Functional Analysis,
Approximation and
Computation
7 (1) (2015), 85–93

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/faac

Quotient operators: new generation of linear operators

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Abstract. In this paper we attempt to investigate some algebraic and topological properties of quotient operators acting on Hilbert space, and to give a characterization of Fredholm quotient operator and its index.

1. Introduction

Throughout this paper, let $\mathcal{B}(H)$ denote the algebra of all bounded operators acting on a complex Hilbert space H equipped with the inner product $\langle .; . \rangle$ and the associated norm $\|.\|$. Let I and P_M denote respectively the identity operator on H and the orthogonal projection onto a subspace M of H. For T closed densely defined linear operator on H, we define T^* to be the adjoint of T and we denote by $D(T)$, $N(T)$ and $R(T)$ the domain, the null space and range of T, respectively. Also, let $\alpha(T) = \dim N(T)$ and $\beta(T) = \operatorname{codim} R(T)$ to be respectively the nullity and the defect of T. We recall that the ascent $p(T)$ and the descent $q(T)$ of the operator T are respectively defined by

$$p(T) = \inf\{p \in \mathbb{N} \,;\, N(T^p) = N(T^{p+1})\}$$

and

$$q(T) = \inf\{q \in \mathbb{N} \,;\, R(T^q) = R(T^{q+1})\}$$

(if no such integer exists, then $p(T) = \infty$, respectively $q(T) = \infty$.). If the ascent and the descent of T are finite, then they are equal. For $T \in \mathcal{B}(H)$, if $R(T)$ is closed and $\alpha(T) < \infty$ (resp. $\beta(T) < \infty$), then T is called an upper semi-Fredholm (resp. lower semi-Fredholm) operator. The operator T is called semi-fredholm if it is either upper or lower semi-Fredholm. For a semi-Fredholm operator T, its index ind$T$ is defined as ind$T = \alpha(T) - \beta(T)$, T is called Fredholm operator if both of $\alpha(T)$ and $\beta(T)$ are finite. Recall that if $S, T \in \mathcal{B}(H)$

2010 *Mathematics Subject Classification*. 47A05, 47A13.
Keywords. Quotient operator; closed range; generalized inverse; normal operator; nilpotent; idempotent; compact and Fredholm.
Received: 17 July 2014; Accepted: 17 April 2015
Communicated by Snežana Živković Zlatanović
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are two Fredholm operators, then the product ST is also and $\text{ind}(ST) = \text{ind}S + \text{ind}T$. The operator $T \in \mathcal{B}(H)$ is said to be compact if $T(M)$ is relatively compact in H for every bounded subset $M \subset H$. Equivalently, for every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in H, there exists a converging subsequence of $(Tx_n)_{n\in\mathbb{N}}$ in H.
Let A and B be two bounded non-zero (linear) operators on H with kernel condition $N(A) \subset N(B)$. Then the quotient B/A (of bounded operators A and B) is defined by the mapping $Ax \longmapsto Bx$, $x \in H$. $R(A)$, $R(B)$ and $G(A,B) = \{(Ax, Bx);\ x \in H\}$ are respectively the domain, the range and the graph of B/A. We note that the quotient of two bounded operators is not necessarily bounded. For the reader's convenience, let us summarize all what has been obtained about this class of operators.
It was shown in [11] that the sum and the product of two quotients are again represented as quotients, and that if a quotient is densely defined its adjoint is also represented as quotient. In fact, Let B/A and D/C be two quotient operators, then:

1. $B/A + D/C = (BC' + DA')/E$, such that A', C' and E are bounded operators such that $R(E) = R(A) \cap R(C)$, and $AC' = E$ and $CA' = E$.
2. $B/A \times D/C = BM/CN$, where N, M are bounded operators verifying $R(N) = D^{-1}(R(A))$ and $AM = DN$.
3. If B/A is densely defined quotient, then $(B/A)^* = B_*/A_*$ where $A_*, B_* \in \mathcal{B}(H)$ such that $R(A_*) = B^{*^{-1}}(R(A^*))$ and $A^*X = B^*A_*$.

