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Analytic Continuation of Functions and Uniformization of Riemann Surfaces

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Abstract. Analytic functions on Riemann surfaces may be represented through an automorphic function on a covering. The convergence depends on the finiteness of the Poincare series of the uniformizing group. It is known also that the harmonic measure of the ideal boundary is related to the convergence of the Fuchsian group. The condition of a finite series representation of the function on effectively closed infinite-genus surfaces with an ideal boundary of zero harmonic measure requires a summation over elements of the Schottky group. Since there is a range in the parameter space such that the series converge, a solution to the problem of analytic continuation will allow the function to be defined over a larger domain in moduli space.

1. Introduction

Superstring amplitudes have been defined over closed Riemann surfaces of arbitrary finite genus, and the summation of these terms yields a finite prediction for the cross-section. The sum of the supermoduli space integrals over the closed Riemann surfaces vanishes, and therefore, it does not represent an observable physical effect. The series expansion of $\langle 0_{out}|S|0_{in}\rangle$ begins with a phase $e^{-i\alpha}$, which is followed by the vanishing vacuum diagrams at finite order, and a contribution from Dirichlet boundaries or infinite-genus Riemann surfaces would be required for unitarity of the scattering matrix.

The string scattering processes may be generalized to include effectively closed infinite-genus surfaces and Dirichlet boundaries for the physical consistency of the theory. The formula for the correlation function on an infinite-genus surface has been defined through an integral over the expectation value of a product of vertex operators over the surface. It would be followed by an integral over the moduli space of a class of infinite-genus surfaces. By contrast with finite genus, string perturbation theory may be restricted to the class of surfaces with ideal boundaries of zero harmonic measure, since it describes the direct limit of the sequence of closed surfaces of finite genus. The integral over this space will require an expression that is valid over the entire class of surfaces. Scattering amplitudes have been derived previously for surfaces uniformized by groups with convergent Poincare series [4], To calculate the moduli space integral, it may be necessary to continue analytically the function given by the Poincare series to regions where the series diverges.

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2. The Poincare Series for a Hexagonal Configuration of Isometric Circles

The Schottky group is defined to be the free group of g generators T_n, $n = 1, ..., g$ and the inverses. The hexagonal configuration consists of equally spaced isometric circles, which number $6\ell - 6$ in the ℓ^{th} level, defined by the limits

$$\epsilon_0 \leq |K_n| \leq \epsilon_0' \qquad \frac{\delta_0}{\sqrt{g}} \leq |\xi_{1n} - \xi_{2n}| \leq \frac{\delta_0'}{\sqrt{g}} \tag{2.1}$$

which is consistent with the genus-independent cut-off on the length of closed geodesics in the regularization of the bosonic string partition function [8]. Separating the elements according to the number of generators in the product, the Poincare series is

$$\sum_{\alpha \neq I} |\gamma_\alpha|^{-2} = 2\sum_{n=1}^{g} |\gamma_\alpha|^{-2} + \sum_{V_\alpha = T^{\pm 1}_{n_{\ell_1}} T^{\pm 1}_{n_{\ell_2}}} |\gamma_\alpha|^{-2} + ..., \tag{2.2}$$

$$2\sum_{n=1}^{g} |\gamma_n|^{-2} < 2g \cdot \frac{\epsilon_0}{(1-\epsilon_0')^2} \frac{\delta_0'^2}{g} = 2\frac{\epsilon_0'}{(1-\epsilon_0')^2}\delta_0'^2 \tag{2.3}$$

The following ratios

$$\frac{|\gamma_{\alpha_1}|^{-2}}{|\gamma_{T_{n_{\ell_2}}}|^{-2}} = |\gamma_{n_{\ell_1}}|^{-2} \left| \frac{\delta_{n_{\ell_1}}}{\gamma_{n_{\ell_1}}} + \frac{\alpha_{n_{\ell_2}}}{\gamma_{n_{\ell_2}}} \right|^{-2} \quad when\ V_{\alpha_1} = T_{n_{\ell_1}} T_{n_{\ell_2}} \tag{2.4}$$

$$\frac{|\gamma_{\alpha_2}|^{-2}}{|\gamma_{T_{n_{\ell_2}}}|^{-2}} = |\gamma_{n_{\ell_1}}|^{-2} \left| -\frac{\alpha_{n_{\ell_1}}}{\gamma_{n_{\ell_1}}} + \frac{\alpha_{n_{\ell_2}}}{\gamma_{n_{\ell_2}}} \right|^{-2} \quad when\ V_{\alpha_2} = T^{-1}_{n_{\ell_1}} T_{n_{\ell_2}}$$

$$\frac{|\gamma_{\alpha_3}|^{-2}}{|\gamma_{T_{n_{\ell_2}}}|^{-2}} = |\gamma_{n_{\ell_1}}|^{-2} \left| \frac{\delta_{n_{\ell_1}}}{\gamma_{n_{\ell_1}}} - \frac{\delta_{n_{\ell_2}}}{\gamma_{n_{\ell_2}}} \right|^{-2} \quad when\ V_{\alpha_3} = T_{n_{\ell_1}} T^{-1}_{n_{\ell_2}}$$

$$\frac{|\gamma_{\alpha_4}|^{-2}}{|\gamma_{T_{n_{\ell_2}}}|^{-2}} = |\gamma_{n_{\ell_1}}|^{-2} \left| \frac{\alpha_{n_{\ell_1}}}{\gamma_{n_{\ell_1}}} + \frac{\delta_{n_{\ell_2}}}{\gamma_{n_{\ell_2}}} \right|^{-2} \quad when\ V_{\alpha_4} = T^{-1}_{n_{\ell_1}} T^{-1}_{n_{\ell_2}}$$

$$\frac{|\gamma_\alpha|^{-2}}{|\gamma_\beta|^{-2}} = |\gamma_{n_\ell}|^{-2} \left| \frac{\delta_{n_\ell}}{\gamma_{n_\ell}} + \frac{\alpha_\beta}{\gamma_\beta} \right|^{-2} \quad when\ V_\alpha = T_{n_\ell} V_\beta$$

are sufficient to derive an upper bound for the sum over products of two elements in the Schottky group.

Theorem 2.1 The upper bound for the sum over the product of two elements of the squares of the inverse radii of the isometric circles of the Schottky group is a double sum of a quadratic function of level numbers in a hexagonal configuration with the multipliers of $O(1)$ and the distance between the fixed points of $O\left(\frac{1}{\sqrt{g}}\right)$.

Proof.

When the isometric circles $I_{T_{n_\ell}}$ and $I_{T^{-1}_{n_\ell}}$ are spaced by a distance of $O\left(\frac{1}{\sqrt{g}}\right)$, with $\ell \equiv \ell_{T^{-1}_{n_\ell}}$ and $\ell + \ell_0 \equiv \ell_{T_{n_{\ell_2}}}$,

$$\left| \frac{\alpha_{T_{n_{\ell_1}}}}{\gamma_{T_{n_{\ell_1}}}} + \frac{\delta_{T_{n_{\ell_2}}}}{\gamma_{T_{n_{\ell_2}}}} \right| > [\ell_{T_{n_{\ell_2}}} - \ell_{T^{-1}_{n_{\ell_1}}}] \frac{\delta_0}{\sqrt{g}} = \ell_0 \frac{\delta_0}{\sqrt{g}} \tag{2.5}$$

When $|\xi_{1_{n_\ell}} - \xi_{2_{n_\ell}}| \leq \frac{\delta_0'}{\sqrt{g}}$,

$$\left|\frac{\delta_{T_{n_{\ell_1}}}}{\gamma_{T_{n_{\ell_1}}}} + \frac{\alpha_{T_{n_{\ell_1}}}}{\gamma_{T_{n_{\ell_1}}}}\right| \leq \frac{\delta_0'}{\sqrt{g}} + 2|\gamma_n|^{-1} \leq \frac{\delta_0'}{\sqrt{g}}\left[1 + \frac{2\epsilon_0'^{\frac{1}{2}}}{1-\epsilon_0'}\right] \tag{2.6}$$

$$\left|\frac{\delta_{T_{n_{\ell_2}}}}{\gamma_{T_{n_{\ell_2}}}} + \frac{\alpha_{T_{n_{\ell_2}}}}{\gamma_{T_{n_{\ell_2}}}}\right| \leq \frac{\delta_0'}{\sqrt{g}}\left[1 + \frac{2\epsilon_0'^{\frac{1}{2}}}{1-\epsilon_0'}\right]$$

