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On Finite Rank Perturbation of Diagonalizable Operators

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Abstract. Let H be a Hilbert space. In this paper we give a necessary and sufficient condition for a $\lambda \in \mathbb{C}$ to be an eigenvalue of the linear operator $T = D + \sum_{i=1}^{n} u_i \otimes v_i$, where D is a diagonalizable operator and $u_i, v_i \in H$, $i = 1, \ldots, n$.

1. Introduction and Preliminaries

Throughout this paper, let H denote a separable (complex) Hilbert space, and $B(H)$ the C^*-algebra of all bounded linear operators on H. We say that an operator $D \in B(H)$ is diagonalizable if there exists an orthonormal basis $\{e_n\}$ for H and a bounded sequence $\{\lambda_n\}$ such that $D(e_n) = \lambda_n e_n$ for all $n \in \mathbb{N}$. For $W \subset B(H)$ we denote by W' the set of all operators which commute with elements of W and set $W'' = (W')'$. For any $u, v \in H$ the rank one operator $u \otimes v$ is defined by $(u \otimes v)(x) = \langle x, v\rangle u$. Let us recall that a norm-closed subspace M of H is called a nontrivial hyperinvariant subspace for T if $\{0\} \neq M \neq H$ and it is an invariant subspace for every operator $S \in \{T\}'$. We use the matrix representation for bounded linear operators on a separable Hilbert space; i.e., if $T \in B(H)$ and $\{e_n\}$ is an orthonormal basis for a separable Hilbert space H, then an infinite matrix (a_{ij}) represents T when $Tx = \sum_i (\sum_j a_{ij} x_j) e_j$ for all $x = \sum_{i=1}^{\infty} x_i e_i \in H$. In this case, we have $\sum_i |a_{ij}|^2 \leq c$ and $\sum_j |a_{ij}|^2 \leq c$ for some $c > 0$, in this case the matrix operator $T = (a_{ij})$ is Hilbert-Schmidt operator if $\sum_{i,j} |a_{ij}|^2 < \infty$. For more details see [6, Theorem 6.21, Theorem 5.6].
Our study motivated by the following problem;

Does every finite rank perturbation of a diagonalizable operator have a nontrivial hyperinvariant subspace?

This problem has been considered in several papers and solved in some special cases [1-5]. In [2] it was shown that if an operator $T \notin \mathbb{C}1$ has the form $T = D + u \otimes v$, where D is a diagonalizable operator and the Fourier coefficients $\{\alpha_k\}$ and $\{\beta_k\}$ of u and v with respect to the orthonormal basis which diagonalizes D satisfy $\sum_{k=1}^{\infty} (|\alpha_k|^{\frac{2}{3}} + |\beta_k|^{\frac{2}{3}}) < \infty$, then T has a nontrivial hyperinvariant subspace. In [1], it was shown that if $T = D + \sum_{i=1}^{n} u_i \otimes v_i$, where D is a diagonalizable operator with respect to the orthonormal basis $\{e_n\}$ and, moreover, the Fourier coefficients of u_i, v_i belong to the space l^1 , then T has a nontrivial hyperinvariant subspace. It is clear that, if λ is an eigenvalue of T, then $\mathcal{N}(T - \lambda)$ is a nontrivial hyperinvariant subspace.

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In [5] it was shown that if the operator $T = D + u \otimes v$ is not a normal operator and for some $n_0 \in \mathbb{N}$, $\alpha_{n_0} = 0$ or $\beta_{n_0} = 0$(α_n and β_n are the Fourier coefficients of u, v, respectively), then T^*, the adjoint of T, has an eigenvalue. Also it is shown that, if all Fourier coefficient of u and v are non-zero and at least one eigenvalue of D has multiplicity larger than 1, then T has an eigenvalue. The following theorem was proved in the same paper:

Theorem 1.1. *Let $u, v \in H$ and D be a diagonalizable operator. Suppose that $T = D + u \otimes v$ and $\mu \in \sigma(D)$. Then $\mu \in \rho(T)$ if and only if the following conditions are satisfied:*
(i) μ is an isolated eigenvalue of D, λ_{n_0}, of multiplicity one,
(ii) $\beta_{n_0} = \langle v, e_{n_0}\rangle \neq 0$ and $\alpha_{n_0} = \langle u, e_{n_0}\rangle \neq 0$.