In [14] W. E. Kaufman showed that a linear operator T on H is closed if and only if T is represented as a quotient B/A such that $R(A^*) + R(B^*) = \{A^*x + B^*y : x; y \in H\}$ is closed in H. So that every closed operator is included in the class of quotients. Moreover, he proved in [15],[16] that the quotient operator is only what was called semiclosed one (for this notion see[15],[19]), and that if T is a densely defined closed operator, then T is represented as $T = B/(I - B^*B)^{1/2}$ using a unique pure contraction B, i.e., an operator such that $\|Bx\| < \|x\|$ for all nonzero x in H. Let Γ defined by $\Gamma(B) = B/(I - B^*B)^{1/2}$. Let $C_0(H)$ be the set of all pure contractions and $C(H)$ the set of all closed and densely defined linear operators on H. Then W. Kaufman showed that there is a one-to-one correspondence between $C_0(H)$ and $C(H)$ via Γ. The function Γ is also used to reformulate questions about unbounded operators in terms of bounded ones.

- In [14] and [16], Kaufman proved that the map Γ preserves many properties of operators: self-adjontness, nonnegative conditions, normality and quasinormality.
- In [9] Hirasawa showed that a pure contraction B is hyponormal if and only if $T = B/(I - B^*B)^{1/2}$ is formally hyponormal , and if B is quasinormal then $T^n = B^n/(I - B^*B)^{n/2}$ is quasinormal for all integers $n \geq 2$.

The aim of this paper is to characterize some algebraic, topological and Fredholm properties of quotient operator in general case and of closed densely defined operator represented as quotient using Γ. In fact, our work is organized as follow:
In the second section, we study some algebraic and topological properties of quotient operators such that the boundedness, commpactness, invertibility and normality. The EP character, the powers and limits of quotient operators are also established. The third section is consecrated to the characterization of Fredholm quotient operator and the calculus of its index.
Throughout this paper, B/A is quotient of two non-zero bounded operators $A, B \in \mathcal{B}(H)$.

2. Some algebraic and topological properties of quotient operator

2.1. Bounded, compact quotient operator

First, we recall the Douglas majorization lemma.

Lemma 2.1. *[6] Let $A, B \in \mathcal{B}(H)$. Then the following conditions are equivalent:*

1. *$R(B) \subset R(A)$.*
2. *$BB^* \leq \lambda AA^*$.*
3. *There exists a bounded operator $X \in \mathcal{B}(H)$ such that $B = AX$.*

If one of these conditions holds, then there exists a unique operator $D \in \mathcal{B}(H)$ such that $AD = B$ and $R(D) \subseteq \overline{R(A^*)}$. D is called Douglas solution of the equation $AX = B$.

For a quotient operator B/A, we can easily deduce from this lemma the following:

Corollary 2.2. *B/A is bounded if and only if $R(B^*) \subset R(A^*)$.*

Proof. Since $R(B^*) \subset R(A^*)$, there exists $X^* \in \mathcal{B}(H)$ such that $A^*X^* = B^*$, in an other words, $B = XA$ where $(X^*)^* = X$. Hence, X is a bounded extension of B/A. So, B/A is bounded on H.

The converse implication is deduced directly from the fact that $B = (B/A)A$. □

Obviously, if $R(B^*) \subset R(A^*)$, then $N(A) \subset N(B)$. Thus, for what conditions on A and B we have the converse implication?

As answer of this question, we have the following result due to Barnes [2].

Proposition 2.3. *For two bounded operators $A, B \in \mathcal{B}(H)$ such that A has closed range and $N(A) \subset N(B)$, we have $R(B^*) \subset R(A^*)$.*

This proposition imply immediately the following theorem.

Theorem 2.4. *If B/A is quotient operator with closed domain, that is, $R(A)$ is closed in H, then B/A is bounded.*

We note that this condition is sufficient and not necessary in general. For this, we have the following example.

Example 2.5. *We consider on $l^2(\mathbb{N})$ the operator A defined by*

$$Ax = (\frac{1}{2}x_1, \frac{1}{3}x_2, ..., \frac{1}{n+1}x_n, ...).$$

Then, A is compact since $\lim_{n\to\infty} \frac{1}{n+1} = 0$ and $R(A)$ is dense and not closed in $l^2(\mathbb{N})$. A is invertible with inverse A^{-1} defined as unbounded quotient I/A by

$$(I/A)(y) = A^{-1}(y) = (2y_1, 3y_3, 4y_3, ..., (n+1)y_n, ...).$$

By virtue of [[2], Proposition 5, p156] and the fact that $R(B^*) \subset R(A^*)$, which is equivalent to $\|Bx\| \leq \lambda\|Ax\|$ for all $x \in H$ and λ positive, we have the following result.

