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Weighted boundedness of multilinear operators for the extreme cases

Dazhao Chen[a], **Lanzhe Liu**[b]

[a]*College of Mathematics, Department of Science and Information Science, Shaoyang University, Hunan Shaoyang, 422000, P. R. China*
[b]*College of Mathematics, Department of Science and Information Science, Shaoyang University, Hunan Shaoyang, 422000, P. R. China*

Abstract. In this paper, the weighted boundedness of some multilinear operators related to the Littlewood-Paley operator and Marcinkiewicz operator for the extreme cases are obtained.

1. Definitions and Theorems

In this paper, we will consider a class of multilinear operators related to some integral operators, whose definitions are following.

Denote that $\Gamma(x) = \{(y,t) \in R^{n+1}_+ : |x-y| < t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. Let m be a positive integer and A be a function on R^n. Set

$$R_{m+1}(A;x,y) = A(x) - \sum_{|\alpha|\le m} \frac{1}{\alpha!} D^\alpha A(y)(x-y)^\alpha,$$

$$Q_{m+1}(A;x,y) = R_m(A;x,y) - \sum_{|\alpha|=m} D^\alpha A(x)(x-y)^\alpha.$$

Definition 1. Let $\varepsilon > 0$ and ψ be a fixed function which satisfies the following properties:
(1) $\int \psi(x)dx = 0$,
(2) $|\psi(x)| \le C(1+|x|)^{-(n+1)}$,
(3) $|\psi(x+y) - \psi(x)| \le C|y|^\varepsilon (1+|x|)^{-(n+1+\varepsilon)}$ when $2|y| < |x|$.
The multilinear Littlewood-Paley operator is defined by

$$g^A_S(f)(x) = \left[\int\int_{\Gamma(x)} |F^A_t(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$F^A_t(f)(x,y) = \int_{R^n} \frac{R_{m+1}(A;x,z)}{|x-z|^m} \psi_t(y-z)f(z)dz$$

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Email addresses: chendazhao27@sina.com (Dazhao Chen), lanzheliu@163.com (Lanzhe Liu)

and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$. Set $F_t(f)(y) = f * \psi_t(y)$. We also define that

$$g_S(f)(x) = \left(\int\int_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2},$$

which is the Littlewood-Paley S operator (see [14]).

We also consider the variant of g_S^A, which is defined by

$$\tilde{g}_S^A(f)(x) = \left[\int\int_{\Gamma(x)} |\tilde{F}_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$\tilde{F}_t^A(f)(x,y) = \int_{R^n} \frac{Q_{m+1}(A;x,z)}{|x-z|^m}\psi_t(y-z)f(z)dz.$$

Let H be the Hilbert space $H = \left\{h : \|h\| = \left(\int\int_{R_+^{n+1}} |h(t)|^2 dydt/t^{n+1}\right)^{1/2} < \infty\right\}$. Then for each fixed $x \in R^n$, $F_t^A(f)(x,y)$ may be viewed as a mapping from $(0,+\infty)$ to H, and it is clear that

$$g_S^A(f)(x) = \left\|\chi_{\Gamma(x)}F_t^A(f)(x,y)\right\|, \ \tilde{g}_S^A(f)(x) = \left\|\chi_{\Gamma(x)}\tilde{F}_t^A(f)(x,y)\right\|, \ g_S(f)(x) = \left\|\chi_{\Gamma(x)}F_t(f)(y)\right\|.$$

Definition 2. Let $0 < \gamma \le 1$ and Ω be homogeneous of degree zero on R^n such that $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$. Assume that $\Omega \in Lip_\gamma(S^{n-1})$, that is there exists a constant $M > 0$ such that for any $x, y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \le M|x-y|^\gamma$. The multilinear Marcinkiewicz operator and its variant are defined by

$$\mu_S^A(f)(x) = \left[\int\int_{\Gamma(x)} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}}\right]^{1/2}$$

and

$$\tilde{\mu}_S^A(f)(x) = \left[\int\int_{\Gamma(x)} |\tilde{F}_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}}\right]^{1/2},$$

where

$$F_t^A(f)(x,y) = \int_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n-1}} \frac{R_{m+1}(A;x,z)}{|x-z|^m} f(z)dz$$

and

$$\tilde{F}_t^A(f)(x,y) = \int_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n-1}} \frac{Q_{m+1}(A;x,z)}{|x-z|^m} f(z)dz.$$

Set

$$F_t(f)(y) = \int_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n-1}} f(z)dz.$$

We also define that

$$\mu_S(f)(x) = \left(\int\int_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}}\right)^{1/2},$$

which is the Marcinkiewicz operator (see [15]).

Let H be the Hilbert space $H = \left\{h : \|h\| = \left(\int\int_{R_+^{n+1}} |h(t)|^2 dydt/t^{n+3}\right)^{1/2} < \infty\right\}$. Then for each fixed $x \in R^n$, $F_t^A(f)(x,y)$ may be viewed as a mapping from $(0,+\infty)$ to H, and it is clear that

$$\mu_S^A(f)(x) = \left\|\chi_{\Gamma(x)}F_t^A(f)(x,y)\right\|, \ \tilde{\mu}_S^A(f)(x) = \left\|\chi_{\Gamma(x)}\tilde{F}_t^A(f)(x,y)\right\|, \ \mu_S(f)(x) = \left\|\chi_{\Gamma(x)}F_t(f)(y)\right\|.$$

Note that when $m = 0$, g_S^A and μ_S^A are just the commutators of F_t and A (see [11][12][15][16]). Let T be the Calderon-Zygmund singular integral operator, a classical result of Coifman, Rochberg and Weiss ([7]) states that the commutator $[b, T] = T(bf) - bTf$ (where $b \in BMO(R^n)$) is bounded on $L^p(R^n)$ for $1 < p < \infty$, Chanillo ([2]) proves a similar result when T is replaced by the fractional integral operator. In [10], the boundedness properties of the commutators for the extreme values of p are obtained. It is well known that multilinear operator, as a non-trivial extension of commutator, is of great interest in harmonic analysis and has been widely studied by many authors (see [3-6][8]). The purpose of this paper is to discuss the endpoint estimates of the multilinear operators g_S^A and μ_S^A.