Our aim in this paper is to generalize this theorem to the operator $T = D + \sum_{k=1}^{n} u_i \otimes v_i$ where D is diagonalizable operator and $u_i, v_i \in H$ for all $i = 1, \dots, n$.

2. Main Results

Throughout this work, let H be a separable Hilbert space and D be a diagonalizable operator with $De_n = \lambda_n e_n$ for all $n = 1, \dots, n$. In this case the set of eigenvalues of D is $\{\lambda_n; n \in \mathbb{N}\}$. First note that if $\{u_1, u_2, \dots, u_n\}$ or $\{v_1, v_2, \dots, v_n\}$ is linearly dependent then $T = D + \sum_{i=1}^{n} u_i \otimes v_i = D + \sum_{i=1}^{k} u'_i \otimes v'_i$ for some u'_i, v'_i in H and $k < n$, so without loss of generality, we assume that $\{u_1, \dots, u_n\}$ and $\{v_1, \dots, v_n\}$ are linearly independent.

Theorem 2.1. *Let $T = D + \sum_{i=1}^{n} u_i \otimes v_i$, $\lambda \in \sigma(D)$ and $u_i = \sum_{k=1}^{\infty} \alpha_{ik} e_k$, $v_i = \sum_{k=1}^{\infty} \beta_{ik} e_k$. Then $\lambda \in \rho(T)$ or $\lambda \in \rho(T^*)$ if and only if λ is an isolated eigenvalue of D with multiplicity m with respect to eigenvectors $e_{n_1}, e_{n_2}, \dots, e_{n_m}$, such that*

$$\left\{\begin{pmatrix} \alpha_{1n_1} \\ \alpha_{1n_2} \\ \vdots \\ \alpha_{1n_m} \end{pmatrix}, \begin{pmatrix} \alpha_{2n_1} \\ \alpha_{2n_2} \\ \vdots \\ \alpha_{2n_m} \end{pmatrix}, \dots, \begin{pmatrix} \alpha_{nn_1} \\ \alpha_{nn_2} \\ \vdots \\ \alpha_{nn_m} \end{pmatrix}\right\} \tag{1}$$

and

$$\left\{\begin{pmatrix} \beta_{1n_1} \\ \beta_{2n_1} \\ \vdots \\ \beta_{nn_1} \end{pmatrix}, \begin{pmatrix} \beta_{1n_2} \\ \beta_{2n_2} \\ \vdots \\ \beta_{nn_2} \end{pmatrix}, \dots, \begin{pmatrix} \beta_{1n_m} \\ \beta_{2n_m} \\ \vdots \\ \beta_{nn_m} \end{pmatrix}\right\} \tag{2}$$

are linearly independent.

Proof. Suppose that λ is an isolated eigenvalue of D with multiplicity m and $\lambda = \lambda_{n_1} = \cdots = \lambda_{n_m}$, such that (1) and (2) are linearly independent. We show that $\lambda \notin \sigma(T)$. Since $D - \lambda$ and so $T - \lambda$ are Fredholm operators of index zero, it suffices to prove that $\lambda \notin \sigma_p(T)$. Suppose that $(T - \lambda)x = 0$. It follows that

$$(D - \lambda)x + \sum_{i=1}^{n} \langle x, v_i\rangle u_i = 0. \tag{3}$$

Considering n_k-th component of the matrix representation in (3), we get

$$\sum_{i=1}^{n} \langle x, v_i\rangle \alpha_{in_k} = 0, \quad k = 1, \dots, n$$

and thus

$$\langle x, v_1\rangle \begin{pmatrix} \alpha_{1n_1} \\ \vdots \\ \alpha_{1n_m} \end{pmatrix} + \cdots + \langle x, v_n\rangle \begin{pmatrix} \alpha_{nn_1} \\ \vdots \\ \alpha_{nn_m} \end{pmatrix} = 0\,. \tag{4}$$

