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On the convergence of modified S-iteration process for generalized asymptotically quasi-nonexpansive mappings in CAT(0) spaces

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Abstract. In this paper, we give the sufficient condition of modified S-iteration process to converge to fixed point for generalized asymptotically quasi-nonexpansive mappings in the framework of CAT(0) spaces. Also we establish some strong convergence theorems of the said iteration process and mapping under semi-compactness and condition (A) which are weaker than completely continuous condition. Our results extend and improve many known results from the existing literature.

1. Introduction

A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces (see [2]), $\mathbb{R}$-trees (see [15]), Euclidean buildings (see [3]), the complex Hilbert ball with a hyperbolic metric (see [10]), and many others. For a thorough discussion of these spaces and of the fundamental role they play in geometry, we refer the reader to Bridson and Haefliger [2].

Fixed point theory in a CAT(0) space has been first studied by Kirk (see [16, 17]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. It is worth mentioning that the results in CAT(0) spaces can be applied to any CAT(k) space with $k \leq 0$ since any CAT(k) space is a CAT(k') space for every $k' \geq k$ (see,e.g., [2]).

The Mann iteration process is defined by the sequence $\{x_n\}$,

$$\begin{cases} x_1 \in K, \\ \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n \geq 1, \end{cases} \tag{1}$$

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where $\{\alpha_n\}$ is a sequence in (0,1).

Further, the Ishikawa iteration process is defined by the sequence $\{x_n\}$,

$$\begin{cases} x_1 \in K, \\ \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ \qquad y_n = (1-\beta_n)x_n + \beta_n T x_n, \quad n \geq 1, \end{cases} \tag{2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in (0,1). This iteration process reduces to the Mann iteration process when $\beta_n = 0$ for all $n \geq 1$.

In 2007, Agarwal, O'Regan and Sahu [1] introduced the S-iteration process in a Banach space,

$$\begin{cases} x_1 \in K, \\ \quad x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n T y_n, \\ \qquad y_n = (1-\beta_n)x_n + \beta_n T x_n, \quad n \geq 1, \end{cases} \tag{3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in (0,1). They showed that their process independent of those of Mann and Ishikawa and converges faster than both of these (see [1, Proposition 3.1]).

Schu [24], in 1991, considered the modified Mann iteration process which is a generalization of the Mann iteration process,

$$\begin{cases} x_1 \in K, \\ \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 1, \end{cases} \tag{4}$$

where $\{\alpha_n\}$ is a sequence in (0,1).

Tan and Xu [29], in 1994, studied the modified Ishikawa iteration process which is a generalization of the Ishikawa iteration process,

$$\begin{cases} x_1 \in K, \\ \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n, \\ \qquad y_n = (1-\beta_n)x_n + \beta_n T^n x_n, \quad n \geq 1, \end{cases} \tag{5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in (0,1). This iteration process reduces to the modified Mann iteration process when $\beta_n = 0$ for all $n \geq 1$.

In 2007, Agarwal, O'Regan and Sahu [1] introduced the modified S-iteration process in a Banach space,

$$\begin{cases} x_1 \in K, \\ \quad x_{n+1} = (1-\alpha_n)T^n x_n + \alpha_n T^n y_n, \\ \qquad y_n = (1-\beta_n)x_n + \beta_n T^n x_n, \quad n \geq 1, \end{cases} \tag{6}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in (0,1). Note that (6) is independent of (5) (and hence of (4)). Also (6) reduces to (3) when $T^n = T$ for all $n \geq 1$.

In 2009, Imnang and Suantai [11] have studied the multi-step iteration process for a finite family of generalized asymptotically quasi-nonexpansive mappings and gave a necessary and sufficient condition for the said scheme and mappings to converge to the common fixed points and also they established some strong convergence theorems in the framework of uniformly convex Banach spaces.