First, let us introduce some notations (see [9][13]). Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For a cube Q and a locally integrable function f, let $f_Q = |Q|^{-1}\int_Q f(x)dx$ and $f^\#(x) = \sup_{x\in Q}|Q|^{-1}\int_Q |f(y) - f_Q|dy$. Moreover, for a weight function w, f is said to belong to $BMO(w)$ if $f^\# \in L^\infty(w)$ and define $\|f\|_{BMO(w)} = \|f^\#\|_{L^\infty(w)}$, if $w = 1$, we denote that $BMO(w) = BMO(R^n)$; Also, we give the concepts of the atom and weighted H^1 space. A function a is called a $H^1(w)$ atom if there exists a cube Q such that a is supported on Q, $\|a\|_{L^\infty(w)} \le w(Q)^{-1}$ and $\int a(x)dx = 0$. It is well known that the weighted Hardy space $H^1(w)$ has the atomic decomposition characterization (see [1][9]).

The A_1 weight is defined by (see [8])

$$A_1 = \{0 < w \in L^p_{loc}(R^n) : M(w)(x) \le Cw(x), a.e.\}.$$

We shall prove the following theorems in Section 2.

Theorem 1. Let $D^\alpha A \in BMO(R^n)$ for $|\alpha| = m$ and $w \in A_1$.

(i) If for any $H^1(w)$-atom a supported on certain cube Q and $u \in 3Q \setminus 2Q$, there is

$$\int_{(4Q)^c}\left\|\chi_{\Gamma(x)}\sum_{|\alpha|=m}\frac{1}{\alpha!}\frac{(x-u)^\alpha}{|x-u|^m}\psi_t(y-u)\int_Q D^\alpha A(z)a(z)dz\right\|w(x)dx \le C,$$

then g_S^A is bounded from $H^1(w)$ to $L^1(w)$;

(ii) If for any cube Q and $u \in 3Q \setminus 2Q$, there is

$$\frac{1}{w(Q)}\int_Q\left\|\chi_{\Gamma(x)}\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A(x) - (D^\alpha A)_Q)\int_{(4Q)^c}\frac{(u-z)^\alpha}{|u-z|^m}\psi_t(u-z)f(z)dz\right\|w(x)dx \le C\|f\|_{L^\infty(w)},$$

then $\tilde{g}_S^A$ is bounded from $L^\infty(w)$ to $BMO(w)$.

Theorem 2. Let $D^\alpha A \in BMO(R^n)$ for $|\alpha| = m$ and $w \in A_1$.

(i) If for any $H^1(w)$-atom a supported on certain cube Q and $u \in 3Q \setminus 2Q$, there is

$$\int_{(4Q)^c}\left\|\chi_{\Gamma(x)}\sum_{|\alpha|=m}\frac{1}{\alpha!}\frac{(x-u)^\alpha}{|x-u|^m}\frac{\Omega(y-u)}{|y-u|^{n-1}}\chi_{\Gamma(y)}(u,t)\int_Q D^\alpha A(z)a(z)dz\right\|w(x)dx \le C,$$

then μ_S^A is bounded from $H^1(w)$ to $L^1(w)$;

(ii) If for any cube Q and $u \in 3Q \setminus 2Q$, there is

$$\begin{aligned}&\frac{1}{w(Q)}\int_Q\left\|\chi_{\Gamma(x)}\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A(x) - (D^\alpha A)_Q)\int_{(4Q)^c}\frac{(u-z)^\alpha}{|u-z|^m}\frac{\Omega(y-z)\chi_{\Gamma(y)}(z,t)}{|y-z|^{n-1}}f(z)dz\right\|w(x)dx\\ \le\;& C\|f\|_{L^\infty(w)},\end{aligned}$$

then $\tilde{\mu}_S^A$ is bounded from $L^\infty(w)$ to $BMO(w)$.

2. Proofs of Theorems

We begin with two lemmas.

Lemma 1. (see [6]) Let A be a function on R^n and $D^\alpha A \in L^q(R^n)$ for $|\alpha| = m$ and some $q > n$. Then

$$|R_m(A;x,y)| \le C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}(x,y)$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

Lemma 2. Let $w \in A_1$, $1 < p < \infty$ and $D^\alpha A \in BMO(R^n)$ for $|\alpha| = m$. Then g_S^A and μ_S^A are all bounded on $L^p(w)$.

