singular values $\{s_k(L)\}_{k=1}^{\infty}$ of a compact operator $L \in B(L^2(\mathbb{R}^d))$ are the eigenvalues of the positive self-adjoint operator $\sqrt{L^*L}$. For $p = 1$, the trace class S_1 is the space of all compact operators whose singular values enjoy $\sum_{k=1}^{\infty} |s_k(L)| < \infty$. More generally, for $0 < p < \infty$, the Schatten class S_p is the space of all compact operators whose singular values lie in l^p. In particular, S_2 is the space of Hilbert-Schmidt operators and for consistency, we define $S_\infty := B(L^2(\mathbb{R}^d))$ to be the space of bounded operators on $L^2(\mathbb{R}^d)$.

To prove the main result of this section, we shall use a Schatten-class result for the Weyl calculus in terms of modulation spaces, see [29]. References to the proof of the following well known theorem can be found in [11].

Theorem 5.3. *Let σ be the Weyl symbol of L_σ.*

(1) If $\sigma \in M^1(\mathbb{R}^{2d})$ then $\|L_\sigma\|_{S_1} \lesssim \|\sigma\|_{M^1}$.

(2) If $\sigma \in M^p(\mathbb{R}^{2d})$, $1 \le p \le 2$, then $\|L_\sigma\|_{S_p} \lesssim \|\sigma\|_{M^p}$.

(3) If $\sigma \in M^{p,p'}(\mathbb{R}^{2d})$, $2 \le p \le \infty$, then $\|L_\sigma\|_{S_p} \lesssim \|\sigma\|_{M^{p,p'}}$.

By Theorem 5.3 (3) (see also [25, Theorem 14.5.2]) if $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$, then L_σ is bounded on $M^{p,q}(\mathbb{R}^d)$, $1 \le p, q \le \infty$. This result has a long history starting with the Calderon-Vaillancourt theorem on boundedness of pseudodifferential operators with smooth and bounded symbols on $L^2(\mathbb{R}^d)$, [5]. It is extended by Sjöstrand in [46] where $M^{\infty,1}$ is used as appropriate symbol class. Sjöstrand's results were thereafter further extended in [25–27, 52–54].

By using the result based on sharp convolution estimates from [58], the relation between the Weyl pseudodifferential operators and localization operators Theorem 5.2, and convolution results for modulation spaces Theorem 4.4, we obtain continuity results for $A_a^{\varphi_1,\varphi_2}$ for different choices of windows and symbol.

Proposition 5.4. *Let the assumptions of Theorem 4.4 hold. If $\varphi_j \in M^{p_j}_{t_j}(\mathbb{R}^d)$, $j = 1, 2$, and $a \in M^{\infty,r}_{u,v}(\mathbb{R}^{2d})$ where $1 \le r \le p_0$, $u \ge t_0$ and $v \ge d\mathsf{R}(\mathrm{p})$ with $v > d\mathsf{R}(\mathrm{p})$ when $\mathsf{R}(\mathrm{p}) > 0$, then $A_a^{\varphi_1,\varphi_2}$ is bounded on $M^{p,q}(\mathbb{R}^d)$, for all $1 \le p, q \le \infty$ and the operator norm satisfies the uniform estimate*

$$\|A_a^{\varphi_1,\varphi_2}\|_{op} \lesssim \|a\|_{M^{\infty,r}_{u,v}} \|\varphi_1\|_{M^{p_1}_{t_1}} \|\varphi_2\|_{M^{p_2}_{t_2}}.$$