Proposition 2.6. *Let B/A bounded quotient operator such that $R(B)$ is closed in H and $N(A) = N(B)$, then $R(A)$ is closed in H.*

Note that the conditions of this proposition (i.e $R(B)$ is closed in H and $N(A) = N(B)$) will be recouped when we introduce the inverse of quotient operator.

Recall from [1] that if $R(A)$ is closed, then there exists a unique bounded operator $A^\dagger$ called the Moore Penrose generalized inverse of A satisfying the following identities:

$$\begin{aligned} &AA^\dagger A = A;\ A^\dagger AA^\dagger = A^\dagger;\ (A^\dagger A)^* = A^\dagger A; \\ &(AA^\dagger)^* = AA^\dagger;\ A^\dagger A = P_{R(A^\dagger)}; AA^\dagger = P_{R(A)}. \end{aligned} \quad (1)$$

Hence, it will be very useful if we arrive to express the quotient operator B/A using $A^\dagger$. For this, we have the following corollary.

Corollary 2.7. *If B/A is quotient operator with closed domain, then $B/A = BA^\dagger$. Furthermore, if A is invertible, then $B/A = BA^{-1}$.*

Proof. It follows immediately from the properties of $A^\dagger$. □

The fact that the quotient is a semiclosed operator and reciprocally, leads us to say that:

Corollary 2.8. *A quotient operator B/A is bounded from $(R(A), \langle ., .\rangle_{B/A})$ to H, where $\langle ., .\rangle_{B/A}$ is the quotient inner product defined by:*

$$\begin{aligned}\langle f, g\rangle_{B/A} &= \langle f, g\rangle + \langle (B/A)f, (B/A)g\rangle \\ &= \langle Ax, Ay\rangle + \langle Bx, By\rangle \text{ for all } \quad f = Ax, g = Ay \in R(A).\end{aligned}$$

According to the definition of a compact operator given in the introduction, we characterize the compact quotient operator B/A where $N(A) \subsetneq N(B)$, (which exclude the case $A = B$ i.e $B/A = I$ on $R(A)$) as follow:

Proposition 2.9. *If B/A is compact, then B is compact. Conversely, if $R(A)$ is closed in H and B is compact, then B/A is also compact.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in H. We have in one hand, $Bx_n = (B/A)Ax_n$ for all $n \in \mathbb{N}$. In other hand, since B/A is compact, then, $((B/A)Ax_n)_{n\in\mathbb{N}}$ has a converging subsequence. In other words, $(Bx_n)_{n\in\mathbb{N}}$ has converging subsequence. Consequently, B is compact on H.

Conversely, the assumption $R(A)$ is closed in H assure the boundedness of B/A. Now, let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in H. Since A is bounded in H, the sequence $(Ax_n)_{n\in\mathbb{N}}$ is also bounded in H. By the compactness of B, the sequence $((B/A)Ax_n)_{n\in\mathbb{N}}$ which is exactly the sequence $(Bx_n)_{n\in\mathbb{N}}$ has a converging subsequence in H. Hence, B/A is compact on H. □

Remark 2.10. *Note that if $R(A)$ is not required to be closed, then B/A need not to be compact when B is compact; take the compact operator A of the first example and consider the quotient A/A.*

Corollary 2.11. *If $N(A) \subsetneq N(B)$, A is of finite range and B is compact, then B/A is compact also.*

Remark 2.12. *In general, the quotient of two compact operators is not necessarily compact.*

2.2. *Inverse of quotient operator*

Our intention in this paragraph, is to prove the following theorem concerning the invertibility of quotient operator.

Theorem 2.13. *Let B/A be a quotient operator on H.*

1. *If $N(A) = N(B)$, then B/A is invertible and $(B/A)^{-1} = A/B$.*
2. *If $N(A) = N(B)$ and $R(B)$ is closed in H, then B/A has a bounded inverse A/B.*
3. *If $R(A)$ and $R(B)$ are closed in H, then the following statements are equivalent*
   (a) *$B^\dagger B$ commutes with $A^\dagger(A^\dagger)^*$ and $A^\dagger A$ commutes with B^*B.*
   (b) *$(B/A)^\dagger(B/A) = AB^\dagger BA^\dagger$ and $(B/A)(B/A)^\dagger = BA^\dagger AB^\dagger$.*
   (c) *$(B/A)^\dagger = AB^\dagger$.*

Proof. 1. The operator A/B is well defined from the condition $N(A) = N(B)$.
Since the domain of A/B is $R(B)$, we notice that the compositions

$$(A/B)(B/A) : \ Au \longrightarrow Bu \longrightarrow Au \ \text{ for all } u \in H$$

$$(B/A)(A/B) : \ Bv \longrightarrow Av \longrightarrow Bv \ \text{ for all } v \in H$$

give the desired equality.