Proof. For g_S^A, by Minkowski inequality and the condition of ψ, we get

$$\begin{aligned} g_S^A(f)(x) &\le \int_{R^n} \frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^m} \left(\int_{\Gamma(x)} |\psi_t(y-z)|^2 \frac{dydt}{t^{1+n}} \right)^{1/2} dz \\ &\le C \int_{R^n} \frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^m} \left(\int_0^\infty \int_{|x-y|\le t} \frac{t^{-2n}}{(1+|y-z|/t)^{2n+2}} \frac{dydt}{t^{1+n}} \right)^{1/2} dz \\ &\le C \int_{R^n} \frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^m} \left(\int_0^\infty \int_{|x-y|\le t} \frac{2^{2n+2} t^{1-n}}{(2t+|y-z|)^{2n+2}} dydt \right)^{1/2} dz, \end{aligned}$$

noting that $2t + |y-z| \ge 2t + |x-z| - |x-y| \ge t + |x-z|$ when $|x-y| \le t$ and

$$\int_0^\infty \frac{tdt}{(t+|x-z|)^{2n+2}} = C|x-z|^{-2n},$$

we obtain

$$g_S^A(f)(x) \le C \int_{R^n} \frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^m} \left(\int_0^\infty \frac{tdt}{(t+|x-z|)^{2n+2}} \right)^{1/2} dz = C \int_{R^n} \frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^{m+n}} dz.$$

For μ_S^A, note that $|x-z| \le 2t$, $|y-z| \ge |x-z| - t \ge |x-z| - 3t$ when $|x-y| \le t$, $|y-z| \le t$, we get

$$\begin{aligned} \mu_S^A(f)(x) &\le \int_{R^n} \left[\int\int_{|x-y|\le t} \left(\frac{|\Omega(y-z)||R_{m+1}(A;x,z)||f(z)|}{|y-z|^{n-1}|x-z|^m} \right)^2 \chi_{\Gamma(z)}(y,t) \frac{dydt}{t^{n+3}} \right]^{1/2} dz \\ &\le C \int_{R^n} \frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^m} \left[\int\int_{|x-y|\le t} \frac{\chi_{\Gamma(z)}(y,t) t^{-n-3}}{(|x-z|-3t)^{2n-2}} dydt \right]^{1/2} dz \\ &\le C \int_{R_n} \frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^{m+3/2}} \left[\int_{|x-z|/2}^\infty \frac{dt}{(|x-z|-3t)^{2n-2}} \right]^{1/2} dz \\ &\le C \int_{R_n} \frac{|R_{m+1}(A;x,z)|}{|x-z|^{m+n}} |f(z)| dz. \end{aligned}$$

Thus, the lemma follows from [8].

Proof of Theorem 1(i). It suffices to show that there exists a constant $C > 0$ such that for every $H^1(w)$-atom a with supp$a \subset Q = Q(x_0, d)$, there is

$$\|g_S^A(a)\|_{L^1(w)} \le C.$$

Let $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A)_Q x^\alpha$, then $R_m(A;x,y) = R_m(\tilde{A};x,y)$ and $D^\alpha \tilde{A} = D^\alpha A - (D^\alpha A)_Q$ for all α with $|\alpha| = m$. We write, by the vanishing moment of a and for $u \in 3Q \setminus 2Q$,

$$
\begin{aligned}
& g_S^A(a)(x) = \left\|\chi_{\Gamma(x)}(y,t)F_t^A(a)(x,y)\right\| \le \chi_{4Q}(x)\left\|\chi_{\Gamma(x)}(y,t)F_t^A(a)(x,y)\right\| \\
& +\chi_{(4Q)^c}(x)\left\|\chi_{\Gamma(x)}(y,t)\int_{R^n}\left[\frac{R_m(\tilde{A};x,z)\psi_t(y-z)}{|x-z|^m} - \frac{R_m(\tilde{A};x,u)\psi_t(y-u)}{|x-u|^m}\right]a(z)dz\right\| \\
& +\chi_{(4Q)^c}(x)\left\|\chi_{\Gamma(x)}(y,t)\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n}\left[\frac{\psi_t(y-z)(x-z)^\alpha}{|x-z|^m} - \frac{\psi_t(y-u)(x-u)^\alpha}{|x-u|^m}\right]D^\alpha\tilde{A}(z)a(z)dz\right\| \\
& +\chi_{(4Q)^c}(x)\left\|\chi_{\Gamma(x)}(y,t)\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n}\frac{(x-u)^\alpha}{|x-u|^m}\psi_t(y-u)D^\alpha\tilde{A}(z)a(z)dz\right\| \\
= \;& I_1(x) + I_2(x,u) + I_3(x,u) + I_4(x,u).
\end{aligned}
$$

By the $L^p(w)$-boundedness of g_S^A for $1 < p < \infty$ (see Lemma 2), we get

$$
\int_{R^n} I_1(x)w(x)dx = \int_{4Q} g_S^A(a)(x)w(x)dx \le \|g_S^A(a)\|_{L^p(w)} w(4Q)^{1-1/p} \le C\|a\|_{L^p(w)} w(Q)^{1-1/p} \le C.
$$

For $I_2(x,u)$, we write

$$
\begin{aligned}
& \frac{R_m(\tilde{A};x,z)\psi_t(y-z)}{|x-z|^m} - \frac{R_m(\tilde{A};x,u)\psi_t(y-u)}{|x-u|^m} = \left[\frac{1}{|x-z|^m} - \frac{1}{|x-u|^m}\right]R_m(\tilde{A};x,z)\psi_t(y-z) \\
& +(\psi_t(y-z) - \psi_t(y-u))\frac{R_m(\tilde{A};x,z)}{|x-u|^m} + \frac{\psi_t(y-u)}{|x-u|^m}[R_m(\tilde{A};x,z) - R_m(\tilde{A};x,u)].
\end{aligned}
$$