Proof. The proof is given in [51]. We repeat it here to emphasize the interplay between the integral transforms and convolution properties of modulation spaces. Let $\varphi_j \in M^{p_j}_{t_j}(\mathbb{R}^d)$, $j = 1, 2$. Then $W(\varphi_2, \varphi_1) \in M^{1,p_0'}_{-t_0,0}(\mathbb{R}^{2d})$ by Proposition 4.7. This fact, together with Proposition 4.4 *(2)* implies that

$$a * W(\varphi_2, \varphi_1) \in M^{\tilde{p},1}(\mathbb{R}^{2d}), \quad \tilde{p} \ge 2,$$

if we show that the involved parameters satisfy the conditions of the theorem. On the one hand, for the Lebesgue parameters it is easy to see that $\tilde{p} \ge 2$ is equivalent to $\mathsf{R}(\mathrm{p}) = \mathsf{R}(p, \infty, 1) \in [0, 1/2]$, and that $1 \le r \le p_0$ is equivalent to $\mathsf{R}(\mathrm{q}) = \mathsf{R}(\infty, r, p_0') \le 1$. On the other hand, by the choice of the weight parameters u and v it follows that $a * W(\varphi_2, \varphi_1) \in M^{\tilde{p},1}(\mathbb{R}^{2d})$, $\tilde{p} \ge 2$.

In particular, if $\tilde{p} = \infty$ then $a * W(\varphi_2, \varphi_1) \in M^{\infty,1}(\mathbb{R}^{2d})$. ¿From Theorem 5.2 (see also [25, Theorem 14.5.2]) it follows that $A_a^{\varphi_1,\varphi_2}$ is bounded on $M^{p,q}(\mathbb{R}^d)$, $1 \le p, q \le \infty$.

The operator norm estimate also follows from [25, Theorem 14.5.2]. □

Remark 5.5. *When $p_1 = p_2 = 1$, $r = p_0 = \infty$ and $t_1 = t_2 = -t_0 = s \ge 0$, $u = -s$, $v = 0$ we recover the celebrated Cordero-Gröchenig Theorem, [11, Theorem 3.2], in the case of polynomial weights, with the uniform estimate*

$$\|A_a^{\varphi_1,\varphi_2}\|_{op} \lesssim \|a\|_{M^{\infty}_{-s,0}} \|\varphi_1\|_{M^1_s} \|\varphi_2\|_{M^1_s}$$

in our notation.

5.2. Schatten-von Neumann properties

In this subsection we use known results on Weyl pseudodifferential operators with symbol σ, their connection to localization operators from Theorem 5.2, and convolution properties of modulation spaces.

We first give theorems in terms of modulation spaces with polynomial weights. The Schatten-von Neumann properties in Theorems 5.6 and 5.8 are formulated in the spirit of [11], see also [52, 53]. Note that more general weights are considered in [54, 55], leading to different type of results.

Theorem 5.6. *Let the assumptions of Proposition 4.4 hold,* $1 \leq q \leq \infty$*, and let* $v \geq d\mathsf{R}(\mathrm{p})$ *with* $v > d\mathsf{R}(\mathrm{p})$ *when* $\mathsf{R}(\mathrm{p}) > 0$.

(1) If $1 \leq p \leq 2$ *and* $p \leq r \leq 2p/(2-p)$ *then the mapping* $(a, \varphi_1, \varphi_2) \mapsto A_a^{\varphi_1,\varphi_2}$ *is bounded from* $M^{r,q}_{-s,v} \times M^1_s \times M^p_s$, *into* S_p*, that is*

$$\|A_a^{\varphi_1,\varphi_2}\|_{S_p} \lesssim \|a\|_{M^{r,q}_{0,t}} \|\varphi_1\|_{M^1_s} \|\varphi_2\|_{M^p_s}.$$

(2) If $2 \leq p \leq \infty$ *and* $p \leq r$ *then the mapping* $(a, \varphi_1, \varphi_2) \mapsto A_a^{\varphi_1,\varphi_2}$ *is bounded from* $M^{r_1,r_2}_{u,v} \times M^1_s \times M^{p'}_s$*, into* S_p*, that is*

$$\|A_a^{\varphi_1,\varphi_2}\|_{S_p} \lesssim \|a\|_{M^{r,q}_{0,t}} \|\varphi_1\|_{M^1_s} \|\varphi_2\|_{M^{p'}_s}.$$

Proof. *(1)* By Proposition 4.7 it follows that $W(\varphi_2, \varphi_1) \in M^{1,p_w}_{-t_0,0}(\mathbb{R}^{2d})$, with $t_0 \geq -s$ and $p_w \in [2p/(p+2), p]$. Therefore $W(\varphi_2, \varphi_1) \in M^{1,p}_{s,0}(\mathbb{R}^{2d})$.

