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Convergence and stability results for a class of asymptotically quasi-nonexpansive mappings in the intermediate sense

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Abstract. In this paper, we study the notion of a class of asymptotically quasi-nonexpansive mappings in the intermediate sense and define a k-step iterative sequence for approximating common fixed point in a class of above mentioned mappings. Also, we study its stability in real Banach spaces. The results presented in this paper are improvement and generalization of several known corresponding results in the existing literature (see, e.g., [3], [4], [9], [12]-[14], [17], [20]-[26]).

1. Introduction

Let E be an arbitrary real Banach space and K be a nonempty closed and convex subset E.

Definition 1.1. *Let $T: K \to K$ be a mapping. Suppose that, for any $x_0 \in E$,*

$$x_{n+1} = f(T, x_n) \tag{1}$$

yields a sequence of points $\{x_n\}$ in K, where f denotes the iterative process involving T and x_n. Suppose that $F(T) = \{x \in K : Tx = x\} \neq \emptyset$ and $\{x_n\}$ converges strongly to $p \in F(T)$. Let $\{y_n\}$ be a sequence in K and $\{\varepsilon_n\}$ be a sequence in $[0, \infty)$ defined by

$$\varepsilon_n = \|y_{n+1} - f(T, y_n)\|. \tag{2}$$

If $\lim_{n\to\infty} \varepsilon_n = 0$ implies that $\lim_{n\to\infty} y_n = p$, then the iterative process defined by (1) is said to be T-stable with respect to T (see, for example, [6]-[8], [15], [16], [18], [19] and references therein).

We say that the iterative process $\{x_n\}$ defined by (1) is almost T-stable or almost stable with respect to T if $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ implies that $\lim_{n\to\infty} y_n = p$ (see [16]). It is easy to see from above that an iterative process which is T-stable is almost T-stable. The example in [16] showed that an iterative process which is almost T-stable need not be T-stable.

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Stability results for many iterative processes for some kinds of nonlinear mappings have been shown in recent papers by many authors (see, for example, [1], [2], [6]-[8], [15], [16], [18], [19] and the references therein).

The concept of quasi-nonexpansive mapping was initiated by Tricomi in 1941 for real function. The concept of an asymptotically nonexpansive mapping and an asymptotically nonexpansive type mapping were introduced by Goebel and Kirk [5] and Kirk [11], respectively, which are closely related to the theory of fixed points in Banach spaces. In 2003, Sahu and Jung [20] studied Ishikawa and Mann iteration process in Banach spaces and they proved some weak and strong convergence theorems for asymptotically quasi-nonexpansive type mapping. In 2006, Shahzad and Udomene [22] gave the necessary and sufficient condition for convergence of common fixed point of two-step modified Ishikawa iterative sequence for two asymptotically quasi-nonexpansive mappings in real Banach space. Recently, Yao and Liou [25] gave the notion of asymptotically quasi-nonexpansive mappings in the intermediate sense and gave necessary and sufficient condition for the iterative sequence to converge to the common fixed points for two asymptotically quasi-nonexpansive mappings in the intermediate sense in the framework of real Banach spaces. The iterative approximating problem of fixed points for asymptotically nonexpansive mappings or asymptotically quasi-nonexpansive mappings have been studied by many authors (see, for example, [3], [4], [9], [12]-[14], [17], [20]-[26] and the references therein).

In this paper, we study the notion of a class of asymptotically quasi-nonexpansive mappings in the intermediate sense and define a k-step iterative sequence for approximating common fixed point in a class of said mappings. Also, we study its stability in real Banach spaces. Our results extend, improve and unify the corresponding results of [3], [4], [9], [12]-[14], [17], [20]-[26].

2. Preliminaries

Definition 2.1. *Let E be a real Banach space and K be a nonempty closed and convex subset of E. Let $T\colon K \to K$ be a mapping. Denote $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in K : Tx = x\}$.*

(1) T is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \tag{3}$$

for all $x, y \in K$.