Note that $|x-z| \sim |x-u| \sim |x-x_0|$ for $z \in Q$ and $x \in R^n \setminus 4Q$. By Lemma 1 and the following inequality (see [13])

$$
|b_{Q_1} - b_{Q_2}| \le C\log(|Q_2|/|Q_1|)\|b\|_{BMO} \text{ for } Q_1 \subset Q_2,
$$

we know that, for $z \in Q$ and $x \in 2^{k+1}Q \setminus 2^kQ$,

$$
\begin{aligned}
|R_m(\tilde{A};x,z)| &\le C|x-z|^m \sum_{|\alpha|=m}(\|D^\alpha A\|_{BMO} + |(D^\alpha A)_{Q(x,z)} - (D^\alpha A)_Q|) \\
&\le Ck|x-z|^m \sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}.
\end{aligned}
$$

And by the formula (see [6]):

$$
R_m(\tilde{A};x,z) - R_m(\tilde{A};x,u) = \sum_{|\beta|<m}\frac{1}{\beta!}R_{m-|\beta|}(D^\beta\tilde{A};z,u)(x-z)^\beta
$$

and Lemma 1, we have

$$
|R_m(\tilde{A};x,z) - R_m(\tilde{A};x,u)| \le C\sum_{|\beta|<m}\sum_{|\alpha|=m}|z-u|^{m-|\beta|}|x-z|^{|\beta|}\|D^\alpha A\|_{BMO}.
$$

Thus, similar to the proof of Lemma 2, we get

$$\begin{aligned}
&\int_{R^n} I_2(x,u)w(x)dx \\
\leq\ & C\sum_{k=2}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}\int_Q\left(\frac{|z-u|}{|x-z|^{m+n+1}}+\frac{|z-u|^{\varepsilon}}{|x-z|^{m+n+\varepsilon}}\right)|R_m(\tilde{A};x,z)||a(z)|dzw(x)dx \\
&+\sum_{|\alpha|=m}\|D^{\alpha}A\|_{BMO}\sum_{k=2}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}\int_Q\frac{|z-u|}{|x-z|^{n+1}}|a(z)|dzw(x)dx \\
\leq\ & C\sum_{|\alpha|=m}\|D^{\alpha}A\|_{BMO}\sum_{k=2}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}\int_Q k\left(\frac{|z-u|}{|x-z|^{n+1}}+\frac{|z-u|^{\varepsilon}}{|x-z|^{n+\varepsilon}}\right)|a(z)|dzw(x)dx \\
\leq\ & C\sum_{|\alpha|=m}\|D^{\alpha}A\|_{BMO}\sum_{k=2}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}k\left(\frac{d}{(2^kd)^{n+1}}+\frac{d^{\varepsilon}}{(2^kd)^{n+\varepsilon}}\right)\|a\|_{L^{\infty}(w)}|Q|w(x)dx \\
\leq\ & C\sum_{|\alpha|=m}\|D^{\alpha}A\|_{BMO}\sum_{k=2}^{\infty}k(2^{-k}+2^{-\varepsilon k})\frac{w(2^{k+1}Q)}{|2^{k+1}Q|}\frac{|Q|}{w(Q)}.
\end{aligned}$$

Note that if $w\in A_1$, then $\frac{w(Q_2)}{|Q_2|}\frac{|Q_1|}{w(Q_1)}\leq C$ for all cubes Q_1, Q_2 with $Q_1\subset Q_2$ (see [9][13]). Thus

$$\int_{R^n} I_2(x,u)w(x)dx\leq C\sum_{|\alpha|=m}\|D^{\alpha}A\|_{BMO}\sum_{k=2}^{\infty}k(2^{-k}+2^{-\varepsilon k})\leq C.$$

For $I_3(x,u)$, similar to the proof of Lemma 2 and $I_2(x,u)$, we obtain

$$\begin{aligned}
&\int_{(4Q)^c} I_3(x,u)w(x)dx \\
\leq\ & C\sum_{|\alpha|=m}\sum_{k=2}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}\int_Q\left(\frac{|z-u|}{|x-z|^{n+1}}+\frac{|z-u|^{\varepsilon}}{|x-z|^{n+\varepsilon}}\right)|D^{\alpha}\tilde{A}(z)||a(z)|dzw(x)dx \\
\leq\ & C\sum_{|\alpha|=m}\sum_{k=2}^{\infty}\left(\frac{d}{(2^kd)^{n+1}}+\frac{d^{\varepsilon}}{(2^kd)^{n+\varepsilon}}\right)\left(\frac{1}{|Q|}\int_Q|D^{\alpha}\tilde{A}(z)|dz\right)\|a\|_{L^{\infty}(w)}|Q|w(2^{k+1}Q) \\
\leq\ & C\sum_{|\alpha|=m}\|D^{\alpha}A\|_{BMO}\sum_{k=2}^{\infty}(2^{-k}+2^{-\varepsilon k})\frac{w(2^{k+1}Q)}{|2^{k+1}Q|}\frac{|Q|}{w(Q)}\leq C.
\end{aligned}$$