By hypothesis, for every $i = 1, \ldots, n$, we have $\langle x, v_i\rangle = 0$. Now, (3) implies that $(D - \lambda)x = 0$, so $x = \sum_{k=1}^{m} c_k e_{n_k}$ for some scalars c_k. Therefore, by (3), we have

$$\sum_i \sum_k c_k \bar{\beta}_{in_k} u_i = 0.$$

Since $\{u_1, \ldots, u_n\}$ is a linearly independent set, hence, taking complex conjugates

$$c_1 \overline{\begin{pmatrix} \beta_{1n_1} \\ \beta_{2n_1} \\ \vdots \\ \beta_{nn_1} \end{pmatrix}} + \cdots + c_m \overline{\begin{pmatrix} \beta_{1n_m} \\ \beta_{2n_m} \\ \vdots \\ \beta_{nn_m} \end{pmatrix}} = 0.$$

This yields that $c_k = 0$, for $k = 1, \ldots, n$, and therefore $x = 0$.

Conversely, assume that the second statement of theorem does not hold. Thus we have three cases: (1) λ is not eigenvalue of D; (2) λ is an eigenvalue of D, but it is not isolated; (3) λ is an eigenvalue of D with multiplicity m and at least one of (1) and (2) are linearly dependent.

If the cases (1) and (2) hold, then there exists a sequence (λ_{j_k}) of eigenvalues of D such that converges to λ. Since

$$(T - \lambda)e_{j_k} = (\lambda_{j_k} - \lambda)e_{j_k} + \sum_{i=1}^{n} \langle e_{j_k}, v_i\rangle u_i,$$

we have

$$\|(T - \lambda)e_{j_k}\| \le |\lambda_{j_k} - \lambda| + \sum_{i=1}^{n} |\langle e_{j_k}, v_i\rangle| \|u_i\|,$$

which converges to zero and shows that $\lambda \in \sigma(T)$. If the case (3) is valid, and we assume that (2) is linearly dependent and $\lambda = \lambda_{n_1} = \lambda_{n_2} = \cdots = \lambda_{n_m}$, then

$$(T - \lambda)e_{n_k} = \sum_i \langle e_{n_k}, v_i\rangle u_i = \sum_i \bar{\beta}_{in_k} u_i \tag{5}$$

On the other hand, by using the fact that (2) is linearly dependent, there exist scalars c_k not simultaneously zero such that

$$\sum_k c_k \bar{\beta}_{in_k} = 0\ ,\ i = 1, \ldots, m$$

Therefore $(T - \lambda)(c_1 e_{n_1} + \cdots + c_m e_{n_m}) = 0$. Hence λ is an eigenvalues of T.

Similarly we can show that λ is an eigenvalue of T^* whenever (1) is linearly dependent. □

Remark 2.2. *If λ satisfies in the hypothesis of the above theorem, then $\mathcal{N}(T - \lambda)$ is a nontrivial hyperinvariant subspace of T. The above theorem is a generalization of Propositions 2.1, 2.2 and 2.3 in [5], for the finite rank perturbation of diagonalzable operators.*

Let W be the set of scalars λ, such that u_i and v_i belong to $\mathrm{Im}(D - \lambda) \cap \mathrm{Im}(D^* - \bar{\lambda})$, for every $i = 1, \cdots, n$. Assume that $u \in H$ and $u = \sum_{k=1}^{\infty} \alpha_k e_k$ and $\lambda \in W$. Then $(D - \lambda)^{-1} : \mathrm{Im}(D - \lambda) \longrightarrow H$ is well-defined and $u \in \mathrm{Im}(D - \lambda)$ if and only if $\sum_{k=1}^{\infty} \frac{|\alpha_k|^2}{|\lambda_k - \lambda|^2} < \infty$.

Now we have the following theorem.

Theorem 2.3. *Suppose that $\lambda \in W$. Then $\lambda \notin \sigma_p(T)$ if and only if*

$$A(\lambda) = \begin{pmatrix} 1 + \langle (D-\lambda)^{-1}u_1, v_1\rangle & \dots & \langle (D-\lambda)^{-1}u_n, v_1\rangle \\ \vdots & \ddots & \vdots \\ \langle (D-\lambda)^{-1}u_1, v_n\rangle & \dots & 1 + \langle (D-\lambda)^{-1}u_n, v_n\rangle \end{pmatrix} \tag{6}$$

is invertible.