(2) T is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\|Tx - p\| \leq \|x - p\| \tag{4}$$

for all $x \in K$ and $p \in F(T)$.

(3) T is said to be asymptotically nonexpansive if there exists a sequence $\{u_n\} \subset [0, \infty)$ with $u_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq (1 + u_n)\|x - y\| \tag{5}$$

for all $x, y \in K$ and $n \geq 0$.

(4) T is said to be asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{u_n\} \subset [0, \infty)$ with $u_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - p\| \leq (1 + u_n)\|x - p\| \tag{6}$$

for all $x \in K$, $p \in F(T)$ and $n \geq 0$.

(5) T is said to be asymptotically nonexpansive type [11], if

$$\limsup_{n\to\infty}\Big\{\sup_{x,y\in K}\Big(\|T^n x - T^n y\| - \|x-y\|\Big)\Big\} \quad\leq\quad 0. \tag{7}$$

(6) T is said to be asymptotically quasi-nonexpansive type [20], if $F(T)\neq\emptyset$ and

$$\limsup_{n\to\infty}\Big\{\sup_{x\in K,\ p\in F(T)}\Big(\|T^n x - p\| - \|x-p\|\Big)\Big\} \quad\leq\quad 0. \tag{8}$$

(7) T is said to be asymptotically quasi-nonexpansive mapping in the intermediate sense [25] provided that T is uniformly continuous and

$$\limsup_{n\to\infty}\Big\{\sup_{x\in K,\ p\in F(T)}\Big(\|T^n x - p\| - \|x-p\|\Big)\Big\} \leq 0. \tag{9}$$

From the above definitions, it follows that asymptotically nonexpansive mapping must be asymptotically quasi-nonexpansive and asymptotically quasi-nonexpansive mapping in the intermediate sense. But the converse does not hold as the following example.

Example 2.2. *Let $X = \mathbb{R}$ be a normed linear space and $K = [0,1]$. For each $x \in K$, we define*

$$T(x) = \begin{cases} k\,x, & \text{if } x \neq 0,\\ 0, & \text{if } x = 0,\end{cases}$$

where $0 < k < 1$. Then

$$|T^n x - T^n y| = k^n|x-y| \leq |x-y|$$

for all $x, y \in K$ and $n \in \mathbb{N}$.

Thus T is an asymptotically nonexpansive mapping with constant sequence $\{1\}$ and

$$\begin{aligned}\limsup_{n\to\infty}\Big\{|T^n x - T^n y| - |x-y|\Big\} &= \limsup_{n\to\infty}\Big\{k^n|x-y| - |x-y|\Big\}\\ &\leq 0,\end{aligned}$$

because $\lim_{n\to\infty} k^n = 0$ as $0 < k < 1$ and for all $x, y \in K$, $n \in \mathbb{N}$. Hence T is an asymptotically nonexpansive mapping in the intermediate sense. Since $F(T) = \{0\} \neq \emptyset$, and so T is an asymptotically quasi-nonexpansive mapping in the intermediate sense.

Example 2.3. *(see [10]) Let $X = \mathbb{R}$, $K = [-\frac{1}{\pi}, \frac{1}{\pi}]$ and $|\lambda| < 1$. For each $x \in K$, define*

$$T(x) = \begin{cases} \lambda\ x\ sin(\frac{1}{x}), & \text{if } x \neq 0,\\ 0, & \text{if } x = 0.\end{cases}$$

Then T is an asymptotically nonexpansive mapping in the intermediate sense but it is not asymptotically nonexpansive mapping.

Throughout this paper, let E be a real Banach space and K be a nonempty closed convex subset of E. Let $T_1, T_2, \ldots, T_k \colon K \to K$ be mappings, $F(T_i)$ the set of fixed points of T_i, k is a positive integer. Let m and n denote the nonnegative integers.