Very recently, Şahin and Başarir [22] modified the iteration process (6) in a CAT(0) space as follows:

Let K be a nonempty closed convex subset of a complete CAT(0) space X and $T\colon K \to K$ be an asymptotically quasi-nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is a sequence generated iteratively by

$$\begin{cases} x_1 \in K, \\ \quad x_{n+1} = (1-\alpha_n)T^n x_n \oplus \alpha_n T^n y_n, \\ \quad\quad y_n = (1-\beta_n)x_n \oplus \beta_n T^n x_n, \quad n \geq 1, \end{cases} \tag{7}$$

where and throughout the paper $\{\alpha_n\}$, $\{\beta_n\}$ are the sequences such that $0 \leq \alpha_n, \beta_n \leq 1$ for all $n \geq 1$. They studied modified S-iteration process for asymptotically quasi-nonexpansive mappings in the CAT(0) space and established some strong convergence results under some suitable conditions which generalize some results of Khan and Abbas [13].

The aim of this paper is to study the modified S-iteration process (7) for generalized asymptotically quasi-nonexpansive mapping and give the sufficient condition to converge to a fixed point in the framework of CAT(0) spaces and also establish some strong convergence results under some suitable conditions. Our results can be applied to an S-iteration process since the modified S-iteration process reduces to the S-iteration process when $n = 1$.

2. Preliminaries and lemmas

In order to prove the main results of this paper, we need the following definitions, concepts and lemmas.

Let (X, d) be a metric space and K be its nonempty subset. Let $T\colon K \to K$ be a mapping. A point $x \in K$ is called a fixed point of T if $Tx = x$. We will also denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in K : Tx = x\}$.

The concept of quasi-nonexpansive mapping was introduced by Diaz and Metcalf [7] in 1967, the concept of asymptotically nonexpansive mapping was introduced by Goebel and Kirk [9] in 1972. The iterative approximation problems for asymptotically quasi-nonexpansive mapping and asymptotically quasi-nonexpansive type mapping were studied by many authors in a Banach space and a CAT(0) space (see, e.g. [6, 8, 14, 18, 19, 21, 23, 26, 27]).

Definition 2.1. *Let (X, d) be a metric space and K be its nonempty subset. Then $T\colon K \to K$ said to be*

(1) nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in K$;

(2) asymptotically nonexpansive [9] if there exists a sequence $\{u_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} u_n = 0$ such that $d(T^n x, T^n y) \leq (1 + u_n)\, d(x, y)$ for all $x, y \in K$ and $n \geq 1$;

(3) quasi-nonexpansive if $d(Tx, p) \leq d(x, p)$ for all $x \in K$ and $p \in F(T)$;

(4) asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{u_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} u_n = 0$ such that $d(T^n x, p) \leq (1 + u_n)\, d(x, p)$ for all $x \in K$, $p \in F(T)$ and $n \geq 1$;

(5) generalized asymptotically quasi-nonexpansive [11] if $F(T) \neq \emptyset$ and there exist sequences $\{u_n\}, \{s_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} u_n = 0 = \lim_{n\to\infty} s_n$ such that $d(T^n x, p) \leq (1 + u_n)\, d(x, p) + s_n$ for all $x \in K$, $p \in F(T)$ and $n \geq 1$;

(6) T is said to be asymptotically nonexpansive mapping in the intermediate sense [4] provided that T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in K} \Big(d(T^n x, T^n y) - d(x,y)\Big) \le 0;$$

(7) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that $d(T^n x, T^n y) \le L\, d(x,y)$ for all $x, y \in K$ and $n \ge 1$;

(8) semi-compact if for a sequence $\{x_n\}$ in K with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in K$.

If we take $s_n = 0$ for all $n \ge 1$ in definition (5), then T is known as an asymptotically quasi-nonexpansive mapping.

Remark 2.2. *Let T be asymptotically nonexpansive mapping in the intermediate sense. Put $c_n = \sup_{x,\,y\in K}\Big(d(T^n x, T^n y) - d(x,y)\Big) \vee 0, \forall n \ge 1$.*

If $F(T) \ne \emptyset$, we obtain that $d(T^n x, p) \le d(x,p) + c_n$ for all $x \in K$ and all $p \in F(T)$. Since $\lim_{n\to\infty} c_n = 0$, therefore T is generalized asymptotically quasi-nonexpansive mapping.

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) *segment* joining x and y. We say X is (i) a *geodesic space* if any two points of X are joined by a geodesic and (ii) a *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$, which we will denote by $[x, y]$, called the segment joining x to y.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists (see [2]).

CAT(0) space

A geodesic metric space is said to be a $CAT(0)$ space if all geodesic triangles of appropriate size satisfy the following $CAT(0)$ comparison axiom.