By this inequality,

$$\left|\frac{\delta_{T_{n_{\ell_1}}}}{\gamma_{T_{n_{\ell_1}}}} + \frac{\alpha_{T_{n_{\ell_2}}}}{\gamma_{T_{n_{\ell_2}}}}\right| > \left|\frac{\alpha_{T_{n_{\ell_1}}}}{\gamma_{T_{n_{\ell_1}}}} + \frac{\delta_{T_{n_{\ell_2}}}}{\gamma_{T_{n_{\ell_2}}}}\right| - \left|\frac{\delta_{T_{n_{\ell_1}}}}{\gamma_{T_{n_{\ell_1}}}} + \frac{\alpha_{T_{n_{\ell_1}}}}{\gamma_{T_{n_{\ell 1}}}}\right| - \left|\frac{\delta_{T_{n_{\ell_2}}}}{\gamma_{T_{n_{\ell_2}}}} + \frac{\alpha_{T_{n_{\ell_2}}}}{\gamma_{T_{n_{\ell 2}}}}\right|$$

$$> \ell_0\frac{\delta_0}{\sqrt{g}} - 2\frac{\delta_0'}{\sqrt{g}}\left[1 + \frac{2\epsilon_0'^{\frac{1}{2}}}{1-\epsilon_0'}\right].$$

With this bound

$$\left|\frac{\delta_{T_{n_{\ell_1}}}}{\gamma_{T_{n_{\ell_1}}}} + \frac{\alpha_{T_{n_{\ell_2}}}}{\gamma_{T_{n_{\ell_2}}}}\right|^{-2} \leq \frac{g}{\delta_0^2} \qquad \textit{when } \ell_0 \leq \left\{2\frac{\delta_0'}{\delta_0}\left(1 + \frac{2\epsilon_0'^{\frac{1}{2}}}{1-\epsilon_0'}\right)\right\} \tag{2.7}$$

$$\left|\frac{\delta_{T_{n_{\ell_1}}}}{\gamma_{T_{n_{\ell_1}}}} + \frac{\alpha_{T_{n_{\ell_2}}}}{\gamma_{T_{n_{\ell_2}}}}\right|^{-2} \leq \frac{g}{\delta_0^2}\left[\ell_0 - 2\frac{\delta_0'}{\delta_0}\left(1 + \frac{2\epsilon_0'^{\frac{1}{2}}}{1-\epsilon_0'}\right)^{-2}\right] \qquad \ell_0 \geq \left\{2\frac{\delta_0'}{\delta_0}\left(1 + \frac{2\epsilon_0'^{\frac{1}{2}}}{1-\epsilon_0'}\right)\right\} + 1.$$

Counting the number of circles at level ℓ, and the total number of levels, the upper bound for the sum over elements V_α, that are products of two generators, is

$$\sum_{V_\alpha = T^{\pm 1}_{n_{\ell_1}} T^{\pm 1}_{n_{\ell_2}}} |\gamma_\alpha|^{-2} < 36\frac{\epsilon_0'^2}{(1-\epsilon_0')^4}\frac{\delta_0'^4}{\delta_0^2}\frac{1}{g}\sum_{\ell=1}^{\lfloor -\frac{1}{2}+\frac{1}{6}\sqrt{9+24g}\rfloor} \ell^2$$

$$+36\frac{\epsilon_0'^2}{(1-\epsilon_0')^4}\frac{\delta_0'^4}{\delta_0^2}\sum_{\ell_0=1}^{\left\{2\frac{\delta_0'}{\delta_0}\left[1+\frac{2\epsilon_0'^{\frac{1}{2}}}{1-\epsilon_0'}\right]\right\}}\sum_{\ell=1}^{\lfloor -\frac{1}{2}+\frac{1}{6}\sqrt{9+24g}\rfloor - \ell_0} \ell(\ell+\ell_0)$$

$$+36\frac{\epsilon_0'^2}{(1-\epsilon_0)'^4}\frac{\delta_0'^4}{\delta_0^2}\frac{1}{g}\sum_{\ell_0=1}^{\left\{2\frac{\delta_0'}{\delta_0}\left[1+\frac{2\epsilon_0'^{\frac{1}{2}}}{1-\epsilon_0'}\right\}\right.}\sum_{\ell=1}^{\ell_0}\ell\left[\ell + \lfloor -\frac{1}{2}+\frac{1}{6}\sqrt{9+24g} - \ell_0\rfloor\right]$$

$$+36\frac{\epsilon_0'^2}{(1-\epsilon_0)'^4}\frac{\delta_0'^4}{\delta_0^2}\frac{1}{g}\sum_{\ell_0=\left\{2\frac{\delta_0'}{\delta_0}\left[1+\frac{2\epsilon_0'^{\frac{1}{2}}}{1-\epsilon_0'}\right]\right\}}^{\lfloor -\frac{1}{2}+\frac{1}{6}\sqrt{9+24g}\rfloor - 1}\sum_{\ell=1}^{\lfloor -\frac{1}{2}+\frac{1}{6}\sqrt{9+24g}\rfloor - \ell_0}\frac{\ell(\ell+\ell_0)}{\left[\ell_0 - 2\frac{\delta_0'}{\delta_0}\left[1+\frac{2\epsilon_0'^{\frac{1}{2}}}{1-\epsilon_0'}\right]\right]^2}$$

$$+36\frac{\epsilon_0'^2}{(1-\epsilon_0)'^4}\frac{\delta_0'^4}{\delta_0^2}\frac{1}{g}\sum_{\ell_0=\left\{2\frac{\delta_0'}{\delta_0}\left[1+\frac{2\epsilon_0'^{\frac{1}{2}}}{1-\epsilon_0'}\right]\right\}}^{\lfloor-\frac{1}{2}+\frac{1}{6}\sqrt{9+24g}\rfloor-1}\sum_{\ell=1}^{\ell_0}\frac{\ell\left(\ell+\lfloor-\frac{1}{2}+\frac{1}{6}\sqrt{9+24g}\rfloor-\ell_0\right)}{\left[\ell_0-2\frac{\delta_0'}{\delta_0}\left[1+\frac{2\epsilon_0'^{\frac{1}{2}}}{1-\epsilon_0'}\right]\right]^2}. \tag{2.8}$$

□

The second technique for obtaining a bound makes use of the slight alteration in the sums upon counting levels with respect to a point located away from the center for the hexagonal configuration. Instead of bounding the distance between the centre of $I_{T_{n_{\ell_1}}}$ and the center of $I_{T^{-1}_{n_{\ell_2}}}$, where $I_{T^{-1}_{n_{\ell_1}}}$ is located at level ℓ and $I_{T_{n_{\ell_2}}}$ belongs to level $\ell+\ell_0$, by $\ell_0\frac{\delta_0}{\sqrt{g}}-2\frac{\delta_0'}{\sqrt{g}}\left[1+\frac{2\epsilon_0'^{\frac{1}{2}}}{1-\epsilon_0'}\right]$, or $\frac{\delta_0}{\sqrt{g}}$ when $\ell_0\leq\left\{2\frac{\delta_0'}{\delta_0}\left[1+\frac{2\epsilon_0'^{\frac{1}{2}}}{1-\epsilon_0'}\right]\right\}$. it is useful for the distances at a level $\ell+\ell_0$ increase as the circles are counted from the closest circle at that level to that furthest away. As there are $6[(\ell+\ell_0)-1]$ circles included at this level, it is better to replace that factor in the numerator by a sum over inverse square distances estimating $\left|\frac{\delta_{T_{n_{\ell_1}}}}{\gamma_{T_{n_{\ell_1}}}}+\frac{\alpha_{T_{n_{\ell_2}}}}{\gamma_{T_{n_{\ell_2}}}}\right|$. Summing over the circles at level $\ell+\ell_0$, the distances range from approximately $\ell_0\frac{\delta_0}{\sqrt{g}}-2\frac{\delta_0'}{\delta_0}\left[1+\frac{2\epsilon_0'^{\frac{1}{2}}}{1-\epsilon_0'}\right]$ to $(2\ell+\ell_0)\frac{\delta_0}{\sqrt{g}}-2\frac{\delta_0'}{\sqrt{g}}\left[1+\frac{2\epsilon_0'^{\frac{1}{2}}}{1-\epsilon_0'}\right]$.