Definition 2.4. *Let E be a real Banach space and K be a nonempty closed convex subset of E. $T_1, T_2, \ldots, T_k \colon K \to K$ are said to be a class of asymptotically quasi-nonexpansive mappings in the intermediate sense provided that $T_i (i = 1, 2, \ldots, k)$ is uniformly continuous and, for each $i \in \{1, 2, \ldots, k\}$,*

$$\limsup_{n\to\infty}\Big\{\sup_{x\in K,\, p\in F=\cap_{i=1}^k F(T_i)}\Big(\|T_i^n x - p\| - \|x-p\|\Big)\Big\} \leq 0. \tag{10}$$

Remark 2.5. *It is easy to see that the notion of a class of asymptotically quasi-nonexpansive mappings in the intermediate sense defined by Definition 2.4 reduces to that of asymptotically quasi-nonexpansive mapping in the intermediate sense defined by Definition 2.1(7) when* $T_1 = T_2 = \cdots = T_k = T$.

In the sequel, we need the following lemma.

Lemma 2.6. *(see [23]) Let* $\{a_n\}$, $\{b_n\}$ *be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq a_n + b_n, \ n \geq 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$. *Then*

(a) $\lim_{n\to\infty} a_n$ *exists.*

(b) If $\liminf_{n\to\infty} a_n = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

3. Main Results

Theorem 3.1. *Let E be a real Banach space, K a nonempty closed convex subset of E. Let* $T_1, T_2, \ldots, T_k \colon K \to K$ *be a class of asymptotically quasi-nonexpansive mappings in the intermediate sense defined by Definition 2.4. Assume that, for each* $i \in \{1, 2, \ldots, k\}$, *there exist* L_i *and* $\gamma_i > 0$ *such that*

$$\|T_i x - q\| \ \leq \ L_i \|x - q\|^{\gamma_i}, \ \forall x \in K, \ \forall q \in F = \cap_{i=1}^{k} F(T_i). \tag{11}$$

Put, for each $i \in \{1, 2, \ldots, k\}$

$$D_n \ = \ \max\Big\{\max_{1\leq i\leq k} \sup_{x\in K,\, q\in F} \Big(\|T_i^n x - q\| - \|x - q\|\Big), 0\Big\}, \tag{12}$$

such that $\sum_{n=0}^{\infty} D_n < \infty$. *For any given* $x_0 \in K$, *define the k-step iterative sequence* $\{x_n\}$ *by*

$$\begin{aligned}
z_{k-1,n} &= (1-\alpha_{k,n})x_n + \alpha_{k,n} T_k^n x_n, \\
z_{k-2,n} &= (1-\alpha_{k-1,n})x_n + \alpha_{k-1,n} T_{k-1}^n z_{k-1,n}, \\
&\ \vdots \\
z_{1,n} &= (1-\alpha_{2,n})x_n + \alpha_{2,n} T_2^n z_{2,n}, \\
x_{n+1} &= (1-\alpha_{1,n})x_n + \alpha_{1,n} T_1^n z_{1,n}, \ n \geq 0,
\end{aligned} \tag{13}$$

where $\{\alpha_{i,n}\}$ *is a sequence in* $[0,1]$ *for each* $i \in \{1, 2, \ldots, k\}$. *Suppose that* $\{y_n\}$ *is a sequence in K and define a sequence* $\{\varepsilon_n\}$ *of positive real numbers by*

$$\begin{aligned}
w_{k-1,n} &= (1-\alpha_{k,n})y_n + \alpha_{k,n} T_k^n y_n, \\
w_{k-2,n} &= (1-\alpha_{k-1,n})y_n + \alpha_{k-1,n} T_{k-1}^n w_{k-1,n}, \\
&\ \vdots \\
w_{1,n} &= (1-\alpha_{2,n})y_n + \alpha_{2,n} T_2^n w_{2,n}, \\
\varepsilon_n &= \|y_{n+1} - (1-\alpha_{1,n})y_n - \alpha_{1,n} T_1^n w_{1,n}\|, \ n \geq 0.
\end{aligned} \tag{14}$$

If $F = \cap_{i=1}^{k} F(T_i) \neq \emptyset$, *then we have the following:*

(i) $\{x_n\}$ *converges strongly to a common fixed point q of* $T_1, T_2, \ldots, T_k$ *in K if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$, *where* $d(x_n, F) = \inf_{p^*\in F} \|x_n - p^*\|$.