2. It can be shown from 1 and the Proposition 2.6.
3. The fact that $R(A)$ and $R(B)$ are both closed in H ensures the existence of $A^\dagger$, $B^\dagger$ and $(B/A)^\dagger$ since $R(B/A) = R(B)$, we have also $B/A = BA^\dagger$. The equivalence between (a), (b) and (c) yield from [[10], Corollary 3.11].

□

Corollary 2.14. *The quotient operator B/A has an everywhere defined and bounded inverse if the operator B is invertible and*

$$(B/A)^{-1} = A/B = AB^{-1}.$$

For a closed densely defined operator $T \in C(H)$, we have the following theorem.

Theorem 2.15. *Let $T \in C(H)$ represented by $T = \Gamma(B) = B/A = BA^{-1}$ with $B \in C_0(H)$ and $A = (I - B^*B)^{1/2}$. If $B^\dagger$ exists, then $T^\dagger$ exists and $T^\dagger = AB^\dagger$.*

Proof. Let $S = AB^\dagger$ be a bounded operator partially defined on $R(B)$. By the definition of $\Gamma(B)$, we have $D(T) = R(A)$ and $R(T) = R(B)$. Since B^*B commutes with $B^\dagger B$, and $AA^{-1}x = x$ for any $x \in R(A)$, we have

$$STx = AB^\dagger BA^{-1}x = B^\dagger BAA^{-1}x = B^\dagger Bx,$$

for any $x \in D(T)$. On the other hand, for any $y \in R(B)$,

$$TSy = BA^{-1}AB^\dagger y = BB^\dagger y.$$

Thus, $S = T^\dagger$ by virtue of the uniqueness of $T^\dagger$. $\square$

2.3. Normal, hyponormal and EP quotient operator

In the following, we investigate the normality of a densely defined quotient operator B/A. The quotient operator B/A is formally hyponormal (resp. formally normal) if

$$\begin{cases} D(B/A) \subset D((B/A)^*) \\ \|(B/A)^*f\| \leq \|B/Af\| \quad (\textit{resp.}\ \|(B/A)^*f\| = \|B/Af\|) \text{ for } f \in R(A), \end{cases}$$

which is equivalent to

$$\begin{cases} R(A) \subset R(A_*) \\ \|(B_*/A_*)f\| \leq \quad \|B/Af\| \quad (\textit{resp.}\ \|(B_*/A_*)f\| = \|(B/A)f\|) \text{ for } f \in R(A). \end{cases}$$

Using the Douglas majorization lemma, the following result yields.

Theorem 2.16. *If B/A is densely defined quotient, then B/A is formally hyponormal (resp. formally normal) if and only if there exists $X \in \mathcal{B}(H)$ such that*

$$\begin{cases} A = A_*X \\ X^*B_*^*B_*X \leq B^*B; \quad (\textit{resp.}\ X^*B_*^*B_*X = B^*B). \end{cases} \tag{2}$$

Corollary 2.17. *B/A is hyponormal (resp. normal) if and only if there exists an invertible operator $X \in \mathcal{B}(H)$ such that* (2) *holds.*

Note that it was shown in [12] that if A, B are normal operators commuting with each other, then B/A is normal.

Now, for closed densely defined operator $T = \Gamma(B) = B/(I-B^*B)^{1/2} = B/A$ such that B is a pure contraction in $C_0(H)$, A and A_* are the associated defect operators $(I - B^*B)^{1/2}$ and $(I - BB^*)^{1/2}$, respectively. Note that since B is a pure contraction, A and A_* are one-to-one and $\Gamma(B^*) = B^*/A_*$. Recall the following relations proved in [14]: $R(A) = D(T)$, $T^* = B^*/A_*$, $A = (I + T^*T)^{-1/2}$, and thus $T^*T = A^{-2} - I$.
The following theorem characterizes normal unbounded operators with closed ranges on arbitrary Hilbert spaces. Recall that a normal operator B has closed range if and only if it has the group inverse $B^\#$ (for this notion see [1]).