Theorem 2.2. The sum over the product of two elements included in the Poincare series for the hexagonal configuration of isometric circles of the Schottky group, with $\epsilon_0\leq|K_n|\leq\epsilon_0'$ and $\frac{\delta_0}{\sqrt{g}}\leq|\xi_{1n}-\xi_{2n}|\leq\frac{\delta_0'}{\sqrt{g}}$, has the upper bound

$$\sum_{V_\alpha=T^{\pm1}_{n_{\ell_1}}T^{\pm1}_{n_{\ell_2}}}|\gamma_\alpha|^{-2}<36\frac{\epsilon_0'^2}{(1-\epsilon_0')^4}\frac{\delta_0'^4}{\delta_0^2}\left[2-\sqrt{\frac{9}{g^2}+\frac{24}{g}}+\frac{3}{g}\right] \tag{2.9}$$

$$\left[\psi\left(-\left\lfloor\frac{1}{2}+\frac{1}{6}\sqrt{9+24g}\right\rfloor\right)\rfloor-\psi(1)\right].$$

Proof. In the calculation of the sum over the inverse square distances, the terms could be derived from an equal division of the range $\left[\ell_0\frac{\delta_0}{\sqrt{g}}-2\frac{\delta_0'}{\sqrt{g}}\left[1+\frac{2\epsilon_0'^{\frac{1}{2}}}{1-\epsilon_0'}\right],(2\ell+\ell_0)\frac{\delta_0}{\sqrt{g}}-2\frac{\delta_0'}{\sqrt{g}}\left[1+\frac{2\epsilon_0'^{\frac{1}{2}}}{1-\epsilon_0'}\right]\right]$, although a more accurate estimate involves the angles at which the lines from the other circles at level $\ell+\ell_0$ to the closest circle at level $\ell+\ell_0$ intersect the perpendicular from the selected circle at level ℓ. In a hexagonal configuration, the gradation in the angle is equal to $\frac{\frac{\pi}{3}}{3(\ell-1)}$ at level ℓ. Thus, the angle θ^j between the points at levels ℓ and $\ell+\ell_0$ is given by $\frac{j\pi}{3(\ell-1)}$, $j=0,1,2,...,6[(\ell+\ell_0)-1]$. Thus, the distance $d^{(j)}_{\ell,\ell+\ell_+0}$ is equal to the absolute value of the vector separating the two points,

$$d^{(j)}_{\ell,\ell+\ell_0}=\sqrt{d^2_{\ell+\ell_0}+d^2_\ell-2d_\ell d_{\ell+\ell_0}cos\left(\frac{j\pi}{3[(\ell+\ell_0)-1]}\right)} \tag{2.10}$$

As the minimum value for the radius of the isometric circle is $\frac{\epsilon_0^{\frac{1}{2}}}{1+\epsilon_0}\frac{\delta_0}{\sqrt{g}}$, the distance d_ℓ is greater than $(\ell-1)\frac{\delta_0}{\sqrt{g}}\left[1+\frac{2\epsilon_0^{\frac{1}{2}}}{1+\epsilon_0}\right]-\frac{\delta_0}{\sqrt{g}}\frac{\epsilon_0^{\frac{1}{2}}}{1+\epsilon_0}$. The minimum value for $d^{(j)}_{\ell,\ell+\ell_0}$ as the index runs from 0 to $6(\ell-1)$ is $d_{\ell+\ell_0}-d_\ell$, and this is bounded below by $(\ell_0-1)\frac{\delta_0}{\sqrt{g}}+(2\ell_0-3)\frac{\delta_0}{\sqrt{g}}\frac{\epsilon_0^{\frac{1}{2}}}{1+\epsilon_0}$.

The distance also equals

$$d^{(j)}_{\ell,\ell+\ell_0} = \sqrt{2}(d_\ell d_{\ell_0})^{\frac{1}{2}}\left[1+\frac{(d_{\ell+\ell_0}-d_\ell)^2}{2d_\ell d_{\ell_0} cos\left(\frac{j\pi}{2[(\ell+\ell_0)-1]}\right)}\right] \tag{2.11}$$

$$= (d_\ell d_{\ell+\ell_0})^{\frac{1}{2}}\frac{j\pi}{3[(\ell+\ell_0)-1]}$$

$$\sqrt{1-\frac{1}{12}\left(\frac{j\pi}{3[(\ell+\ell_0)-1]}\right)^2+...+\left(\frac{3[(\ell+\ell_0)-1]}{j\pi}\right)^2\frac{(d_{\ell+\ell_0}-d_\ell)^2}{d_\ell d_{\ell+\ell_0}}}.$$

Since

$$\left(\frac{\ell+\ell_0-1}{j}\right)^2\frac{(d_{\ell+\ell_0}-d_\ell)^2}{d_\ell d_{\ell+\ell_0}} \sim \frac{(\ell+\ell_0-1)(\ell_0-1)^2}{j^2(\ell-1)} \tag{2.12}$$

$$d^{(j)}_{\ell,\ell+\ell_0} \simeq (d_\ell d_{\ell+\ell_0})^{\frac{1}{2}}\left(\frac{j\pi}{3(\ell+\ell_0-1)}\right) \tag{2.13}$$

$$\sqrt{1+\frac{1}{12}\left(\frac{j\pi}{3(\ell+\ell_0-1)}\right)^2+...+\frac{3}{\pi}\frac{(\ell+\ell_0-1)(\ell_0-1)^2}{j^2(\ell-1)}}$$

The radical may be evaluated under several conditions. When $\ell_0 \gg \ell$ and $j \ll \ell+\ell_0-1$, then

$$\sqrt{1+\frac{1}{12}\left(\frac{j\pi}{3(\ell+\ell_0-1)}\right)^2+...+\frac{3}{\pi}\frac{(\ell+\ell_0-1)(\ell_0-1)^2}{j^2(\ell-1)}} \simeq \frac{3}{j\pi}\frac{(\ell_0-1)^{\frac{3}{2}}}{(\ell-1)^{\frac{1}{2}}}. \tag{2.14}$$

If $\ell_0 \gg \ell$ and $j = \lambda_j(\ell+\ell_0-1)$, $\lambda_j = O(1)$, then

$$\sqrt{1+\frac{1}{12}\left(\frac{j\pi}{3(\ell+\ell_0-1)}\right)^2+...+\left(\frac{3}{\pi}\right)^2\frac{(\ell+\ell_0-1)(\ell_0-1)^2}{j^2(\ell-1)}} \simeq \sqrt{1+\left(\frac{3}{\pi}\right)^2\frac{(\ell_0-1)^2}{j^2}}. \tag{2.15}$$

If $\ell_0 \ll \ell$ and $j = \lambda_j(\ell+\ell_0-1)$,

$$\sqrt{1+\frac{1}{12}\left(\frac{j\pi}{3(\ell+\ell_0-1)}\right)^2+...+\left(\frac{3}{\pi}\right)^2\frac{(\ell+\ell_0-1)(\ell_0-1)^2}{j^2(\ell-1)}} \tag{2.16}$$

$$= \sqrt{1+\frac{\lambda_j^2}{12}\left(\frac{\pi}{3}\right)^2-\frac{\lambda_j^4}{360}\left(\frac{\pi}{3}\right)^4+\frac{\lambda_j^6}{20160}\left(\frac{\pi}{3}\right)^6}$$

When $\ell_0 = \kappa_{\ell_0}\ell$ and $j = \lambda_j[\ell_0+\ell-1]$, then

$$\sqrt{1+\frac{1}{12}\left(\frac{j\pi}{3(\ell+\ell_0-1)}\right)^2+...+\left(\frac{3}{\pi}\right)^2\frac{(\ell+\ell_0-1)(\ell_0-1)^2}{j^2(\ell-1)}} \tag{2.17}$$

$$= \sqrt{1-\frac{\lambda_j^2}{12}\left(\frac{\pi}{3}\right)^2+\frac{\lambda_j^2}{360}\left(\frac{\pi}{3}\right)^4-\frac{\lambda_j^6}{20160}\left(\frac{\pi}{3}\right)^6+\left(\frac{3}{\pi}\right)^2\frac{\kappa_{\ell_0}^2}{\lambda_j^2(1+\kappa_{\ell_0})}}$$

Each of these factors multiply $(d_\ell d_{\ell+\ell_0})^{\frac{1}{2}} \frac{j\pi}{3(\ell+\ell_0-1)}$, which increases linearly with j.