This and Proposition 4.4 *(2)* imply $a * W(\varphi_2, \varphi_1) \in M^p(\mathbb{R}^{2d})$. The result now follows from Theorem 5.3 (2).

(2) By Proposition 4.7 it follows that $W(\varphi_2, \varphi_1) \in M^{1,p_w}_{-t_0,0}(\mathbb{R}^{2d})$, with $t_0 \geq -s$ and $p_w \in [p', 2p'/(p'+2), p]$. Therefore $W(\varphi_2, \varphi_1) \in M^{1,p'}_{s,0}(\mathbb{R}^{2d})$.

The statement follows from Proposition 4.4 *(2)* and Theorem 5.3 (3), similarly to the previous case. □

Remark 5.7. *A particular choice:* $r = p$, $q = \infty$ *and* $v = 0$ *gives [11, Theorem 3.4].*

We finish with necessary conditions whose proofs follow easily from the proofs of Theorems 4.3 and 4.4 in [11] and are therefore omitted.

Theorem 5.8. *Let the assumptions of Proposition 4.4 hold and let* $a \in \mathcal{S}'(\mathbb{R}^{2d})$.

(1) If there exists a constant $C = C(a) > 0$ *depending only on the symbol a such that*

$$\|A_a^{\varphi_1,\varphi_2}\|_{S_\infty} \leq C \|\varphi_1\|_{M^{p_1}_{t_1}} \|\varphi_2\|_{M^{p_2}_{t_2}},$$

for all $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$*, then* $a \in M^{\infty,r}_{u,v}(\mathbb{R}^{2d})$ *where* $1 \leq r \leq p_0$*,* $u \geq t_0$ *and* $v \geq d\mathsf{R}(\mathrm{p})$ *with* $v > d\mathsf{R}(\mathrm{p})$ *when* $\mathsf{R}(\mathrm{p}) > 0$.

(2) If there exists a constant $C = C(a) > 0$ *depending only on the symbol a such that*

$$\|A_a^{\varphi_1,\varphi_2}\|_{S_2} \leq C \|\varphi_1\|_{M^1} \|\varphi_2\|_{M^1}$$

for all $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$*, then* $a \in M^{2,\infty}(\mathbb{R}^{2d})$.

We finish the paper with a trace-class result in the context of ultradistributions. To that end we need a result on the behavior of the STFT as follows.

Proposition 5.9. *Let there be given sequence* $(N_q)_{q \in \mathbb{N}_0}$ *such that (9) and (10) hold. Let* $u \in (\mathcal{S}^{\{N_q\}}_{\{N_q\}})'$ $(u \in (\Sigma^{(N_q)}_{(N_q)})'$*, respectively) such that it can be extended continuously to* $\mathcal{E}^{\{N_q\}}(\Omega)$ $(\mathcal{E}^{(N_q)}(\Omega)$*, respectively) for some open bounded set* $\Omega \in \mathbb{R}^d$*. If* $\varphi \in \mathcal{S}^{\{N_q\}}_{\{N_q\}}(\mathbb{R}^d)$ *(if* $\varphi \in \Sigma^{(N_q)}_{(N_q)}(\mathbb{R}^d)$*, respectively) then*

$$|V_\varphi u(x, \omega)| \lesssim e^{-N(a|x|)} e^{N(\tilde{a}|\omega|)}, \tag{29}$$

for some $a, \tilde{a} > 0$ *(resp. for every* $a, \tilde{a} > 0$*).*

We remark that the proof of Proposition 5.9 is based on a representation theorem for quasianalytic ultradistributions based on the parametrix of the heat kernel operators, we refer to [15].