(ii) $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ *and* $\liminf_{n\to\infty} d(y_n, F) = 0$ *imply that* $\{y_n\}$ *converges strongly to a common fixed point q of* $T_1, T_2, \ldots, T_k$ *in K.*

(iii) If $\{y_n\}$ *converges strongly to a common fixed point q of* $T_1, T_2, \ldots, T_k$ *in K, then* $\lim_{n\to\infty} \varepsilon_n = 0$.

In order to prove Theorem 3.1, we first give the following lemma.

Lemma 3.2. *Assume that all the assumptions in Theorem 3.1 hold and* $\sum_{n=0}^{\infty} \varepsilon_n < \infty$. *Then*

(i)

$$\|y_{n+1} - q\| \leq \|y_n - q\| + \Big(\sum_{p=1}^{k} \delta_n^p\Big) D_n + \varepsilon_n, \ \forall q \in F;$$

(ii)

$$\|y_m - q\| \leq \|y_n - q\| + \sum_{s=n}^{m-1} \varepsilon_s + \Big(\sum_{p=1}^{k} \delta_n^p\Big) \sum_{s=n}^{m-1} D_s, \ \forall q \in F, \ m > n;$$

(iii)

$$\lim_{n\to\infty} d(y_n, F) \text{ exists.}$$

Proof. Take any $q \in F$, and since $\{w_{i,n}\} \subset K$, it follows from (12), (13) and (14) that

$$\begin{aligned} \|y_{n+1} - q\| &\leq \varepsilon_n + \|(1 - \alpha_{1,n})(y_n - q) + \alpha_{1,n}(T^n w_{1,n} - q)\| \\ &\leq (1 - \alpha_{1,n})\|y_n - q\| + \alpha_{1,n}\|T^n w_{1,n} - q\| + \varepsilon_n \\ &\leq (1 - \alpha_{1,n})\|y_n - q\| + \alpha_{1,n}\|w_{1,n} - q\| + \alpha_{1,n} D_n \end{aligned} \tag{15}$$

and

$$\begin{aligned} \|w_{1,n} - q\| &= \|(1 - \alpha_{2,n})(y_n - q) + \alpha_{2,n}(T_2^n w_{2,n} - q)\| \\ &\leq (1 - \alpha_{2,n})\|y_n - q\| + \alpha_{2,n}\|T_2^n w_{2,n} - q\| \\ &\leq (1 - \alpha_{2,n})\|y_n - q\| + \alpha_{2,n}(\|w_{2,n} - q\| + D_n) \\ &\leq (1 - \alpha_{2,n})\|y_n - q\| + \alpha_{2,n}\|w_{2,n} - q\| + \alpha_{2,n} D_n. \end{aligned} \tag{16}$$

Continuing in this way, we can deduce that

$$\begin{aligned} \|w_{i,n} - q\| &= \|(1 - \alpha_{i+1,n})(y_n - q) + \alpha_{i+1,n}(T_{i+1}^n w_{i+1,n} - q)\| \\ &\leq (1 - \alpha_{i+1,n})\|y_n - q\| + \alpha_{i+1,n}\|T_{i+1}^n w_{i+1,n} - q\| \\ &\leq (1 - \alpha_{i+1,n})\|y_n - q\| + \alpha_{i+1,n}(\|w_{i+1,n} - q\| + D_n) \\ &\leq (1 - \alpha_{i+1,n})\|y_n - q\| + \alpha_{i+1,n}\|w_{i+1,n} - q\| + \alpha_{i+1,n} D_n. \end{aligned} \tag{17}$$