Thus, using the condition of $I_4(x,u)$, we obtain

$$\int_{R^n} g_S^A(a)(x)w(x)dx\leq C.$$

(ii). It is only to prove that there exists a constant C_Q such that

$$\frac{1}{w(Q)}\int_Q|\tilde{g}_S^A(f)(x)-C_Q|w(x)dx\leq C\|f\|_{L^{\infty}(w)}$$

holds for any cube Q. By the following equality

$$Q_{m+1}(A;x,z)=R_{m+1}(A;x,z)+\sum_{|\alpha|=m}\frac{1}{\alpha!}(x-z)^{\alpha}(D^{\alpha}A(x)-D^{\alpha}A(z)),$$

we have, similar to the proof of Lemma 2,

$$\tilde{g}_S^A(a)(x) \le g_S^A(a)(x) + C \sum_{|\alpha|=m} \int_{R^n} \frac{|D^\alpha A(x) - D^\alpha A(z)|}{|x-z|^n} |a(z)| dz,$$

thus, $\tilde{g}_S^A$ is $L^p(w)$-bounded by Lemma 2 and [2]. For any cube $Q = Q(x_0, d)$, let $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A)_{\tilde{Q}} x^\alpha$. We write, for $f = f\chi_{4Q} + f\chi_{(4Q)^c} = f_1 + f_2$ and $u \in 3Q \setminus 2Q$,

$$\begin{aligned}
&\left|\tilde{g}_S^A(f)(x) - g_S\left(\frac{R_m(\tilde{A}; x_0, \cdot)}{|x_0 - \cdot|^m} f_2\right)(x_0)\right| = \left|\left\|\chi_{\Gamma(x)} \tilde{F}_t^A(f)(x,y)\right\| - \left\|\chi_{\Gamma(x_0)} F_t\left(\frac{R_m(\tilde{A}; x_0, \cdot)}{|x_0 - \cdot|^m} f_2\right)(y)\right\|\right| \\
\le\ & \left\|\chi_{\Gamma(x)}(y,t) \tilde{F}_t^A(f)(x,y) - \chi_{\Gamma(x_0)}(y,t) F_t\left(\frac{R_m(\tilde{A}; x_0, \cdot)}{|x_0 - \cdot|^m} f_2\right)(y)\right\| \le \left\|\chi_{\Gamma(x)}(y,t) \tilde{F}_t^A(f_1)(x,y)\right\| \\
& + \left\|\left[\chi_{\Gamma(x)}(y,t) \int_{R^n} \frac{R_m(\tilde{A}; x, z)}{|x-z|^m} \psi_t(y-z) - \chi_{\Gamma(x_0)}(y,t) \int_{R^n} \frac{R_m(\tilde{A}; x_0, z)}{|x_0 - z|^m} \psi_t(y-z)\right] f_2(z) dz\right\| \\
& + \left\|\chi_{\Gamma(x)}(y,t) \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A(x) - (D^\alpha A)_Q) \int_{R^n} \left[\frac{\psi_t(y-z)(x-z)^\alpha}{|x-z|^m} - \frac{\psi_t(u-z)(u-z)^\alpha}{|u-z|^m}\right] f_2(z) dz\right\| \\
& + \left\|\chi_{\Gamma(x)}(y,t) \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A(x) - (D^\alpha A)_Q) \int_{R^n} \frac{(u-z)^\alpha}{|u-z|^m} \psi_t(u-z) f_2(z) dz\right\| \\
=\ & J_1(x) + J_2(x) + J_3(x,u) + J_4(x,u).
\end{aligned}$$

By the $L^p(w)$-boundedness of $\tilde{g}_S^A$ for $1 < p < \infty$, we get

$$\frac{1}{w(Q)} \int_Q J_1(x) w(x) dx \le w(Q)^{-1/p} \|\tilde{g}_S^A(f_1)\|_{L^p(w)} \le C w(Q)^{-1/p} \|f_1\|_{L^p(w)} \le C\|f\|_{L^\infty(w)}.$$

For $J_2(x)$, note that

$$\begin{aligned}
& \left\|\int_{R^n} (\chi_{\Gamma(x)}(y,t) - \chi_{\Gamma(x_0)}(y,t)) \frac{R_m(\tilde{A}; x_0, z)\psi_t(y-z)}{|x_0 - z|^m} f_2(z) dz\right\| \\
\le\ & C \int_{R^n} \frac{|f_2(z)||R_m(\tilde{A}(x_0, z)|}{|x_0 - z|^m} \left(\int\int_{R_+^{n+1}} |\chi_{\Gamma(x)}(y,t) - \chi_{\Gamma(x_0)}(y,t)|^2 |\psi_t(y-z)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2} dz \\
\le\ & C \int_{R^n} \frac{|f_2(z)||R_m(\tilde{A}(x_0, z)|}{|x_0 - z|^m} \left|\int\int_{\Gamma(x)} \frac{t^{1-n} dydt}{(t + |y-z|)^{2n+2}} - \int\int_{\Gamma(x_0)} \frac{t^{1-n} dydt}{(t + |y-z|)^{2n+2}}\right|^{1/2} dz \\
\le\ & C \int_{R^n} \frac{|f_2(z)||R_m(\tilde{A}(x_0, z)|}{|x_0 - z|^m} \left(\int\int_{|y| \le t} \left|\frac{1}{(t + |x+y-z|)^{2n+2}} - \frac{1}{(t + |x_0 + y - z|)^{2n+2}}\right| \frac{dydt}{t^{n-1}}\right)^{1/2} dz \\
\le\ & C \int_{R^n} \frac{|f_2(z)||R_m(\tilde{A}(x_0, z)|)}{|x_0 - z|^m} \left(\int\int_{|y| \le t} \frac{|x - x_0| t^{1-n} dydt}{(t + |x+y-z|)^{2n+3}}\right)^{1/2} dz \\
\le\ & C \int_{R^n} \frac{|f_2(z)||R_m(\tilde{A}(x_0, z)|)}{|x_0 - z|^m} \frac{|x - x_0|^{1/2}}{|x_0 - z|^{n+1/2}} dz,
\end{aligned}$$

similar to the proof of Lemma 2 and $I_2(x,u)$, we get

$$\begin{aligned}
&\frac{1}{w(Q)}\int_Q J_2(x)w(x)dx\\
\le\ & C\sum_{|\alpha|=m}\frac{\|D^\alpha A\|_{BMO}}{w(Q)}\int_Q\sum_{k=2}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}k\left(\frac{|x-x_0|}{|x_0-z|^{n+1}}+\frac{|x-x_0|^{1/2}}{|x_0-z|^{n+1/2}}+\frac{|x-x_0|^{\varepsilon}}{|x_0-z|^{n+\varepsilon}}\right)|f(z)|dzw(x)dx\\
\le\ & C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\|f\|_{L^\infty(w)}\sum_{k=1}^{\infty}k(2^{-k}+2^{-k/2}+2^{-\varepsilon k})\\
\le\ & C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\|f\|_{L^\infty(w)}.
\end{aligned}$$