Let $\triangle$ be a geodesic triangle in X, and let $\overline{\triangle} \subset \mathbb{R}^2$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the $CAT(0)$ inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x,y) \quad \le \quad d_{\mathbb{R}^2}(\overline{x}, \overline{y}). \tag{8}$$

Complete $CAT(0)$ spaces are often called *Hadamard spaces* (see [12]). If x, y_1, y_2 are points of a $CAT(0)$ space and y_0 is the midpoint of the segment $[y_1, y_2]$ which we will denote by $(y_1 \oplus y_2)/2$, then the $CAT(0)$ inequality implies

$$d^2\Big(x, \frac{y_1 \oplus y_2}{2}\Big) \quad \le \quad \frac{1}{2}\, d^2(x, y_1) + \frac{1}{2}\, d^2(x, y_2) - \frac{1}{4}\, d^2(y_1, y_2). \tag{9}$$

The inequality (9) is the (CN) inequality of Bruhat and Tits [5].

Let us recall that a geodesic metric space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality (see [2, page 163]).

A subset C of a $CAT(0)$ space X is convex if for any $x, y \in C$, we have $[x, y] \subset C$.

Lemma 2.3. *(See [20]) Let X be a CAT(0) space.*

(i) For $x,\ y \in X$ and $t \in [0,1]$, there exists a unique point $z \in [x,\ y]$ such that

$$d(x,\ z) = t\,d(x,\ y) \quad \text{and} \quad d(y,\ z) = (1-t)\,d(x,\ y). \quad (A)$$

We use the notation $(1-t)x \oplus ty$ for the unique point z satisfying (A).

(ii) For $x,\ y \in X$ and $t \in [0,1]$, we have

$$d((1-t)x \oplus ty, z) \le (1-t)d(x,z) + td(y,z).$$

Lemma 2.4. *(See [28]) Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of non-negative real numbers satisfying the inequality*

$$a_{n+1} \le (1+\delta_n)a_n + b_n, \qquad n \ge 1.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence converging to zero, then $\lim_{n\to\infty} a_n = 0$.

Example 2.5. *(A generalized asymptotically quasi-nonexpansive mapping whose fixed point set is not closed). Let $X = \mathbb{R}$, $K = [-1,1]$ and $d(x,y) = |x-y|$ be the usual metric on X. Let $T\colon K \to K$ be a mapping defined by*

$$\begin{cases} T(x) = x, & \text{if } x \in [-1,0) \\ \quad\;\; = \dfrac{1}{2}, & \text{if } x = 0 \\ \quad\;\; = x^2, & \text{if } x \in (0,1]. \end{cases}$$

Then T is generalized asymptotically nonexpansive mapping and T is discontinuous at $x = 0$ and hence T is not Lipschitzian. Also notice that $F(T) = [-1,0)$ is not closed. We prove that

$$|T^n x - T^n y| \quad \le \quad |x-y| + \frac{1}{2^{2^n}} \quad (*)$$

for all $x, y \in [-1,1]$ and $n \ge 1$. The inequality above holds trivially if $x = y = 0$ or $x, y \in [-1,0)$. Then it suffices to consider the following cases.

Case 1 $(x, y \in (0,1])$. Then

$$|T^n x - T^n y| = |x^{2^n} - y^{2^n}| \le \frac{1}{2^{2^n}}.$$

Case 2 $(x \in [-1,0)$ and $y = 0)$. Then

$$|T^n x - T^n y| = \left|x - \frac{1}{2^{2^n}}\right| \le |x-y| + \frac{1}{2^{2^n}}.$$

Case 3 $(x \in [-1,0)$ and $y \in (0,1])$. Then

$$|T^n x - T^n y| = |x - y^{2^n}| \le |x-y|.$$

Case 4 $(x = 0$ and $y \in (0,1])$. Then

$$|T^n x - T^n y| = \left|\frac{1}{2^{2^n}} - y^{2^n}\right| \le |x-y| + \frac{1}{2^{2^n}}.$$

Hence the condition $(*)$ holds. This completes the proof.

Now, we give the sufficient condition that guarantees the closedness of the fixed point set of a generalized asymptotically quasi-nonexpansive mapping.