Since $\{y_n\} \subset K$, we have

$$\begin{aligned} \|w_{k-1,n} - q\| &= \|(1 - \alpha_{k,n})(y_n - q) + \alpha_{k,n}(T_k^n y_n - q)\| \\ &\leq (1 - \alpha_{k,n})\|y_n - q\| + \alpha_{k,n}\|T_k^n y_n - q\| \\ &\leq (1 - \alpha_{k,n})\|y_n - q\| + \alpha_{k,n}(\|y_n - q\| + D_n) \\ &\leq \|y_n - q\| + \alpha_{k,n} D_n. \end{aligned} \tag{18}$$

Substituting (17) and (18) into (15), for any $q \in F$, we have

$$\begin{aligned} \|y_{n+1} - q\| &\leq \|y_n - q\| + \varepsilon_n + \alpha_{1,n} D_n \\ &\quad + \alpha_{1,n}\alpha_{2,n} D_n + \cdots + \alpha_{1,n}\alpha_{2,n} \ldots \alpha_{k,n} D_n \\ &\leq \|y_n - q\| + \varepsilon_n + \Big\{ \prod_{i=1}^{1} \alpha_{i,n} \Big\} D_n \\ &\quad + \Big\{ \prod_{i=1}^{2} \alpha_{i,n} \Big\} D_n + \cdots + \Big\{ \prod_{i=1}^{k} \alpha_{i,n} \Big\} D_n \\ &\leq \|y_n - q\| + \varepsilon_n + \delta_n^1 D_n + \delta_n^2 D_n + \cdots + \delta_n^k D_n \\ &\leq \|y_n - q\| + \varepsilon_n + \Big(\sum_{p=1}^{k} \delta_n^p \Big) D_n \end{aligned} \tag{19}$$

for all $n \geq 0$, where $\delta_n^l = \prod_{i=1}^{l} \alpha_{i,n}$. Hence the conclusion (i) holds. From the conclusion (i), we have

$$\begin{aligned} \|y_m - q\| &\leq \|y_{m-1} - q\| + \varepsilon_{m-1} + \Big(\sum_{p=1}^{k} \delta_n^p \Big) D_{m-1} \\ &\leq \|y_{m-2} - q\| + \varepsilon_{m-2} + \varepsilon_{m-1} + \Big(\sum_{p=1}^{k} \delta_n^p \Big) D_{m-2} \\ &\quad + \Big(\sum_{p=1}^{k} \delta_n^p \Big) D_{m-1} \\ &\leq \ldots \\ &\leq \ldots \\ &\leq \|y_n - q\| + \sum_{s=n}^{m-1} \varepsilon_s + \Big(\sum_{p=1}^{k} \delta_n^p \Big) \sum_{s=n}^{m-1} D_s \end{aligned} \tag{20}$$

for all $q \in F$ and $m > n$. Thus the conclusion (ii) holds. Again, it follows from conclusion (i) that

$$d(y_{n+1}, F) \leq d(y_n, F) + \varepsilon_n + \Big(\sum_{p=1}^{k} \delta_n^p \Big) D_n. \tag{21}$$

Since by hypothesis

$$\sum_{n=0}^{\infty} D_n < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} \varepsilon_n < \infty,$$

we have

$$\sum_{n=0}^{\infty} \Big(\varepsilon_n + Q D_n \Big) < \infty, \text{ where } Q = \Big(\sum_{p=1}^{k} \delta_n^p \Big) > 0.$$

Therefore, from Lemma 2.1 we know that $\lim_{n\to\infty} d(y_n, F)$ exists. Thus, the conclusion (iii) holds. This completes the proof of Lemma 3.2. □