When $j = 3(\ell + \ell_0 - 1)$, $\lambda_j = 3$, and if $\ell_0 \ll \ell$, the distance is almost equal to

$$(d_\ell d_{\ell+\ell_0})^{\frac{1}{2}} \pi \sqrt{1 - \frac{\pi^2}{12} + \frac{\pi^4}{360} - \frac{\pi^6}{20160}} \simeq 1.727 (d_\ell d_{\ell+\ell_0})^{\frac{1}{2}} \tag{2.18}$$

If $\ell_0 = \kappa_{\ell_0} \ell$, $\lambda_j = 3$, the distance would equal

$$(d_\ell d_{\ell+\ell_0})^{\frac{1}{2}} \pi \sqrt{0.3014 + \frac{\kappa_{\ell_0}^2}{\pi^2 (1 + \kappa_{\ell_0})}}. \tag{2.19}$$

The geometric mean $(d_\ell d_{\ell+\ell_0})^{\frac{1}{2}}$ is less than the arithmetic mean $\frac{d_\ell + d_{\ell_0}}{2}$, which would account for the balancing of the increase of the square-root factor resulting from a larger κ_{ℓ_0}.

When $\ell_0 \gg \ell$,

$$\frac{(d_\ell d_{\ell+\ell_0})^{\frac{1}{2}} \pi \cdot \ell_0 - 1}{\pi (\ell + \ell_0 - 1)^{\frac{1}{2}} (\ell - 1)^{\frac{1}{2}}} \simeq [\ell(\ell + \ell_0 - 1)]^{\frac{1}{2}} \frac{\delta_0}{\sqrt{g}} \frac{\ell_0 - 1}{(\ell + \ell_0 - 1)^{\frac{1}{2}} (\ell - 1)^{\frac{1}{2}}} \simeq (\ell_0 - 1) \frac{\delta_0}{\sqrt{g}} \tag{2.20}$$

which has a similar form to the exact formula

$$d^{(j)}_{\ell,\ell+\ell_0} \Big|_{j=3(\ell+\ell_0-1)} = \sqrt{d^2_{\ell+\ell_0} + d^2_\ell + 2 d_\ell d_{\ell+\ell_0}} = d_{\ell+\ell_0} + d_\ell \simeq (2\ell + \ell_0) \frac{\delta_0}{\sqrt{g}}. \tag{2.21}$$

Summation over the set of isometric circles at level $\ell + \ell_0$ gives

$$\begin{aligned} \sum_{j=0}^{6(\ell+\ell_0)-7} |d_{\ell,\ell+\ell_0}|^{-2} = \frac{9(\ell + \ell_0 - 1)^2}{\pi^2 d_\ell d_{\ell+\ell_0}} \sum_{j=0}^{\lambda_{j'}(\ell+\ell_0-1)} \frac{1}{j^2} \Big[1 - \frac{1}{12} \left(\frac{j\pi}{3(\ell + \ell_0 - 1)} \right)^2 \\ + \left(\frac{3(\ell + \ell_0 - 1)}{j\pi} \right)^2 \frac{(d_{\ell+\ell_0} - d_\ell)^2}{d_\ell d_{\ell+\ell_0}} \Big]^{-1} \\ + \frac{9(\ell + \ell_0 - 1)^2}{\pi^2 d_\ell d_{\ell+\ell_0}} \sum_{\lambda_{j'}(\ell+\ell_0-1)}^{6(\ell+\ell_0)-7} \frac{1}{j^2} \Big[1 - \frac{1}{12} \left(\frac{j\pi}{3(\ell + \ell_0 - 1)} \right)^2 \\ + \left(\frac{3(\ell + \ell_0 - 1)}{j\pi} \right)^2 \frac{(d_{\ell+\ell_0} - d_\ell)^2}{d_\ell d_{\ell+\ell_0}} \Big]^{-1} \end{aligned} \tag{2.22}$$

when $\lambda_{j'} = O(1)$ can be chosen to be less than 1.

When $j \ll \ell + \ell_0 - 1$, the terms dominating the series are

$$\frac{9(\ell + \ell_0 - 1)^2}{\pi^2 d_\ell d_{\ell+\ell_0}} \sum_{j \ll \ell+\ell_0-1} \frac{1}{j^2} \frac{j^2 \pi^2}{9(\ell + \ell_0 - 1)^2} \frac{d_\ell d_{\ell+\ell_0}}{(d_{\ell+\ell_0} - d_\ell)^2} = \sum_{j \ll \ell+\ell_0-1} \frac{1}{(d_{\ell+\ell_0} - d_\ell)^2}. \tag{2.23}$$

When $d_{\ell+\ell_0} - d_\ell \ll (d_\ell d_{\ell+\ell_0})^{\frac{1}{2}}$, the second sum is dominated by

$$\frac{9(\ell + \ell_0 - 1)^2}{\pi^2 d_\ell d_{\ell+\ell_0}} \sum_{\lambda_{j'}(\ell+\ell_0-1)}^{6(\ell+\ell_0)-7} \frac{1}{j^2} \left[1 - \frac{1}{12} \left(\frac{\lambda_j \pi}{3} \right)^2 + ... \right]^{-1}. \tag{2.24}$$

Over the interval $[\lambda_{j'}(\ell + \ell_0 - 1), 6(\ell + \ell_0) - 7]$, the sum can be estimated by considering the integral

$$\begin{aligned} \int \frac{dx}{x^2} \Big[1 - \frac{1}{12} \left(\frac{x\pi}{3(\ell + \ell_0 - 1)} \right)^2 + ... + \left(\frac{3(\ell + \ell_0 - 1)}{x\pi} \right)^2 \left(\frac{(d_{\ell+\ell_0} - d_\ell)^2}{d_\ell d_{\ell+\ell_0}} \right) \Big]^{-1} \\ \simeq \int \frac{dx}{\left[\left(\frac{3(\ell+\ell_0-1)}{\pi} \right)^2 \frac{(d_{\ell+\ell_0} - d_\ell)^2}{d_\ell d_{\ell+\ell_0}} + x^2 - \frac{1}{12} \left(\frac{\pi}{3(\ell+\ell_0-1)} \right)^2 x^4 + \frac{1}{360} \left(\frac{\pi}{3(\ell+\ell_0-1)} \right)^4 x^6 \right]}. \end{aligned} \tag{2.25}$$

As $d^2_{\ell+\ell_0} + d^2_\ell > 2d_\ell d_{\ell+\ell_0}$, it follows that

$$\int \frac{dx}{\left[d^2_{\ell+\ell_0} + d^2_\ell - 2d_\ell d_{\ell+\ell_0} cos\left(\frac{x\pi}{3[\ell+\ell_0-1]}\right)\right]} \tag{2.26}$$
$$= \frac{6[\ell+\ell_0-1]}{\pi}\frac{1}{(d^2_{\ell+\ell_0} - d^2_\ell)} arc\ tan\left[\frac{d_{\ell+\ell_0}+d_\ell}{d_{\ell+\ell_0}-d_\ell} tan \frac{x\pi}{3[\ell+\ell_0-1]}\right].$$

When $\ell_0 \ll \ell$,

$$arc\ tan\left[\frac{d_{\ell+\ell_0}+d_\ell}{d_{\ell+\ell_0}} tan\left(\frac{x\pi}{6[\ell+\ell_0-1]}\right)\right|_{x=6(\ell+\ell_0)-7} \simeq arc\ tan\left[\frac{2\ell+\ell_0}{\ell_0}\frac{\pi}{6[\ell+\ell_0-1]}\right], \tag{2.27}$$

which is less than $arc\ tan\left[\frac{\pi}{3\ell_0}\right]$ as the tangent function is monotonically increasing in the interval $\left[0, \frac{\pi}{2}\right]$.