Theorem 5.10. *Let there be given sequence $(N_q)_{q\in\mathbb{N}_0}$ such that (9) and (10) hold. Let $a \in (\mathcal{S}^{N_q}_{N_q})'(\mathbb{R}^{2d})$ ($a \in (\Sigma^{N_q}_{N_q})'(\mathbb{R}^{2d})$, respectively) such that it can be extended continuously to $\mathcal{E}^{\{N_q\}}(\Omega)$ ($\mathcal{E}^{(N_q)}(\Omega)$, respectively) for some open bounded set $\Omega \subseteq \mathbb{R}^{2d}$. Furthermore, let $\varphi_1, \varphi_2 \in \mathcal{S}^{N_q}_{N_q}(\mathbb{R}^d)$ (resp. $\varphi_1, \varphi_2 \in \Sigma^{N_q}_{N_q}(\mathbb{R}^d)$), then $A^{\varphi_1,\varphi_2}_a$ is a trace-class operator.*

Proof. We repeat the proof from [15] and show the claim only when $a \in (\mathcal{S}^{N_q}_{N_q})'(\mathbb{R}^{2d})$, since the other case is similar. As already mentioned, in the definition of modulation spaces we fix Gaussian window $g(x) = e^{-\pi x^2} \in \mathcal{S}^{1/2}_{1/2}(\mathbb{R}^d) \subset \Sigma^{N_q}_{N_q}(\mathbb{R}^d)$, see [14, Lemma 2.3]. By Proposition 5.9 we have

$$|V_g a(x,\omega)| \le C e^{-N(h|x|)} e^{N(k|\omega|)}$$

for arbitrary $h, k > 0$. Then, for a given $b > 0$, we choose $k < b$ to obtain

$$\sup_{\omega\in\mathbb{R}^d} \int_{\mathbb{R}^d} |V_g a(x,\omega)| e^{-N(b|\omega|)} dx \le \sup_{\omega\in\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-N(h|x|)} e^{N(k|\omega|)} e^{-N(b|\omega|)} dx < \infty.$$

Therefore $a \in M^{1,\infty}_{1\otimes e^{-N(b|\cdot|)}}(\mathbb{R}^{2d})$, where $b > 0$ can be chosen arbitrary.

If $\varphi_1, \varphi_2 \in \mathcal{S}^{N_q}_{N_q}(\mathbb{R}^d)$, then $W(\varphi_2, \varphi_1) \in \mathcal{S}^{N_q}_{N_q}(\mathbb{R}^{2d})$ by Theorem 3.2, and therefore, by Theorem 4.3, there exist $h, k > 0$ such that

$$W(\varphi_2, \varphi_1) \in M^1_{e^{N(h|x|)}\otimes e^{N(k|\omega|)}}(\mathbb{R}^{2d}) \subset M^1_{1\otimes e^{N(k|\omega|)}}(\mathbb{R}^{2d}).$$

Now, we choose $b = k$ and use the convolution relations of Proposition 4.6 to obtain

$$M^{1,\infty}_{1\otimes e^{-N(k|\cdot|)}}(\mathbb{R}^{2d}) * M^1_{1\otimes e^{-N(k|\cdot|)}}(\mathbb{R}^{2d}) \hookrightarrow M^1(\mathbb{R}^{2d}),$$

hence $\sigma = a * W(\varphi_2, \varphi_1) \in M^1(\mathbb{R}^{2d})$. Theorem 5.6 yields the desired result. □

For example, our result holds for $f = \sum_{n\in\mathbb{N}} a_n \delta^{(n)}$, where $|a_n| \le \frac{C_h h^n}{n!^s}$, for every $h > 0$ and corresponding $C_h > 0$, and $s > 1/2$.

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