For $J_3(x,u)$, since $w\in A_1$, w satisfies the reverse of Holder's inequality:

$$\left(\frac{1}{|Q|}\int_Q w(x)^r dx\right)^{1/r}\le\frac{C}{|Q|}\int_Q w(x)dx$$

for all cube Q and some $1<r<\infty$ (see [9][13]), then

$$\begin{aligned}
&\frac{1}{w(Q)}\int_Q J_3(x,u)w(x)dx\\
\le\ & \sum_{|\alpha|=m}\frac{C}{w(Q)}\int_Q|D^\alpha A(x)-(D^\alpha A)_Q|\sum_{k=2}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}\left(\frac{|z-u|}{|x-z|^{n+1}}+\frac{|y-u|^{\varepsilon}}{|x-y|^{n+\varepsilon}}\right)|f(y)|dyw(x)dx\\
\le\ & C\sum_{|\alpha|=m}\left(\frac{1}{|Q|}\int_Q|D^\alpha A(x)-(D^\alpha A)_Q|^{r'}dx\right)^{1/r'}\left(\frac{1}{|Q|}\int_Q w(x)^r dx\right)^{1/r}|Q|w(Q)^{-1}\\
&\times\sum_{k=2}^{\infty}\left(\frac{d}{(2^kd)^{n+1}}+\frac{d^{\varepsilon}}{(2^kd)^{n+\varepsilon}}\right)|2^kQ|\|f\|_{L^\infty(w)}\\
\le\ & C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\sum_{k=2}^{\infty}(2^{-k}+2^{-\varepsilon k})\|f\|_{L^\infty(w)}\le C\|f\|_{L^\infty(w)}.
\end{aligned}$$

Thus, using the condition of $J_4(x,u)$, we obtain

$$\frac{1}{w(Q)}\int_Q\left|\tilde{g}_S^A(f)(x)-g_S\left(\frac{R_m(\tilde{A};x_0,\cdot)}{|x_0-\cdot|^m}f_2\right)(x_0)\right|w(x)dx\le C\|f\|_{L^\infty(w)}.$$

This completes the proof of Theorem 1.

Proof of Theorem 2(i). It suffices to show that there exists a constant $C>0$ such that for every $H^1(w)$-atom a with supp$a\subset Q=Q(x_0,d)$, there is

$$\|\mu_S^A(a)\|_{L^1(w)}\le C.$$

Let $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A)_{\tilde{Q}} x^\alpha$. We write, by the vanishing moment of a and for $u \in 3Q \setminus 2Q$,

$$\begin{aligned}
&\mu_S^A(a)(x) = \left\|\chi_{\Gamma(x)} F_t^A(a)(x,y)\right\| \le \chi_{4Q}(x)\left\|\chi_{\Gamma(x)} F_t^A(a)(x,y)\right\| \\
&+\chi_{(4Q)^c}(x)\left\|\chi_{\Gamma(x)}\int_{R^n}\left[\frac{R_m(\tilde{A};x,z)}{|x-z|^m}\frac{\Omega(y-z)\chi_{\Gamma(y)}(z,t)}{|y-z|^{n-1}} - \frac{R_m(\tilde{A};x,u)}{|x-u|^m}\frac{\Omega(y-u)\chi_{\Gamma(y)}(u,t)}{|y-u|^{n-1}}\right]a(z)dz\right\| \\
&+\chi_{(4Q)^c}(x)\left\|\sum_{|\alpha|=m}\frac{\chi_{\Gamma(x)}}{\alpha!}\int_{R^n}\left[\frac{(x-z)^\alpha}{|x-z|^m}\frac{\Omega(y-z)\chi_{\Gamma(y)}(z,t)}{|y-z|^{n-1}} - \frac{(x-u)^\alpha}{|x-u|^m}\frac{\Omega(y-u)\chi_{\Gamma(y)}(u,t)}{|y-u|^{n-1}}\right]D^\alpha\tilde{A}(z)a(z)dz\right\| \\
&+\chi_{(4Q)^c}(x)\left\|\chi_{\Gamma(x)}\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n}\frac{(x-u)^\alpha}{|x-u|^m}\frac{\Omega(y-u)}{|y-u|^{n-1}}\chi_{\Gamma(y)}(u,t)D^\alpha\tilde{A}(z)a(z)dy\right\| \\
= \;& K_1(x) + K_2(x,u) + K_3(x,u) + K_4(x,u).
\end{aligned}$$

By the $L^p(w)$-boundedness of μ_S^A, we get

$$\int_{R^n} K_1(x)w(x)dx = \int_{4Q}\mu_S^A(a)(x)w(x)dx \le C\|a\|_{L^\infty(w)}w(Q) \le C.$$