When $\ell_0 \gg 1$,

$$\frac{6[\ell+\ell_0-1]}{\pi}\frac{1}{(d^2_{\ell+\ell_0} - d^2_\ell} \cdot arc\ tan\left[\frac{2\ell+\ell_0}{\ell_0}\frac{\pi}{6[\ell+\ell_0-1]} \simeq \frac{2\ell+\ell_0}{\ell_0}\frac{1}{(d^2_{\ell+\ell_0} - d^2_\ell)}. \tag{2.28}$$

Given that $d_{\ell+\ell_0} - d_\ell \simeq \ell_0 \frac{\delta_0}{\sqrt{g}}$, the following bound

$$\sum_{V_\alpha = T^{\pm 1}_{n_{\ell_1}} T^{\pm 1}_{n_{\ell_2}}} |\gamma_\alpha|^{-2} < \frac{36\epsilon'^2_0}{(1-\epsilon'_0)^4}\frac{\delta'^4_0}{g^2} \sum_{\ell_0}^{\lfloor -\frac{1}{2}+\frac{1}{6}\sqrt{9+24g}\rfloor - 1 \sum_{\ell=2}^{\lfloor -\frac{1}{2}+\frac{1}{6}\sqrt{9+24g}\rfloor - 1} \frac{g}{\delta_0^2}\frac{\ell-1}{\ell_0^2} F(\ell,\ell_0)} \tag{2.29}$$

can be given, where $\frac{1}{\ell_0^2} F(\ell, \ell_0)$ is a function determined by

$$2\int_0^{3(\ell+\ell_0-1)} \frac{dx}{\left[d^2_{\ell+\ell_0} + d^2_\ell - 2d_\ell d_{\ell+\ell_0} cos\left(\frac{x\pi}{3(\ell+\ell_0-1)}\right)\right]} = \frac{6(\ell+\ell_0-1)}{(d_{\ell+\ell_0}-d_\ell)(d_{\ell+\ell_0}+d_\ell)} \tag{2.30}$$
$$< \frac{3}{(d_{\ell+\ell_0}-d_\ell)}\frac{\sqrt{g}}{\delta_0} + \frac{3\ell_0}{d^2_{\ell+\ell_0} - d^2_\ell}$$
$$< \frac{3g}{\delta_0^2}\left[\frac{1}{\ell_0} + \frac{1}{2\ell+\ell_0}\right].$$

The dominant contribution to the sum over ℓ is derived from the first term. Since

$$\sum_{\ell=2}^{\lfloor -\frac{1}{2}+\frac{1}{6}\sqrt{9+24g}\rfloor} [\ell-1] = \frac{\lfloor -\frac{1}{2}+\frac{1}{6}\sqrt{9+24g}\rfloor\left(\lfloor -\frac{1}{2}+\frac{1}{6}\sqrt{9+24g}\rfloor - 1\right)}{2} \tag{2.31}$$
$$\simeq \frac{1}{2}\left[1+\frac{2}{3}g - \frac{1}{3}\sqrt{9+24g}\right].$$

the upper bound is nearly

$$36\frac{\epsilon'_0}{[1-\epsilon'_0]^4}\frac{\delta'^4_0}{g^2}\frac{g}{\delta_0^2} \cdot \frac{3}{2}\left[1+\frac{2}{3}g-\frac{1}{3}\sqrt{9+24g}\right] \sum_{\ell_0=0}^{\lfloor -\frac{1}{2}+\frac{1}{6}\sqrt{9+24g}\rfloor} \frac{1}{\ell_0+1} \tag{2.32}$$
$$= 18\frac{\epsilon'^2_0}{(1-\epsilon'_0)^4}\frac{\delta_0 -'^4}{\delta_0^2}\left[2 - \sqrt{\frac{9}{g^2}+24g} + \frac{3}{g}\right]\left[\psi\left(\left\lfloor -\frac{1}{2}+\frac{1}{6}\sqrt{9+24g}\right\rfloor\right) - \psi(1)\right].$$

□

The sum of the magnitudes of the multipliers is $\sum_\alpha{}'|K_\alpha| = \sum_\alpha{}' \frac{|\gamma_\alpha|^{-2}}{|\xi_{1\alpha}+\frac{\delta_\alpha}{\gamma_\alpha}|^2}$. With $\xi_{1\alpha} \in D_{V_\alpha^{-1}\subset D_{T^{-1}_{n_{\ell_1}}}}$ and $-\frac{\delta_\alpha}{\gamma_\alpha} \in D_{V_\alpha} \subset D_{T_{n_{\ell 2}}}$ when $V_\alpha = T_{n_{\ell_1}} T_{n_{\ell_2}}$, $\left|\xi_{1\alpha} + \frac{\delta_\alpha}{\gamma_\alpha}\right| > \ell_0 \frac{\delta_0}{\sqrt{g}}$.

Theorem 2.3 The upper bound for the sum of the magnitudes of the multiplies over the products of two elements in the Schottky group is given by

$$\sum_{V_\alpha = T^{\pm1}_{n_{\ell_1}} T^{\pm1}_{n_{\ell_2}}} |K_\alpha| < \epsilon_0'^2 \left|1 - \frac{\epsilon_0'}{g^{1-2g}}\right|^{-4} \frac{\delta_0'^4}{g^2} \sum_{\ell_0}^{\lfloor -\frac{1}{2}+\frac{1}{6}\sqrt{9+24g}-1\rfloor} \sum_{\ell=2}^{\lfloor -\frac{1}{2}+\frac{1}{6}\sqrt{9+24g}\rfloor -1} \tag{2.33}$$
$$\sum_{j_1=0}^{6(\ell-1)} \sum_{j_2=0}^{6(\ell+\ell_0)-7} [d(I_{T^{-1}_{n_{\ell_1},j_1}}, \{I_{T_{n_{\ell_2}}}\})]^{-2} \left[d^2_{\ell+\ell_0} + d^2_\ell - 2d_\ell d_{\ell+\ell_0} cos\left(\frac{j_2\pi}{3(\ell+\ell_0-1)}\right)\right]^{-1}$$

for the hexagonal configuration of isometric circles with $|K_n| = O(1)$ and $|\xi_{1n} - \xi_{2n}| = O\left(\frac{1}{\sqrt{g}}\right)$, $n = 1, ..., g$.

Proof. The bound on the sum $\sum_\alpha{}'|K_\alpha|$ has the form

$$36 \frac{\epsilon_0'^2}{[1-\epsilon_0']^4} \frac{\delta_0'^4}{\delta_0^2} \left[2 - \sqrt{\frac{9}{g^2} + \frac{24}{g}} + \frac{3}{g}\right] \frac{g}{\delta_0^2} \sum_{\ell_0}^{\lfloor -\frac{1}{2}+\frac{1}{6}\sqrt{9+24g}\rfloor -1} \frac{1}{(\ell_0+1)^3} \tag{2.34}$$

which increases linearly with g as $g \to \infty$. Thus the exponentially decreasing lower bound for $\prod_\alpha{}'|1-K_\alpha|^{-1}$ is valid.