The Proof of Theorem 3.1

The necessity of the conclusion (i) is obvious and the sufficiency follows from conclusion (ii) by setting $\varepsilon_n = 0$ for all $n \geq 0$ in (14) and considering (13). Now, we prove the conclusion (ii) holds. It follows from Lemma 3.2(iii) that $\lim_{n\to\infty} d(y_n, F)$ exists. Since by hypothesis

$$\liminf_{n\to\infty} d(y_n, F) = 0,$$

so by Lemma 2.6, we have

$$\lim_{n\to\infty} d(y_n, F) = 0. \tag{22}$$

First, we have to prove that $\{y_n\}$ is a Cauchy sequence in E. In fact, it follows from (22), the assumptions $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ and $\sum_{n=0}^{\infty} D_n < \infty$, that for any given $\varepsilon > 0$ there exists a positive integer n_1 such that

$$d(y_n, F) < \varepsilon, \ n \geq n_1 \tag{23}$$

$$\sum_{s=n_1}^{\infty} D_s < \frac{\varepsilon}{Q}, \text{ where } Q = \sum_{p=1}^{k} \delta_n^p, \tag{24}$$

and

$$\sum_{n=n_1}^{\infty} \varepsilon_n < \varepsilon. \tag{25}$$

By the definition of infimum, it follows from (23) that for any given $n \geq n_1$ there exists an $q^*(n) \in F$ such that

$$\|y_n - q^*(n)\| < 2\varepsilon. \tag{26}$$

On the other hand, for any $m, n \geq n_1$, without loss of generality $m > n_1$, it follows from Lemma 3.2(ii) that

$$\begin{aligned} \|y_m - y_n\| &\leq \|y_m - q^*(n)\| + \|y_n - q^*(n)\| \\ &\leq \|y_n - q^*(n)\| + Q\sum_{s=n}^{m-1} D_s + \sum_{s=n}^{m-1} \varepsilon_s \\ &\quad + \|y_n - q^*(n)\|, \text{ where } Q = \sum_{p=1}^{k} \delta_n^p, \\ &= 2\|y_n - q^*(n)\| + Q\sum_{s=n}^{m-1} D_s + \sum_{s=n}^{m-1} \varepsilon_s. \end{aligned} \tag{27}$$

Therefore by (24) - (27), for any $m > n \geq n_1$, we have

$$\|y_m - y_n\| < 4\varepsilon + \varepsilon + \varepsilon = 6\varepsilon. \tag{28}$$

This shows that $\{y_n\}$ is a Cauchy sequence in E. Since E is complete, there exists an $z^* \in E$ such that $y_n \to z^*$ as $n \to \infty$.

Now, we prove that z^* is a common fixed point of $T_1, T_2, \ldots, T_k$ in K. Since $y_n \to z^*$ and $d(y_n, F) \to 0$ as $n \to \infty$, for any given $\varepsilon > 0$, there exists a positive integer $n_2 \geq n_1$ such that

$$\|y_n - z^*\| < \varepsilon, \quad d(y_n, F) < \varepsilon, \tag{29}$$

for all $n \geq n_2$. The second inequality in (29) implies that there exists $z_1^* \in F$ such that

$$\|y_{n_2} - z_1^*\| < 2\varepsilon. \tag{30}$$

Thus, from (12), (29) and (30) and for any $n \geq n_2$, we have

$$\begin{aligned} \|T_i^n z^* - z^*\| &\leq \|T_i^n z^* - z_1^*\| + \|z^* - z_1^*\| \\ &\leq D_n + 2\|z^* - z_1^*\| \\ &\leq D_n + 2\Big(\|z^* - y_{n_2}\| + \|z_1^* - y_{n_2}\|\Big) \\ &< D_n + 2(\varepsilon + 2\varepsilon) = D_n + 6\varepsilon = \varepsilon_1, \end{aligned} \tag{31}$$

where $\varepsilon_1 = D_n + 6\varepsilon$, since $D_n \to 0$ as $n \to \infty$ and $\varepsilon > 0$, it follows that $\varepsilon_1 > 0$. The inequality (31) implies that $T_i^n z^* \to z^*$ as $n \to \infty$. Again since