Theorem 2.18. *Suppose that $B \in C_0(H)$ have a closed range. Then the following statements are equivalent:*

1. $\Gamma(B)$ *is normal;*

2. *there is an unitary X such that $(I - B^*B)^{1/2} = (I - BB^*)^{1/2}X$;*
3. *B is normal;*
4. *$B(BB^*B)^\dagger = (BB^*B)^\dagger B$;*
5. *$B(B^* + B^\dagger) = (B^* + B^\dagger)B$;*
6. *$B^\dagger(B + B^*) = (B + B^*)B^\dagger$;*
7. *$indB \leq 1$ and $B^\# B^* = B^* B^\#$;*
8. *$q(B) < \infty$ and $B^*B(BB^*)^\dagger B^*B = BB^*$;*
9. *$p(B) < \infty$ and $BB^*(B^*B)^\dagger BB^* = B^*B$;*
10. *there exists some $X \in \mathcal{B}(H)$ such that $BB^*X = B^*B$ and $B^*BX = BB^*$;*
11. *$p(B) < \infty$ and there exists some $X \in \mathcal{B}(H)$ such that $BX = B^*$ and $(B^\dagger)^*X = B^\dagger$.*

Proof. (1) $\Leftrightarrow$ (2) follows from [[9], Theorem 3.4], (1) $\Leftrightarrow$ (3) follows from [[16], Theorem 2] and the equivalence of statements (3) − (11) follows from [[5], Theorem 3.1]. □

By [[9], Theorem 3.2] and [[5], Theorem 4.1] we have the following result.

Theorem 2.19. *Let $B \in C_0(H)$ and $BB^* + B^*B$ have closed ranges. Then the following statements are equivalent:*

1. *$\Gamma(B)$ is formally hyponormal;*
2. *there is a contraction X such that $(I - B^*B)^{1/2} = (I - BB^*)^{1/2}X$;*
3. *B is hyponormal;*
4. *$2BB^*(BB^* + B^*B)^\dagger BB^* \leq BB^*$.*

An operator $B \in \mathcal{B}(H)$ is EP operator if $R(B)$ is closed and $BB^\dagger = B^\dagger B$. The class of all normal operators with a closed range is a subclass of EP operators, while the class of all hyponormal operators with a closed range is not a subclass of EP operators. Also, the class of all EP operators is not contained in the set of all hyponormal operators. An elementary observation shows that a closed range operator B is EP if and only if $R(B) = R(B^*)$. Recently, Djordjevi (see [5]) obtained several results characterizing EP operators in arbitrary Hilbert spaces. Using properties of quotient operators, we obtain an extension of results from [[5], Theorem 5.1] for unbounded operators. These characterizations are obtained using the definition of the Moore-Penrose inverse of a quotient operator $\Gamma(B)$ where B is a pure contraction. The following theorem characterizes the Moore-Penrose inverse of a quotient operator $\Gamma(B)$.

The fact that $R(T) = R(B), TT^\dagger = BB^\dagger, B^\dagger B = T^\dagger T$ combined with [[5], Theorem 5.1] yield

Theorem 2.20. *Suppose that $B \in C_0(H)$ has a closed range. Then the following statements are equivalent:*

1. *$T = \Gamma(B)$ is EP operator;*
2. *B is EP operator;*
3. *$BB^\dagger = B^2(B^\dagger)^2$ and $p(B) < \infty$;*
4. *$B^\dagger B = (B^\dagger)^2B^2$ and $q(B) < \infty$;*
5. *$ind(B) \leq 1$ and $BB^\dagger B^*B = B^*BBB^\dagger$;*
6. *$ind(B) \leq 1$ and $B^\dagger BBB^* = BB^*B^\dagger B$;*
7. *$ind(B) \leq 1$ and $BB^\dagger(BB^* - B^*B) = (BB^* - B^*B)BB^\dagger$;*
8. *$ind(B) \leq 1$ and $B^\dagger B(BB^* - B^*B) = (BB^* - B^*B)BB^\dagger$;*
9. *$B^*B^\# B + BB^\# B^* = 2B^*$;*
10. *$B^\dagger B^\# B + BB^\# B^\dagger = 2B^\dagger$;*
11. *$BBB^\dagger + B^\dagger BB = 2B$;*
12. *$BBB^\dagger + (BBB^\dagger)^* = B + B^*$ and $p(B) < \infty$;*
13. *$B^\dagger BB + (B^\dagger BB)^* = B + B^*$ and $q(B) < \infty$.*

2.4. Powers of quotient operator

First, we note that it is necessary to assume that $R(B) \subset R(A)$, so that we can discuss about the powers of the quotient operator B/A.