Proof. First, assume $A(\lambda)$ is invertible and there exists $x \neq 0$ such that $(T-\lambda)x = 0$. Thus we have

$$(D-\lambda)(x + \langle x, v_1\rangle (D-\lambda)^{-1}u_1 + \cdots + \langle x, v_n\rangle (D-\lambda)^{-1}u_n) = 0.$$

This implies that

$$x + \langle x, v_1\rangle (D-\lambda)^{-1}u_1 + \cdots + \langle x, v_n\rangle (D-\lambda)^{-1}u_n = 0$$

which at least one of the $\langle x, v_i\rangle$ is not zero. By the inner product with v_i $(i = 1, \dots, n)$, we get

$$\begin{cases} (1 + \langle (D-\lambda)^{-1}u_1, v_1\rangle)\langle x, v_1\rangle + \cdots + \langle (D-\lambda)^{-1}u_n, v_1\rangle\langle x, v_n\rangle = 0 \\ \vdots \\ \langle (D-\lambda)^{-1}u_1, v_n\rangle\langle x, v_1\rangle + \cdots + (1 + \langle (D-\lambda)^{-1}u_n, v_n)\langle x, v_n\rangle = 0 \end{cases} \tag{7}$$

This is a contradiction to the invertiblity of matrix (6). Hence $\mathcal{N}(T-\lambda) = 0$.

Now suppose that the matrix (6) is not invertible. Then there is a nonzero vector x such that $\langle x, v_1\rangle, \langle x, v_2\rangle, \dots, \langle x, v_n\rangle$ is a solution of the homogeneous systems of equations (7). Hence $y = -\sum_{i=1}^{n}\langle x, v_i\rangle(D-\lambda)^{-1}u_i$ is nonzero and $(I + \sum_{i=1}^{n}(D-\lambda)^{-1}u_i \otimes v_i)y = 0$, whence $(T-\lambda)y = 0$. □

We define the linear operator $\phi : M_n(B(H)) \to H$ by $\phi(a_{ij}) = \sum_{i,j} a_{ij}$, we also set $B(\lambda) = ((D-\lambda)^{-1}u_i \otimes (D^* - \bar{\lambda})^{-1}v_j)$ and $K(\lambda) = \phi(A^{-1}(\lambda)B(\lambda))$, where $A(\lambda)$ was introduced in Theorem 2.3, for every $\lambda \in W$.

Corollary 2.4. *For any f in* Im$(D-\lambda)$ *and $\lambda \in W$ we have*

$$(T-\lambda)\left[(D-\lambda)^{-1} - K(\lambda)\right] f = f.$$

Proof. Let $f \in \mathrm{Im}(D-\lambda)$ and $A^{-1}(\lambda) = (a_{ij})$, then

$$\begin{aligned}
&(T-\lambda)\left[(D-\lambda)^{-1} - K(\lambda)\right] f \\
&= f - \sum_{i,j} a_{ij}\langle f, (D^* - \bar{\lambda})^{-1}v_j\rangle u_i + \sum_{l=1}^{n}\langle f, (D^* - \bar{\lambda})^{-1}v_l\rangle u_l - \sum_{l=1}^{n}\langle K(\lambda)f, v_l\rangle u_l \\
&= f - \sum_{i,j} a_{ij}\langle f, (D^* - \bar{\lambda})^{-1}v_j\rangle u_i + \sum_{l=1}^{n}\langle f, (D^* - \bar{\lambda})^{-1}v_l\rangle u_l \\
&\quad - \sum_{l=1}^{n}\sum_{j=1}^{n}\left\langle \sum_{i=1}^{n} a_{ij}\langle (D-\lambda)^{-1}u_i, v_l\rangle f, (D^* - \bar{\lambda})^{-1}v_j\right\rangle u_l \\
&= f - \sum_{i,j} a_{ij}\langle f, (D^* - \bar{\lambda})^{-1}v_j\rangle u_i + \sum_{l=1}^{n}\langle f, (D^* - \bar{\lambda})^{-1}v_l\rangle u_l \\
&\quad - \sum_{l=1}^{n}\sum_{j=1}^{n}\left\langle (\delta_{lj} - a_{lj})f, (D^* - \bar{\lambda})^{-1}v_j\right\rangle u_l \\
&= f
\end{aligned}$$

□

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