$$\begin{aligned} \|T_i^n z^* - T_i z^*\| &\leq \|T_i^n z^* - z_1^*\| + \|T_i z^* - z_1^*\| \\ &\leq D_n + \|z^* - z_1^*\| + \|T_i z^* - z_1^*\|, \end{aligned} \tag{32}$$

for all $n \geq n_2$, by assumption (11) and using (29) and (30), we have

$$\begin{aligned} \|T_i^n z^* - T_i z^*\| &\leq D_n + \|z^* - z_1^*\| + L_i\|z^* - z_1^*\|^{\gamma_i} \\ &\leq D_n + \|z^* - y_{n_2}\| + \|z_1^* - y_{n_2}\| \\ &\quad + L_i\Big(\|z^* - y_{n_2}\| + \|z_1^* - y_{n_2}\|\Big)^{\gamma_i} \\ &< D_n + 3\varepsilon + L_i(3\varepsilon)^{\gamma_i} = \varepsilon_1' \end{aligned} \tag{33}$$

where $\varepsilon_1' = D_n + 3\varepsilon + L_i(3\varepsilon)^{\gamma_i}$, since $D_n \to 0$ as $n \to \infty$, $\varepsilon > 0$ and $L_i > 0$ for all $i \in \{1, 2, \dots, k\}$, it follows that $\varepsilon_1' > 0$, the inequality (33) shows that $T_i^n z^* \to T_i z^*$ as $n \to \infty$. By the uniqueness and the continuity of limit, we have $T_i z^* = z^*$ for all $i \in \{1, 2, \dots, k\}$, that is, z^* is a common fixed point of $T_1, T_2, \dots, T_k$ in K. Therefore, the conclusion (ii) holds.

From (12) and (14)-(18), we have, for any given $\varepsilon > 0$,

$$\begin{aligned} \varepsilon_n &\leq \|y_{n+1} - z^*\| + \|(1 - \alpha_{1,n})(y_n - z^*) + \alpha_{1,n}(T_1^n w_{1,n} - z^*)\| \\ &\leq \|y_{n+1} - z^*\| + (1 - \alpha_{1,n})\|y_n - z^*\| + \alpha_{1,n}\|T_1^n w_{1,n} - z^*\| \\ &\leq \|y_{n+1} - z^*\| + (1 - \alpha_{1,n})\|y_n - z^*\| + \alpha_{1,n}\Big(\|w_{1,n} - z^*\| + D_n\Big) \\ &\leq \|y_{n+1} - z^*\| + (1 - \alpha_{1,n})\|y_n - z^*\| + \alpha_{1,n}\|w_{1,n} - z^*\| + \alpha_{1,n}D_n \\ &\leq \dots \\ &\leq \dots \\ &\leq \|y_{n+1} - z^*\| + \|y_n - z^*\| + \alpha_{1,n}D_n + \alpha_{1,n}\alpha_{2,n}D_n + \dots \\ &\quad + \alpha_{1,n}\alpha_{2,n}\dots\alpha_{k,n}D_n \\ &\leq \|y_{n+1} - z^*\| + \|y_n - z^*\| + \Big\{\prod_{i=1}^{1} \alpha_{i,n}\Big\}D_n + \Big\{\prod_{i=1}^{2} \alpha_{i,n}\Big\}D_n \\ &\quad + \dots + \Big\{\prod_{i=1}^{k} \alpha_{i,n}\Big\}D_n \\ &\leq \|y_{n+1} - z^*\| + \|y_n - z^*\| + \delta_n^1 D_n + \delta_n^2 D_n + \dots + \delta_n^k D_n \\ &\leq \|y_{n+1} - z^*\| + \|y_n - z^*\| + \Big(\sum_{p=1}^{k} \delta_n^p\Big)D_n \\ &= \|y_{n+1} - z^*\| + \|y_n - z^*\| + Q\,D_n \end{aligned} \tag{34}$$

for all $n \geq 0$, where $\delta_n^l = \prod_{i=1}^{l} \alpha_{i,n}$ and $Q = \Big(\sum_{p=1}^{k} \delta_n^p\Big) > 0$. Since $y_n \to z^*$, and $\sum_{n=0}^{\infty} D_n < \infty$, it follows that $\lim_{n\to\infty} \varepsilon_n = 0$. Thus the conclusion (iii) holds. This completes the proof of Theorem 3.1.