For $K_2(x,u)$, we write

$$\begin{aligned}
&\frac{R_m(\tilde{A};x,z)}{|x-z|^m}\frac{\Omega(y-z)}{|y-z|^{n-1}}\chi_{\Gamma(y)}(z,t) - \frac{R_m(\tilde{A};x,u)}{|x-u|^m}\frac{\Omega(y-u)}{|y-u|^{n-1}}\chi_{\Gamma(y)}(u,t) \\
= \;& (\chi_{\Gamma(y)}(z,t) - \chi_{\Gamma(y)}(u,t))\frac{\Omega(y-z)R_m(\tilde{A};x,z)}{|y-z|^{n-1}|x-z|^m} \\
&+\left[\frac{\Omega(y-z)}{|y-z|^{n-1}} - \frac{\Omega(y-u)}{|y-u|^{n-1}}\right]\frac{R_m(\tilde{A};x,z)}{|x-z|^m}\chi_{\Gamma(y)}(u,t) \\
&+\frac{\Omega(y-u)\chi_{\Gamma(y)}(u,t)}{|y-u|^{n-1}}\left(\frac{R_m(\tilde{A};x,z)}{|x-z|^m} - \frac{R_m(\tilde{A};x,u)}{|x-u|^m}\right).
\end{aligned}$$

By the following inequality (see [15]):

$$\left|\frac{\Omega(y-z)}{|y-z|^{n-1}} - \frac{\Omega(y-u)}{|y-u|^{n-1}}\right| \le C\left(\frac{|z-u|}{|y-z|^n} + \frac{|z-u|^\gamma}{|y-z|^{n-1+\gamma}}\right)$$

and note that

$$\begin{aligned}
&\left\|\chi_{\Gamma(x)}\int_{R^n}(\chi_{\Gamma(y)}(z,t) - \chi_{\Gamma(y)}(u,t))\frac{\Omega(y-z)R_m(\tilde{A};x,z)}{|y-z|^{n-1}|x-z|^m}a(z)dz\right\| \\
\le \;& C\int_{R^n}\frac{|a(z)||R_m(\tilde{A};x,z)|}{|x-z|^m}\left(\int\int_{R_+^{n+1}}\frac{\chi_{\Gamma(x)}(y,t)|\chi_{\Gamma(y)}(z,t) - \chi_{\Gamma(y)}(u,t)|^2}{|y-z|^{2n-2}}\frac{dydt}{t^{n+3}}\right)^{1/2}dz \\
\le \;& C\int_{R^n}\frac{|a(z)||R_m(\tilde{A};x,z)|}{|x-z|^m}\left|\int\int_{\Gamma(x),\,\Gamma(z)}\frac{t^{-n-3}dydt}{|y-z|^{2n-2}} - \int\int_{\Gamma(x),\,\Gamma(u)}\frac{t^{-n-3}dydt}{|y-z|^{2n-2}}\right|^{1/2}dz \\
\le \;& C\int_{R^n}\frac{|a(z)||R_m(\tilde{A};x,z)|}{|x-z|^m}\left(\int\int_{|y|\le t,|x+y-z|\le t}\left|\frac{1}{|x+y-z|^{2n-2}} - \frac{1}{|x+y-u|^{2n-2}}\right|\frac{dydt}{t^{n+3}}\right)^{1/2}dz \\
\le \;& C\int_{R^n}\frac{|a(z)||R_m(\tilde{A};x,z)|}{|x-z|^m}\left(\int\int_{|y|\le t,|x+y-z|\le t}\frac{|u-z|t^{-n-3}dydt}{|x+y-z|^{2n-1}}\right)^{1/2}dz \\
\le \;& C\int_{R^n}\frac{|a(z)||R_m(\tilde{A};x,z)|}{|x-z|^m}\frac{|u-z|^{1/2}}{|x-z|^{n+1/2}}dz,
\end{aligned}$$

similar to the proof of Theorem 1, we obtain

$$\begin{aligned}
&\int_{R^n} K_2(x,u)w(x)dx \\
\leq\ & C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\sum_{k=2}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}\int_Q k\left(\frac{|u-z|}{|x-z|^{n+1}}+\frac{|u-z|^{1/2}}{|x-z|^{n+1/2}}+\frac{|u-z|^{\gamma}}{|x-z|^{n+\gamma}}\right)|a(z)|dzw(x)dx \\
\leq\ & C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\sum_{k=2}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ} k\left(\frac{d}{(2^kd)^{n+1}}+\frac{d^{1/2}}{(2^kd)^{n+1/2}}+\frac{d^{\gamma}}{(2^kd)^{n+\gamma}}\right)\|a\|_{L^\infty(w)}|Q|w(x)dx \\
\leq\ & C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\sum_{k=2}^{\infty}k(2^{-k}+2^{-k/2}+2^{-\gamma k})\frac{w(2^{k+1}Q)}{|2^{k+1}Q|}\frac{|Q|}{w(Q)} \\
\leq\ & C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\sum_{k=2}^{\infty}k(2^{-k}+2^{-k/2}+2^{-\gamma k})\leq C.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&\int_{R^n} K_3(x,u)w(x)dx \\
\leq\ & C\sum_{|\alpha|=m}\sum_{k=2}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}\int_Q \left(\frac{|u-z|}{|x-z|^{n+1}}+\frac{|u-z|^{1/2}}{|x-z|^{n+1/2}}+\frac{|u-z|^{\gamma}}{|x-z|^{n+\gamma}}\right)|D^\alpha\tilde{A}(z)||a(z)|dzw(x)dx \\
\leq\ & C\sum_{|\alpha|=m}\sum_{k=2}^{\infty}\left(\frac{d}{(2^kd)^{n+1}}+\frac{d^{1/2}}{(2^kd)^{n+1/2}}+\frac{d^{\gamma}}{(2^kd)^{n+\gamma}}\right)\left(\frac{1}{|Q|}\int_Q|D^\alpha\tilde{A}(y)|dy\right)\|a\|_{L^\infty(w)}|Q|w(2^{k+1}Q) \\
\leq\ & C.
\end{aligned}$$