Theorem 2.21. *Let B/A be quotient of two commuting bounded operators A and B such that $R(B) \subset R(A)$. Then,*

$$(B/A)^n = B^n/A^n \text{ for all } n \in \mathbb{N}.$$

Proof. We proceed by induction on the values of n using the definition of product of quotient operators. □

The quotient B/A is nilpotent (resp. idempotent) if $R(B) \subset R(A)$ and $(B/A)^2 = 0$ (resp. $(B/A)^2 = B/A$). This imply immediately the following result.

Theorem 2.22. *The quotient operator B/A is nilpotent (resp. idempotent) if the solution X of the Douglas equation $AX = B$ is nilpotent (resp. idempotent).*

It's follows from this theorem that, the quotient of two idempotent operators is idempotent, and that if B is nilpotent then B/A is again nilpotent.

In general, for $n \in \mathbb{N}$, B/A is $n-$nilpotent (resp. $n-$idempotent) if $AX = B$ and $X^n = 0$ (resp. $X^n = X$).

The quotient operator B/A is said to be quasi-nilpotent if it has null spectral radius, that is

$$\lim_{n\to\infty} \|(B/A)^n Ax\|^{1/n} = 0.$$

Theorem 2.23. *The quotient operator B/A is quasi-nilpotent if the solution X of the Douglas equation $AX = B$ is quasi-nilpotent.*

Proof. Since

$$\|(B/A)^n Ax\|^{1/n} = \|AX^n x\|^{1/n} \text{ for all } x \in H.$$

Then, if X is quasi-nilpotent, we have

$$\lim_{n\to\infty} \|(B/A)^n Ax\|^{1/n} = \lim_{n\to\infty} \|A\|^{1/n}\|X^n x\|^{1/n} = 0.$$

□

2.5. Limit of a sequence of quotient operators

In the following theorem, we try to prove that the limit of a converging quotient operators sequence is also quotient operator.

Theorem 2.24. *Let $(B_n/A_n)_{n\in\mathbb{N}}$ be a sequence of quotient operators converging to an operator C with domain $D(C) = \bigcap_{n\in\mathbb{N}} R(A_n) \cap K$, where K is the Hilbert space of all x in H such that $\lim_{n\to\infty}(B_n/A_n)_n x$ exists. Then C is quotient of two bounded operators.*

Proof. First, let $Q_n = B_n/A_n$ for all $n \in \mathbb{N}$. As we have done above (Corollary 2.8), we consider for all $(x, y) \in (D(C))^2$ the inner product:

$$\langle x, y\rangle_C = \langle x, y\rangle_K + \langle Cx, Cy\rangle = \langle x, y\rangle_K + \lim_{n\to\infty}\langle Q_n x, Q_n y\rangle.$$

Now, we show that $(D(C), \|.\|_C)$ is complete.

Let $(x_m)_{m\in\mathbb{N}}$ be a Cauchy sequence in $(D(C), \|.\|_C)$. Clearly, $(x_m)_{m\in\mathbb{N}}$ is Cauchy in K, H and $(R(A_n), \|.\|_{Q_n})$. Hence, $(x_m)_{m\in\mathbb{N}}$ converges to x in $D(C)$. We have from Corollary 2.8

$$\|Q_n x_m - Q_n x\| \underset{m\to\infty}{\longrightarrow} 0.$$

Therefore

$$\|x_m - x\|_C = \|x_m - x\|_K + \lim_{n\to\infty} \|Q_n x_m - Q_n x\| \underset{m\to\infty}{\longrightarrow} 0.$$

So, $(D(C), \|.\|_C)$ is complete.