Proposition 2.6. *Let K be a nonempty subset of a complete $CAT(0)$ space X and $T\colon K \to K$ be a generalized asymptotically quasi-nonexpansive mapping. If $G(T) := \{(x, Tx) : x \in K\}$ is closed, then $F(T)$ is closed.*

Proof. Let $\{p_n\}$ be a sequence in $F(T)$ such that $p_n \to p$ as $n \to \infty$. Since T is a generalized asymptotically quasi-nonexpansive mapping with the sequence $\{(u_n, s_n)\}$, we have

$$\begin{aligned} d(T^n p, p) &\leq d(T^n p, p_n) + d(p_n, p) \\ &\leq (1 + u_n)d(p_n, p) + s_n + d(p_n, p) \\ &= (2 + u_n)d(p_n, p) + s_n \to 0, \text{ as } n \to \infty. \end{aligned}$$

Then $T^n p \to p$, and so $T(T^n p) = T^{n+1}p \to p$. Hence by closedness of $G(T)$, $Tp = p$, that is, $F(T)$ is closed. Thus we conclude that the fixed point set for non Lipschitzian generalized asymptotically quasi-nonexpansive mapping must be closed. This completes the proof. □

3. Main Results

In this section, we establish some strong convergence results of modified S-iteration scheme (7) to converge to a fixed point for generalized asymptotically quasi-nonexpansive mapping in the setting of CAT(0) space.

Theorem 3.1. *Let K be a nonempty closed convex subset of a complete $CAT(0)$ space X and let $T\colon K \to K$ be a generalized asymptotically quasi-nonexpansive mapping with sequences $\{u_n\}, \{s_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} u_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$. Suppose that $F(T) \neq \emptyset$ is closed. Let $\{x_n\}$ be defined by the iteration process (7). If $\liminf_{n\to\infty} d(x_n, F(T)) = 0$ or $\limsup_{n\to\infty} d(x_n, F(T)) = 0$, then the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. Let $p \in F(T)$. From (7), Definition 2.1(5) and Lemma 2.3(ii), we have

$$\begin{aligned} d(y_n, p) &= d((1 - \beta_n)x_n \oplus \beta_n T^n x_n, p) \\ &\leq (1 - \beta_n)d(x_n, p) + \beta_n d(T^n x_n, p) \\ &\leq (1 - \beta_n)d(x_n, p) + \beta_n[(1 + u_n)d(x_n, p) + s_n] \\ &\leq (1 + u_n)d(x_n, p) + s_n. \end{aligned} \tag{10}$$

Again using (7), (10), Definition 2.1(5) and Lemma 2.3(ii), we have

$$\begin{aligned} d(x_{n+1}, p) &= d((1 - \alpha_n)T^n x_n \oplus \alpha_n T^n y_n, p) \\ &\leq (1 - \alpha_n)d(T^n x_n, p) + \alpha_n d(T^n y_n, p) \\ &\leq (1 - \alpha_n)[(1 + u_n)d(x_n, p) + s_n] + \alpha_n[(1 + u_n)d(y_n, p) + s_n] \\ &\leq (1 - \alpha_n)(1 + u_n)d(x_n, p) + \alpha_n(1 + u_n)d(y_n, p) + s_n \\ &\leq (1 - \alpha_n)(1 + u_n)d(x_n, p) + \alpha_n(1 + u_n)[(1 + u_n)d(x_n, p) + s_n] + s_n \\ &\leq (1 + u_n)^2 d(x_n, p) + (2 + u_n)s_n \\ &= (1 + \mu_n)d(x_n, p) + \sigma_n, \end{aligned} \tag{11}$$

where $\mu_n = 2u_n + u_n^2$ and $\sigma_n = (2 + u_n)s_n$. Since by hypothesis of the theorem $\sum_{n=1}^{\infty} u_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$, it follows that $\sum_{n=1}^{\infty} \mu_n < \infty$ and $\sum_{n=1}^{\infty} \sigma_n < \infty$. This gives

$$d(x_{n+1}, F(T)) \leq (1 + \mu_n)d(x_n, F(T)) + \sigma_n. \tag{12}$$

Since by hypothesis $\sum_{n=1}^{\infty} \mu_n < \infty$ and $\sum_{n=1}^{\infty} \sigma_n < \infty$ by Lemma 2.4 and $\liminf_{n\to\infty} d(x_n, F(T)) = 0$ or $\limsup_{n\to\infty} d(x_n, F(T)) = 0$ gives that