Thus, by using the condition of $K_4(x,u)$, we obtain

$$\int_{R^n}\mu_S^A(a)(x)w(x)dx\leq C.$$

(ii). For any cube $Q=Q(x_0,d)$, let $\tilde{A}(x)=A(x)-\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A)_{\tilde{Q}}x^\alpha$. We write, for $f=f\chi_{4Q}+f\chi_{(4Q)^c}=f_1+f_2$ and $u\in 3Q\setminus 2Q$,

$$\begin{aligned}
&\left|\tilde{\mu}_S^A(f)(x)-\mu_S\left(\frac{R_m(\tilde{A};x_0,\cdot)}{|x_0-\cdot|^m}f_2\right)(x_0)\right| = \left|\left\|\chi_{\Gamma(x)}\tilde{F}_t^A(f)(x,y)\right\|-\left\|\chi_{\Gamma(x_0)}F_t\left(\frac{R_m(\tilde{A};x_0,\cdot)}{|x_0-\cdot|^m}f_2\right)(y)\right\|\right| \\
\leq\ & \left\|\chi_{\Gamma(x)}\tilde{F}_t^A(f)(x,y)-\chi_{\Gamma(x_0)}F_t\left(\frac{R_m(\tilde{A};x_0,\cdot)}{|x_0-\cdot|^m}f_2\right)(y)\right\|\leq\left\|\chi_{\Gamma(x)}(y,t)\tilde{F}_t^A(f_1)(x,y)\right\| \\
&+\left\|\chi_{\Gamma(x)}(y,t)\int_{|y-z\leq t}\left[\frac{R_m(\tilde{A};x,z)\Omega(y-z)}{|x-z|^m|y-z|^{n-1}}-\chi_{\Gamma(x_0)}(y,t)\int_{|y-z|\leq t}\frac{R_m(\tilde{A};x_0,z)\Omega(y-z)}{|x_0-z|^m|y-z|^{n-1}}\right]f_2(z)dz\right\| \\
&+\left\|\chi_{\Gamma(x)}(y,t)\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A(x)-(D^\alpha A)_Q)\int_{|y-z|\leq t}\left[\frac{\Omega(y-z)(x-z)^\alpha}{|y-z|^{n-1}|x-z|^m}-\frac{\Omega(y-z)(u-z)^\alpha}{|y-z|^{n-1}|u-z|^m}\right]f_2(z)dz\right\| \\
&+\left\|\chi_{\Gamma(x)}(y,t)\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A(x)-(D^\alpha A)_Q)\int_{|y-z|\leq t}\frac{\Omega(y-z)(u-z)^\alpha}{|y-z|^{n-1}|u-z|^m}f_2(z)dz\right\| \\
=\ & L_1(x)+L_2(x)+L_3(x,u)+L_4(x,u).
\end{aligned}$$

By the $L^p(w)$-boundedness of $\tilde{\mu}_S^A$, we get

$$\frac{1}{w(Q)}\int_Q L_1(x)w(x)dx\leq C\|f\|_{L^\infty(w)}.$$

For $L_2(x)$, we write

$$\chi_{\Gamma(x)}(y,t)\frac{R_m(\tilde{A};x,z)\Omega(y-z)}{|x-z|^m|y-z|^{n-1}}-\chi_{\Gamma(x_0)}(y,t)\frac{R_m(\tilde{A};x_0,z)\Omega(y-z)}{|x_0-z|^m|y-z|^{n-1}}$$
$$=\chi_{\Gamma(x)}(y,t)\left[\frac{R_m(\tilde{A};x,z)}{|x-z|^m}-\frac{R_m(\tilde{A};x_0,z)}{|x_0-z|^m}\right]\frac{\Omega(y-z)}{|y-z|^{n-1}}$$
$$+(\chi_{\Gamma(x)}(y,t)-\chi_{\Gamma(x_0)}(y,t))\frac{R_m(\tilde{A};x_0,z)}{|x_0-z|^m}\frac{\Omega(y-z)}{|y-z|^{n-1}},$$

then, similar to the proof of Lemma 2 and $K_2(x,u)$, we obtain

$$\begin{aligned}&\frac{1}{w(Q)}\int_Q L_2(x)w(x)dx\\ \le\ & C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\sum_{k=2}^\infty\int_{2^{k+1}Q\setminus 2^kQ}k\left(\frac{|x-x_0|}{|x-y|^{n+1}}+\frac{|x-x_0|^{1/2}}{|x-y|^{n+1/2}}+\frac{|x-x_0|^\gamma}{|x-y|^{n+\gamma}}\right)|f(y)|dy\\ \le\ & C\|f\|_{L^\infty(w)}.\end{aligned}$$

Similarly, we get

$$\frac{1}{w(Q)}\int_Q L_3(x,u)w(x)dx\le C\|f\|_{L^\infty(w)}.$$

Thus, by using the condition of $L_4(x,u)$, we obtain

$$\frac{1}{w(Q)}\int_Q\left|\tilde{\mu}_S^A(f)(x)-\mu_S\left(\frac{R_m(\tilde{A};x_0,\cdot)}{|x_0-\cdot|^m}f_2\right)(x_0)\right|w(x)dx\le C\|f\|_{L^\infty(w)}.$$

This completes the proof of Theorem 2.

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