Suppose that the limits for the absolute values of the multipliers and the fixed point distances are $\frac{\epsilon_0}{g^{1-2q}} < |K_n| < \frac{\epsilon_0'}{g^{1-2q}}$, $\frac{\delta_0}{g^q} < |\xi_{1n} - \xi_{2n}| < \frac{\delta_0'}{g^q}$. Then

$$2\sum_{n=1}^{g} |\gamma_n|^{-2} < 2g \frac{\epsilon_0'}{g^{1-2q}} \left|1 - \frac{\epsilon_0'}{g^{1-2q}}\right|^{-2} \frac{\delta_0'^2}{g^{2q}} = 2\epsilon_0' \left|1 - \frac{\epsilon_0'}{g^{1-2q}}\right|^{-2} \delta_0'^2 \tag{2.35}$$
$$\sum_{V_\alpha = T^{\pm1}_{n_{\ell_1}} T^{\pm1}_{n_{\ell_2}}} |\gamma_\alpha|^{-2} < \epsilon_0'^2 \left|1 - \frac{\epsilon_0'}{g^{1-2q}}\right|^{-4} \frac{\delta_0'^4}{g^2} \sum_{V_\alpha = T^{\pm1}_{n_{\ell_1}} T^{\pm1}_{n_{\ell_2}}} \left|\frac{\delta_{T^{\pm1}_{n_{\ell_1}}}}{\gamma_{T^{\pm1}_{n_{\ell_1}}}} + \frac{\alpha_{T^{\pm1}_{n_{\ell_1}}}}{\gamma_{T^{\pm1}_{n_{\ell_1}}}}\right|^{-2}.$$

The bound for the sum $\sum_{V_\alpha = T^{\pm1}_{n_{\ell_1}} T^{\pm1}_{n_{\ell_2}}} \left|\frac{\delta_{T^{\pm1}_{n_{\ell_1}}}}{\gamma_{T^{\pm1}_{n_{\ell_1}}}} + \frac{\alpha_{T^{\pm1}_{n_{\ell_1}}}}{\gamma_{T^{\pm1}_{n_{\ell_1}}}}\right|^{-2}$ can be found through the earlier method with the labels referring to $\ell_{T_{n_{\ell_1}}}$ and $\ell_{T^{-1}_{n_{\ell_2}}}$. The relation to the other sum $\sum_\alpha{}' \frac{|\gamma_\alpha|^{-2}}{}|\xi_\alpha + \frac{\delta_\alpha}{\gamma_\alpha}|^2$ requires the distances $|\xi_{1n_{\ell_1}} - \xi_{2n_{\ell_1}}|$ and $|\xi_{1n_{\ell_2}} - \xi_{2n_{\ell_2}}|$. With the bound

$$\left|\xi_{1\alpha} + \frac{\delta_\alpha}{\gamma_\alpha}\right| > \ell_0 \frac{\delta_0}{\sqrt{g}} - 2\frac{\delta_0'}{g^2}\left[1 + 2\frac{\epsilon_0'^{\frac{1}{2}}}{\left(1 - \frac{\epsilon_0'}{g^{1-2q}}\right)}\right] \qquad \ell_0 = O\left(\}^{\frac{\infty}{\epsilon}-\text{Ш}+\nabla}\right) \tag{2.36}$$

the sum including $\frac{1}{\ell_0}$ has the reduced range $\left[0, O\left(g^{\frac{1}{2}-q}\right)\right]$, and the summand decreases as $\frac{1}{\ell_0^3}$ in the range $\left[O\left(g^{\frac{1}{2}-q+r}\right), \left\lfloor -\frac{1}{2} + \frac{1}{6}\sqrt{9+24g}\right\rfloor - 1\right]$.

Given the labelling $\ell \equiv \ell_{T_{n_{\ell_1}}}$ and $\ell + \ell_0 = \ell_{T^{-1}_{n_{\ell_2}}}$, the lower bounds for the distances between $I_{T_{n_\ell}}$ and $I_{T^{-1}_{n_{\ell_2}}}$ range between $\ell_0 \frac{\delta_0}{\sqrt{g}}$ and $(2\ell + \ell_0)\frac{\delta_0}{\sqrt{g}}$. Fixing the circles $I_{T_{n_{\ell_1}}}$ and $I_{T^{-1}_{n_{\ell_1}}}$ temporarily, and allowing $I_{T^{-1}_{n_{\ell_2}}}$ to be an arbitrary isometric circle at level $\ell + \ell_0$, the least distance between $I_{T^{-1}_{n_{\ell_1}}}$ and the group of cirlces $\{I_{T_{n_{\ell_2}}}\}$ results from a configuration in which the set surrounds $I_{T^{-1}_{n_{\ell_1}}}$. The densest packing of $6(\ell + \ell_0 - 1)$ circles about $I[T^{-1}_{n_{\ell_1}}$ occurs in the range of distances $\left[\frac{\delta_0}{\sqrt{g}}, \left(\left\lfloor -\frac{1}{2} + \frac{1}{6}\sqrt{9+24g}\right\rfloor - 1\right)\frac{\delta_0}{\sqrt{g}}\right] = \left[\frac{\delta_0}{\sqrt{g}}, \left(-\frac{1}{2} + \frac{1}{2}\sqrt{1 + 8(\ell + \ell_0 - 1)}\right)\frac{\delta_0}{\sqrt{g}}\right]$.

The upper bound for $\sum_{\substack{V_\alpha = T^{\pm 1}_{n_{\ell_1}} T^{\pm 1}_{n_{\ell_2}} \\ densest\ packing}} |K_\alpha|$ is

$$
\begin{aligned}
&18 \frac{\epsilon_0'^2}{[1-\epsilon_0'^2]} \frac{1}{3}\frac{1}{6}[1-\epsilon_0'^4]\left|1 - \frac{\epsilon_0'}{g^{1-2q}}\right|^{-4} \frac{\delta_0'^4}{g^2} \sum_{\substack{V_\alpha = T^{\pm 1}_{n_{\ell_1}} T^{\pm 1}_{n_{\ell_2}} \\ densest\ packing}} \frac{\left| \frac{\delta_{T^{\pm 1}_{n_{\ell_1}}}}{\gamma_{T^{\pm 1}_{n_{\ell_1}}}} + \frac{\alpha_{T^{\pm 1}_{n_{\ell_2}}}}{\gamma_{T^{\pm 1}_{n_{\ell_2}}}} \right|^{-2}}{\left| \xi_{1\alpha} + \frac{\delta_\alpha}{\gamma_\alpha}\right|^2} \qquad (2.37) \\
&= \epsilon_0'^2 \left|1 - \frac{\epsilon_0'}{g^{1-2q}}\right|^{-4} \frac{\delta_0'^4}{g^2} \sum_{\substack{V_\alpha = T^{\pm 1}_{n_{\ell_1}} T^{\pm 1}_{n_{\ell_2}} \\ densest\ packing}} \frac{\left| \frac{\delta_{T^{\pm 1}_{n_{\ell_1}}}}{\gamma_{T^{\pm 1}_{n_{\ell_1}}}} + \frac{\alpha_{T^{\pm 1}_{n_{\ell_2}}}}{\gamma_{T^{\pm 1}_{n_{\ell_2}}}} \right|^{-2}}{\left| \xi_{1\alpha} + \frac{\delta_\alpha}{\gamma_\alpha}\right|^2} \\
&= \epsilon_0'^2 \left|1 - \frac{\epsilon_0'}{g^{1-2q}}\right|^{-4} \frac{\delta_0'^4}{g^2} \sum_{\ell_0=0}^{\lfloor -\frac{1}{2}+\frac{1}{6}\sqrt{9+24g}-1\rfloor} \sum_{\ell=2}^{\lfloor -\frac{1}{2}+\frac{1}{6}\sqrt{9+24g}\rfloor -1} \sum_{j_1=0}^{6(\ell-1)} \sum_{j_2=0}^{6(\ell+\ell_0)-7} \\
&\quad [d(I_{T^{-1}_{n_{\ell_1}}, j_1}, \{I_{T_{n_{\ell_2}}}\})]^{-2} \left[d^2_{\ell+\ell_0} + d^2_\ell - 2 d_\ell d_{\ell+\ell_0} cos\left(\frac{j_2 \pi}{3(\ell+\ell_0-1)}\right)\right]^{-1}
\end{aligned}
$$

where the set $\{I_{T_{n_{\ell_2}}}\}$ is indexed by j_2.