$$\lim_{n\to\infty} d(x_n, F(T)) = 0. \tag{13}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence in K. With the help of inequality $1 + x \leq e^x$, $x \geq 0$. For any integer $m \geq 1$, we have from (11) that

$$\begin{aligned}
d(x_{n+m}, p) &\leq (1 + \mu_{n+m-1})d(x_{n+m-1}, p) + \sigma_{n+m-1} \\
&\leq e^{\mu_{n+m-1}} d(x_{n+m-1}, p) + \sigma_{n+m-1} \\
&\leq e^{\mu_{n+m-1}} e^{\mu_{n+m-2}} d(x_{n+m-2}, p) + e^{\mu_{n+m-1}} \sigma_{n+m-2} + \sigma_{n+m-1} \\
&\leq \dots \\
&\leq \left(e^{\sum_{k=n}^{n+m-1} \mu_k}\right) d(x_n, p) + \left(e^{\sum_{k=n}^{n+m-1} \mu_k}\right) \sum_{j=n}^{n+m-1} \sigma_j \\
&\leq \left(e^{\sum_{n=1}^{\infty} \mu_n}\right) d(x_n, p) + \left(e^{\sum_{n=1}^{\infty} \mu_n}\right) \sum_{j=n}^{n+m-1} \sigma_j \\
&= W d(x_n, p) + W \sum_{j=n}^{n+m-1} \sigma_j
\end{aligned} \tag{14}$$

where $W = e^{\sum_{n=1}^{\infty} \mu_n}$.

Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, without loss of generality, we may assume that a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{p_{n_k}\} \subset F(T)$ such that $d(x_{n_k}, p_{n_k}) \to 0$ as $k \to \infty$. Then for any $\varepsilon > 0$, there exists $k_\varepsilon > 0$ such that

$$d(x_{n_k}, p_{n_k}) < \frac{\varepsilon}{4W} \quad \text{and} \quad \sum_{j=n_{k_\varepsilon}}^{\infty} \sigma_j < \frac{\varepsilon}{4W}, \tag{15}$$

for all $k \geq k_\varepsilon$.

For any $m \geq 1$ and for all $n \geq n_{k_\varepsilon}$, by (14) and (15), we have

$$\begin{aligned}
d(x_{n+m}, x_n) &\leq d(x_{n+m}, p_{n_k}) + d(x_n, p_{n_k}) \\
&\leq W d(x_{n_k}, p_{n_k}) + W \sum_{j=n_{k_\varepsilon}}^{\infty} \sigma_j + W d(x_{n_k}, p_{n_k}) + W \sum_{j=n_{k_\varepsilon}}^{\infty} \sigma_j \\
&= 2W d(x_{n_k}, p_{n_k}) + 2W \sum_{j=n_{k_\varepsilon}}^{\infty} \sigma_j \\
&< 2W.\frac{\varepsilon}{4W} + 2W.\frac{\varepsilon}{4W} = \varepsilon.
\end{aligned} \tag{16}$$

This proves that $\{x_n\}$ is a Cauchy sequence in K. Thus, the completeness of X implies that $\{x_n\}$ must be convergent. Assume that $\lim_{n\to\infty} x_n = q$. Since K is closed, therefore $q \in K$. Next, we show that $q \in F(T)$. Now $\lim_{n\to\infty} d(x_n, F(T)) = 0$ gives that $d(q, F(T)) = 0$. Since $F(T)$ is closed, $q \in F(T)$. This completes the proof. □

Theorem 3.2. *Let K be a nonempty closed convex subset of a complete CAT(0) space X and let $T\colon K \to K$ be a generalized asymptotically quasi-nonexpansive mapping with sequences $\{u_n\}, \{s_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} u_n < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$. Suppose that $F(T) \neq \emptyset$ is closed. Let $\{x_n\}$ be defined by the iteration process (7). If T satisfies the following conditions:*

(i) $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

(ii) If the sequence $\{z_n\}$ *in K satisfies* $\lim_{n\to\infty} d(z_n, Tz_n) = 0$, *then* $\liminf_{n\to\infty} d(z_n, F(T)) = 0$ *or* $\limsup_{n\to\infty} d(z_n, F(T)) = 0$.