If we take $T_1 = T_2 = \dots = T_k = T$ in Theorem 3.1, then we obtain the following conclusion.

Theorem 3.3. *Let E be a real Banach space, K a nonempty closed convex subset of E. Let $T\colon K \to K$ be an asymptotically quasi-nonexpansive mapping in the intermediate sense defined by Definition 2.1(7). Assume that there exist constants L and $\gamma > 0$ such that*

$$\|Tx - q\| \leq L\|x - q\|^{\gamma}, \ \forall x \in K, \ \forall q \in F(T).$$

Put

$$G_n = \max\Big\{0, \sup_{x\in K,\, q\in F(T)} \Big(\|T^n x - q\| - \|x - q\|\Big)\Big\},$$

such that $\sum_{n=0}^{\infty} G_n < \infty$. For any given $x_0 \in K$, define the k-step iterative sequence $\{x_n\}$ by

$$\begin{aligned} z_{k-1,n} &= (1 - \alpha_{k,n})x_n + \alpha_{k,n}T^n x_n, \\ z_{k-2,n} &= (1 - \alpha_{k-1,n})x_n + \alpha_{k-1,n}T^n z_{k-1,n}, \\ &\vdots \\ z_{1,n} &= (1 - \alpha_{2,n})x_n + \alpha_{2,n}T^n z_{2,n}, \\ x_{n+1} &= (1 - \alpha_{1,n})x_n + \alpha_{1,n}T^n z_{1,n}, \ n \geq 0, \end{aligned}$$

where $\{\alpha_{i,n}\}$ is a sequence in $[0, 1]$. Suppose that $\{y_n\}$ is a sequence in K and define a sequence $\{\varepsilon_n\}$ of positive real numbers by

$$\begin{aligned} w_{k-1,n} &= (1 - \alpha_{k,n})y_n + \alpha_{k,n}T^n y_n, \\ w_{k-2,n} &= (1 - \alpha_{k-1,n})y_n + \alpha_{k-1,n}T^n w_{k-1,n}, \\ &\vdots \\ w_{1,n} &= (1 - \alpha_{2,n})y_n + \alpha_{2,n}T^n w_{2,n}, \\ \varepsilon_n &= \|y_{n+1} - (1 - \alpha_{1,n})y_n - \alpha_{1,n}T^n w_{1,n}\|, \ n \geq 0. \end{aligned}$$

If $F(T) \neq \emptyset$, then we have the following:

(i) $\{x_n\}$ converges strongly to a fixed point q of T in K if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

(ii) $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ and $\liminf_{n\to\infty} d(y_n, F(T)) = 0$ imply that $\{y_n\}$ converges strongly to a fixed point q of T in K.

(iii) If $\{y_n\}$ converges strongly to a fixed point q of T in K, then $\lim_{n\to\infty} \varepsilon_n = 0$.

Remark 3.4. *Theorem 3.1 and 3.3 extend, improve and unify the corresponding results of [3], [4], [9], [12]-[14], [17], [20]-[26] to the case of more general class of quasi-nonexpansive, asymptotically nonexpansive, asymptotically quasi-nonexpansive, asymptotically quasi-nonexpansive type mappings and k-step iterative sequence considered in this paper.*

4. Conclusion

The class of asymptotically quasi-nonexpansive mappings in the intermediate sense is more general than the class of quasi nonexpansive, asymptotically nonexpansive, asymptotically quasi-nonexpansive and asymptotically quasi-nonexpansive type mappings. Therefore, the results presented in this paper are improvement and generalization of several known results in the current existing literature.

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