□

The double sum may be rewritten as

$$
\sum_{j_1=1}^{6(\ell-1)} \sum_{j_2=0}^{6(\ell+\ell_0)-7} [d(I_{T^{-1}_{n_{\ell_1}}, j_1}, \{I_{T_{n_{\ell_2}}}\})]^{-2} \left[d^2_{\ell+\ell_0} + d^2_\ell - 2 d_\ell d_{\ell+\ell_0} cos\left(\frac{j_2 \pi}{3(\ell+\ell_0-1)}\right)\right]^{-1}
$$

$$= \sum_{j_1=1}^{2\left(\lfloor -\frac{1}{2}+\frac{1}{2}\sqrt{1+8(\ell+\ell_0-1)}\rfloor -1\right)} \sum_{j_2=0}^{6(\ell+\ell_0)-7} [d(I_{T^{-1}_{n_{\ell_1}},j_1}, \{I_{T_{n_{\ell_2}}}\})]^{-2} \tag{2.38}$$

$$\left[d^2_{\ell+\ell_0} + d^2_\ell - 2d_\ell d_{\ell+\ell_0} cos\left(\frac{j_2\pi}{3(\ell+\ell_0-1)}\right)\right]^{-1}$$

$$+ \sum_{j_1=2\left(\lfloor -\frac{1}{2}+\frac{1}{2}\sqrt{1+8(\ell+\ell_0-1)}\rfloor -1\right)}^{6(\ell-1)-2\left(\lfloor -\frac{1}{2}+\frac{1}{2}\sqrt{1+8(\ell+\ell_0-1)}\rfloor -1\right)} \sum_{j_2=0}^{6(\ell+\ell_0)-7} [d(I_{T^{-1}_{n_{\ell_1}},j_1}, \{I_{T_{n_{\ell_2}}}\})]^{-2}$$

$$\left[d^2_{\ell+\ell_0} + d^2_\ell - 2d_\ell d_{\ell+\ell_0} cos\left(\frac{j_2\pi}{3(\ell+\ell_0-1)}\right)\right]^{-1}$$

$$+ \sum_{j_1=6(\ell-1)-2\left(\lfloor -\frac{1}{2}+\frac{1}{2}\sqrt{1+8(\ell+\ell_0-1)}\rfloor -1\right)}^{6(\ell-1)} \sum_{j_2=0}^{6(\ell+\ell_0)-7} [d(I_{T^{-1}_{n_{\ell_1}},j_1}, \{I_{T_{n_{\ell_2}}}\})]^{-2}$$

$$\left[d^2_{\ell+\ell_0} + d^2_\ell - 2d_\ell d_{\ell+\ell_0} cos\left(\frac{j_2\pi}{3(\ell+\ell_0-1)}\right)\right]^{-1}.$$

In the second double sum, $d\left(I_{T^{-1}_{n_{\ell_1}},j_1}, I_{T_{n_{\ell_2}},j_2}\right) \geq \left(\lfloor -\frac{1}{2} + \frac{1}{2}\sqrt{1+8(\ell+\ell_0-1)}\right)\frac{\delta_0}{\sqrt{g}}$ for all j_2. It increases and then decreases nearly linearly for values of j_1 in the range $\left[\left(-\frac{1}{2}+\frac{1}{2}\sqrt{1+8(\ell+\ell_0-1)}\rfloor\right] - 1\right), 6(\ell-1) - 2\left(\lfloor -\frac{1}{2}+\frac{1}{2}\sqrt{1+8(\ell+\ell_0-1)}\rfloor - 1\right)\right]$.

It follows from Theorems 2.2 and 2.3 that the sum of the magnitudes of the multipliers over the products of two elements in the Schottky group is bounded by a polynomial function of g. This polynomial may be demonstrated to be lnear, and the linearity of the bounds may be extended to sums over products of a larger number of elements [5]. The primitive element products then can be bounded through the inequality $\prod_\alpha{}'|1-K_\alpha|^{-1} < exp\left(\sum_\alpha{}'|K_\alpha|\right)$. It follows that the estimates of the major factors in the regularized moduli space integrals with respect to the genus would follow from integrals of singular functions of the multiplliers and fixed-point distances in the neighbourhood of the compactification divisor.

3. Analytic Continuation of the Poincare Series

The region of convergence of the Poincare series $\sum_{\alpha\neq I}|\gamma_\alpha|^{-2}$ for a Schottky group with elements V_α is determined by the convergence fo the series $\sum_n|\gamma_n|^{-2}$. It has been established previously that the latter series is finite only for $r_n = c\frac{1}{n^{\frac{1}{2+\epsilon}}}$ and $\epsilon > 0$. The constant c may be chosen such that the sum over all of the elements of the groups not equal to the identity is finite. Therefore, the domain of convergence of the Poincare series of a Schottky group is an open region in the parameter space $\{\gamma_n\}$.

The complement is a closed space, and it will not be possible to use the method of overlapping neighbourhoods directly to continue analytically the Poincare series to this region. Instead,it is conventional to equate the Poincare series to an automorphic form which may be given an integral representation [10][13]. When the domain of the integral is larger than that of the region of γ_n.

Theorem 3.1. The correlation functions on an infinite-genus surface may be analytically continued beyond the region of the convergence of a group of Schottky type with an infinite number of generators to the entire class of O_G surfaces. This analytic continuation remains valid for a subset of O_{HD} surfaces and it does not exist for Type II surfaces.

Proof. Consequently, there would exist a formula for the Green function and the amplitudes for the surfaces by these Schottky groups. The calculation then may be extended from spheres with an infinite number of handles decreasing in size at a rate $\frac{cc'}{\sqrt{2}\zeta(2q)n^{-q}}$ with $q > \frac{1}{2}$, where c is the lower bounde for $\left|\frac{\alpha_\alpha = \xi_{1n}}{\xi_{2n} - \xi_{1n}}\right|$, c' is an upper bound for $|\xi_{2n} - \xi_{1n}|$ and q is defined by $|K_n|^{-\frac{1}{2}} = cn^q + c_2$ [4].

Generally, the prime form $E(x, y) = \frac{\Theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](\int_x^y d\omega_i|\tau)}{\partial_i\Theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](0|\tau)\omega_i(x)\partial_j\Theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](0|\tau)\omega_j]^{\frac{1}{2}}}$, where ω_i is a holomorphic differential of the first kind normalized by $\int_{A_j} \omega_i = \delta_{ij}$ and $\int_{B_j} \omega_j = \tau_{ij}$, where $\{(A_i, B_j)|i, j = 1, ..., g\}$ is a canonical homology basis and τ is the period matrix, would be used for the Green function on a Riemann surface when it is combined with an exponential factor $exp\Big[- \pi \sum_{i.j} \left(\int_x^y Im\ \omega_j\right) (Im\ \tau)_{ij} \left(\int_x^y Im\ \omega_j\right)$ to give a single-valued function [11]. Its expansion in terms of Schottky group parameters and the formula for the theta function

$$E(x, y) = \frac{(x - y)}{\sqrt{dx\,dy}} \prod_{\alpha \neq I}{}' \frac{(x - V_\alpha y)(y - V_\alpha x)}{(x- = V_\alpha x)(y_{V\alpha} y)} \tag{3.1}$$

$$\Theta\begin{bmatrix} 0 \\ \beta \end{bmatrix}(|\tau) = \prod_\alpha{}' \prod_{n=1}^{\infty} \left[1 - e^{2\pi i N_\alpha} K_\alpha^{n-\frac{1}{2}}\right]\left[1 - e^{-2\pi i N} K_\alpha^{n-\frac{1}{2}}\right](1 - K_\alpha^n),$$

where N_α is the difference between the number of generators and inverses in the product V_α, yields the conventional scattering amplitude [9]. The prime formmay be defined on any surface which admits a theta function $\theta(z|\tau) = \sum_{m \in \mathbb{Z}^g} exp\Big(\frac{1}{2} m\tau m^T$