It follows from a result of Mac-Nerney ([18],Theorem 3), that there exists an operator $A \in \mathcal{B}(H)$ such that $R(A) = D(C)$ and for all $(x, y) \in (D(C))^2$

$$\langle x, y\rangle_C = \langle A^{-1}x, A^{-1}y\rangle.$$

Set $B = CA$. Then we have for all $x \in H$;

$$\|Bx\|^2 \leq \langle Ax, Ax\rangle_C \leq \|x\|^2.$$

Hence, B is bounded on H and C is the quotient B/A. □

3. Fredholm quotient operator

In this section, we shall give a characterisation of Fredholm quotient operator and its index.

Theorem 3.1. *Let B/A be quotient of two bounded operator A and B. Then,*

1. *If B is Fredholm, then B/A is also.*
2. *If A and B are both Fredholm operators, then,*
   (a) B/A is Fredholm operator and $ind(B/A) = indB - indA$.
   (b) $indA^\dagger = -indA$.

Proof. (1) stems directly from the fact that the range and the null space of B/A are respectively $R(B)$ and $N(B/A) = \{Au;\ u \in N(B)\}$.
(2) Since A, B and B/A are Fredholm operators and $B = (B/A)A$. Then,
(a) $\mathrm{ind}B = \mathrm{ind}(B/A) + \mathrm{ind}A \Longrightarrow \mathrm{ind}(B/A) = \mathrm{ind}B - \mathrm{ind}A$.
(b) Since A is Fredholm, we have $R(A)$ is closed, which is equivalent to say that $A^\dagger$ exists and it is also Fredholm. Moreover, $B/A = BA^\dagger$. Hence, we have from (a) $\mathrm{ind}A^\dagger = -\mathrm{ind}A$.
□

Example 3.2. *Let A and B are two bounded operators defined respectively on $l^2(\mathbb{N})$ by*

$$Ax = (0, x_1, x_2, x_3, ...);\ Bx = (x_2, x_3, x_4, ...).$$

A and B are respectively Fredholm operators with $indA = -1$, $indB = 1$. Since A is injective and $N(B)$ consists of elements of the form $(x_1, 0, 0, ...)$, we define the quotient B/A as follows:

$$(B/A)(0, x_1, x_2, x_3, ...) = (x_2, x_3, x_4, ...).$$

It is very easy to see that $N(B/A)$ consists of elements of the form $(x_1, x_2, 0, ...)$ and $R(B/A) = l^2(\mathbb{N})$. Hence, $\alpha(B/A) = 2$ and $\beta(B/A) = 0$. In other words, B/A is Fredholm operator with index

$$ind(B/A) = 2 = indB - indA.$$

The quotient B/A is Weyl operator if it is Fredholm and ind $B/A = 0$. So, let us characterize in the following corollaries some cases when the quotient of two bounded operators is Weyl operator.

Corollary 3.3.
1. *The quotient of two Fredholm operators with the same index is Weyl operator.*
2. *The quotient of two Weyl operators is Weyl operator.*
3. *If B/A is quotient operator where B is Weyl, then B/A is not necessary Weyl operator.*

For closed densely defined operator $T = B/(I - B^*B)^{1/2}$ where B is a pure contraction on H, we have the following theorem.

Theorem 3.4. $T = \Gamma(B)$ *is Fredholm if and only if* B *is Fredholm operator, and*

$$ind\Gamma(B) = indB.$$

Proof. The equivalence between the fact that $\Gamma(B)$ is Fredholm and B is also Fredholm, yields immediately from Theorem 3.1. The index relation was initially proved by Cordes and Labrousse [3], nevertheless this proof is not technic and is too long. We present here a direct and easy demonstration by virtue of the index relation of quotient operator established in Theorem 3.1.
Fisrt, we shall prove that $(I - B^*B)^{1/2}$ is Weyl operator, that is a Fredholm with index zero. In fact, since the perturbation of a Weyl operator by a self-adjoint operator with norm less then 1 is also Weyl operator, we have $(I - B^*B)$ is Weyl operator, and using functional calculus, the square root of $(I - B^*B)$ is also Weyl operator.
Hence, ind $\Gamma(B)$ = ind B − ind $(I - B^*B)^{1/2}$ = ind B. □

Corollary 3.5. $\Gamma(B)$ *is Weyl operator if and only if* B *is Weyl operator.*

Acknowledgement. The authors would like to thank the anonymous referee for their constructive comments and suggestions to improve the quality of the paper.

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