Then the sequence $\{x_n\}$ *converges strongly to a fixed point of T.*

Proof. It follows from the hypothesis that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. From (ii), $\liminf_{n\to\infty} d(x_n, F(T)) = 0$ or $\limsup_{n\to\infty} d(x_n, F(T)) = 0$. Therefore, the sequence $\{x_n\}$ must converge to a fixed point of T by Theorem 3.1. This completes the proof. □

Theorem 3.3. *Let K be a nonempty closed convex subset of a complete CAT(0) space X and let* $T\colon K \to K$ *be uniformly L-Lipschitzian and generalized asymptotically quasi-nonexpansive mapping with sequences* $\{u_n\}, \{s_n\} \subset [0,\infty)$ *such that* $\sum_{n=1}^{\infty} u_n < \infty$ *and* $\sum_{n=1}^{\infty} s_n < \infty$. *Suppose that* $F(T) \neq \emptyset$ *is closed. Let* $\{x_n\}$ *be defined by the iteration process (7). If T is semi-compact and* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, *then the sequence* $\{x_n\}$ *converges to a fixed point of T.*

Proof. From the hypothesis, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Also, since T is semi-compact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to p \in K$. Hence, we have

$$\begin{aligned} d(p, Tp) &\leq d(p, x_{n_j}) + d(x_{n_j}, Tx_{n_j}) + d(Tx_{n_j}, Tp) \\ &\leq (1+L) d(p, x_{n_j}) + d(x_{n_j}, Tx_{n_j}) \to 0. \end{aligned}$$

Thus $p \in F(T)$. By (11),

$$d(x_{n+1}, p) \leq (1+\mu_n) d(x_n, p) + \sigma_n.$$

Since $\sum_{n=1}^{\infty} \mu_n < \infty$ and $\sum_{n=1}^{\infty} \sigma_n < \infty$, by Lemma 2.4, $\lim_{n\to\infty} d(x_n, p)$ exists and $x_{n_j} \to p \in F(T)$ gives that $x_n \to p \in F(T)$. This completes the proof. □

We recall the following definition.

A mapping $T\colon K \to K$, where K is a subset of a normed linear space E, is said to satisfy Condition (A) [25] if there exists a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0,\infty)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in K$ where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T) \neq \emptyset\}$. It is to be noted that Condition (A) is weaker than compactness of the domain K.

As an application of Theorem 3.1, we establish another strong convergence result employing Condition (A) as follows.

Theorem 3.4. *Let K be a nonempty closed convex subset of a complete CAT(0) space X and let* $T\colon K \to K$ *be a generalized asymptotically quasi-nonexpansive mapping with sequences* $\{u_n\}, \{s_n\} \subset [0,\infty)$ *such that* $\sum_{n=1}^{\infty} u_n < \infty$ *and* $\sum_{n=1}^{\infty} s_n < \infty$. *Suppose that* $F(T) \neq \emptyset$ *is closed. Let* $\{x_n\}$ *be defined by the iteration process (7). Assume that* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. *Let T satisfy Condition (A), then the sequence* $\{x_n\}$ *converges strongly to a fixed point of T.*

Proof. Since by hypothesis

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{17}$$

From Condition (A) and (17), we get

$$\lim_{n\to\infty} f\Big(d(x_n, F(T))\Big) \leq \lim_{n\to\infty} d(x_n, Tx_n) = 0,$$

i.e., $\lim_{n\to\infty} f\Big(d(x_n, F(T))\Big) = 0$. Since $f\colon [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(t) > 0$ for all $t \in (0,\infty)$, therefore we have

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

Now all the conditions of Theorem 3.1 are satisfied, therefore by its conclusion $\{x_n\}$ converges strongly to a point of $F(T)$. This completes the proof. $\square$

4. Conclusion

The class of mappings used in this article is more general than that of asymptotically nonexpansive, asymptotically quasi-nonexpansive and asymptotically quasi-nonexpansive in the intermediate sense mappings. Thus the results obtained in this article are good improvement and generalization of some previous works for a CAT(0) space given in the existing literature.

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