$+mz^T\Big)$, $z \in \mathbb{Z}^g$, that converges if $Im\ \tau$ is positive-definite. The positivity of the determinant of $Im\ \tau$ follows from Riemann bilinear relations

$$\langle\omega, \omega^*\rangle = \frac{i}{2} \sum_{i,j} m_i m_j \sum_{k=1}^{g} \Big[\int_{A_k} \omega_i \int_{B_k} \bar{\omega}_j - \int_{A_k} \bar{\omega}_j \int_{B_j} \Big] = \sum_{i.j} m_i (Im\ \tau)_{ij} m_j > 0 \tag{3.2}$$

$$\omega = \sum_i m_i \omega_i$$

for all closed finite-genus surfaces. These relations may be extended directly to O_G surfaces [2] and a subset of the class of O_{HD} surfaces [1] with a choice of the homology basis for which operators on the space of harmonic differentials have a bounded norm. The bilinear relation for ω, $\sigma \in \Gamma_{hse}(\bar{\Sigma})$, where includes a boundary term for bordered surfaces Σ, and it is necessary to establish the vanishing of $\int_{\partial\Sigma_n} v\bar{\sigma}$, where $v(p) = \int_{p_0}^{p}$ in the limit $\partial\Sigma_n \to \partial\Sigma$ and $p_0 \in \partial\sigma$. The vanishing of the linear measure would be considered to be a sufficient criterion for nonsingular differentials at the ideal boundary, and therefore, generalized bilinear relations yielding $det\ (Im\ \tau) > 0$ would be expected to be valid for O_G and O_{HD} surfaces.

The absolute convergence of the theta function has been proven for a class of transcendental hyperelliptic surfaces in O_G. The conditions on the locations of the branch points are chosen such that a set of inequalities is satisfied by the elements of $Im\ \tau$, a quite rapid increase in the magnitudeof the branch points is required

[14]. The eigenvalues of a $g \times g$ matrix satisfy the inequalities

$$min_j\{h_j^{(g-k+1)} - \sigma_j^{(g-k+1)}\} \leq \lambda_k \leq max_j\{h_j^{(k)} + \sigma_j^{(k)}\} \quad (3.3)$$
$$h_j^{(k)} = j^{th}\ diagonal\ element\ of\ a\ principal\ k \times k\ submatrix$$
$$\sigma_j^{(k)} = sum\ of\ absolute\ values\ of'off-diagonal\ elements$$
$$of\ j^{th}\ row\ of\ as\ principal\ k \times k\ submatrix$$

It follows by the symmetry of the period matrix that

$$min_n\Big[(Im\ \tau)_{kk} - \sum_{k\neq\ell} |(Im\ \tau)_{k\ell}|\Big] \leq \lambda_1 \leq (Im\ \tau)_{gg} \quad (3.4)$$
$$\vdots$$
$$(Im\ \tau)_{11} \leq \lambda_g \leq max_k\Big[(Im\ \tau)_{kk} + \sum_{k\neq\ell} |(Im\ \tau)_{k\ell}|\Big]$$

for a matrix with $(Im\ \tau)_{gg} \geq ... \geq (Im\ \tau)_{11}$.

Given that $(Im\ \tau)_{nn} - \sum_{m\neq n} |(Im\ \tau)_{mn}| \geq 0$ for $n = 1, ..., g$, each of the eigenvalues of $Im\ \tau$ would be positive and $det(Im\ \tau) > 0$. The surfacs uniformizied by Schottky groups with isometric circlesof generators the limits in §2 on the multipliers and fixed points do not have period matrices satisfying these inequalities because $ln\left(\frac{1}{|K_n|}\right) = O(1)$. It may be concluded similarly that the lower bounds for the eigenvalues from Eq.(3.3) are not positive even for isometric circles $\frac{\epsilon_0}{g} \leq |K_n| \leq \frac{\epsilon_0'}{g}$, $\delta \leq |\xi_{1n} - \xi_{2n}| \leq \delta_0^2$. Nevertheless, the positivity of the determinant follows from the bilinear relations at finite genus, while an upper bound for ϵ_0' for the category of isometric cirlces in §2 and an expansion of $det(Im\ \tau)$ for Schottky groups with the magnitudes of multipliers decreasing inversely with the genus [6] is sufficient in the limit of infinite genus.

The theta function has been given for Hill's surface with branch points at λ_{2n-1}, $\lambda_{2n} = n^2\pi^2 + O\left(\frac{1}{n^2}\right)$ by summing over the class of vectors $\vec{m}$ yielding finite sums $\sum_i m_i$ [12]. The formula for correlation functions and the scattering amplitudes, therefore, may be extended to infinite-genus surfaces in O_G beyond the previous limits for the spheres with an infinite number of handles uniformized by groups of Schottky type with the decrease of $r_n = |\gamma_n|^{-1}$ requires for convergence of the Poincare series.

The evaluation of scattering cross-sections for reggeon ladder diagrams with an infinite number of handles could be considered. The O_G surfaces may be represented as spheres with a countable number of handles accumulating to a limit point and cannot be conformally transformed to infinite ladder diagrams, since the border arcs reflect a classification as Type II surfaces.

The integral over the ideal boundary of a Type II surface would represent an extra term in the bilinear relation. By transposition, $(Im\ \tau)_{kk} = \langle\omega_k, \omega_k^*\rangle - \int_{\partial\Omega} v_k\bar{\omega}_k$, where v_k is the integral of the holomorphic differential ω_k, and the sign is not definite. Consequently, $det(Im\ \tau)$ is not necessarily positive, and the theta function, the prime form and the correlation functions would not be analytically continued by this method to all Type II surfaces.

□

The regularized integral over a region in the Schottky group parameter space, defined by a genus-independent lower bound for the closed geodesic lengths, has been estimated to be $O\left(2^6\pi^{\frac{33}{2}}\left(\frac{e^2}{4\sqrt{2}\pi}\right)^{\frac{g}{ln\ g}} c^g \frac{g^{\left(2-\frac{1}{ln\ g}\right)g+\frac{1}{2}}}{(ln\ g)^{\left(15-\frac{3}{2\ ln\ g}\right)g-\frac{1}{2}}}\right)$ [7], and it would follows that the Type II surfaces may be required to increase the integral to $(2g)!$, which is characteristic of a cubic field theory [8].

An estimate of the string amplitujdes to all ordersin perturbation theory follows by stochastic quantization of a quantum Hamiltonian yielding a propagator satisfying a Langevin equation [3]. Expansion of the solution gives the dependence with respect to the genus for boundaries of fixed length. Factorization of the amplitudes with a fixed length condition for each component is consistent with the regularization of the the partition function. Specifying that the bordered lengths tend to zero at infinite genus, the regularized partition function may be evaluated.

4. Conclusion

The Green function on the Riemann surface may be found initially through a series representation that requires the sum $\sum_{\alpha \neq I} |\gamma|^{-2}$ to converge. The convergence of the Poincare series is known to be related to topological characteristics of the covering surface such as the Hausdorff dimension of the limit set of the uniformizing group. The region of convergence can be defined by limits on the Schottky group prarameters. The existence of a range of values, for which the Poincare series diverges, does not affect the exponent of convergence. Nevertheless, it wold be expected that there exists an analytic continuation to the remainder of the region of Schottky space with the same infimum for the exponent. This result may be verified through a more general prime form representation of correlation functions defined in terms of the thetat function. The theta function $\vartheta(|\tau)$ converges when $det(Im\ \tau) > 0$, which would follow from a bilinear relation for a noncompact surface. The integral over the ideal boundary in a generalized relation vansihes for surfaces in the class O_G and a subset of O_{HD}, and the positivity of the quadratic form $\sum_{i,j} m_i (Im\ \tau)_{ij} m_j$ follows. Given the existence of a theta function on the entire class of O_G surfaces, the correlation functions on a sphere with an infinite number of handles may be analytically continued to the entire region in Schottky space consisting of configurations of isometric circles with the same rate of decrease of the radii.

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