Functional Analysis,
Approximation and
Computation
15 (2) (2023), 23–27

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/faac

On the mixed identity for normed modules

Robin E. Harte[a], Dragan S. Djordjević[b]

[a]*Trinity College, Dublin, Ireland*
[b]*University of Niš, Faculty of Sciences and Mathematics, 18000 Niš, Serbia*

Abstract. The mixed identity for normed modules shows that the real and the complex duals of a complex Banach space coincide, offers a partial extension of the von Neumann double commutant theorem to reflexive Banach spaces, explains the Arens multiplication on the second dual of a Banach algebra, and has a high old time playing around among the integration spaces

1. Sets and Mappings

Recall the *Cartesian Product* of sets

$$N \times M \equiv \{(n, m) : m \in M, n \in N\} \tag{1.1}$$

which converts functions of two variables to functions of one variable:

$$T(n, m) = T((n, m)) \ . \tag{1.2}$$

Here already there is a mixed identity

$$Map(N \times M, P) \cong Map(M, Map(N, P)), \tag{1.3}$$

for $(n, m) \in N \times M$

$$T((n, m)) = T(m)(n) \ , \tag{1.4}$$

and also a functorial identity :

$$Map(N \times M, P) \cong Map(N, P) \times Map(M, P) \ . \tag{1.5}$$

2020 *Mathematics Subject Classification*. 46H52.
Keywords. Mixed identity; Normed modules; Duals; Arens multiplication; Integration spaces.
Received: 22 September 2023; Accepted: 13 October 2023
Communicated by Dijana Mosić
Research supported by the Ministry of Science, Technological Development and Inovation, Republic of Serbia, grant no. 451-03-47/2023-01/ 200124. The research is done under the project *Linear operators: invertibility, spectra and operator equations*, no. O-30-22 under the Branch of SANU in Niš.
Email addresses: hartere@gmail.com (Robin E. Harte), dragandjordjevic70@gmail.com (Dragan S. Djordjević)

2. Abelian groups

When M and N are abelian groups, or more generally left or right A modules, the cartesian product $N \times M$ gives rise to the *direct sum*

$$N \oplus M \cong \begin{pmatrix} N & O \\ O & M \end{pmatrix} . \tag{2.1}$$

with linear structure given by setting

$$\alpha \begin{pmatrix} n & 0 \\ 0 & m \end{pmatrix} + \alpha' \begin{pmatrix} n' & 0 \\ 0 & m' \end{pmatrix} = \begin{pmatrix} \alpha n + \alpha' n' & 0 \\ 0 & \alpha m + \alpha m' \end{pmatrix} . \tag{2.2}$$

The functorial identity extends to the direct sum, but not the mixed identity; for that we need instead the tensor product:

$$N \otimes M \equiv \{n \otimes m : (n, m) \in N \times M\} , \tag{2.3}$$

which is a certain quotient of the free abelian group or module generated by the cartesian product $N \times M$, specifically that generated by all expressions of the forms

$$(\alpha n + \alpha' n', m) - \alpha(n, m) - \alpha'(n', m) \tag{2.4}$$

or

$$(n, \alpha m + \alpha' m') - \alpha(n, m) - \alpha'(n, m') ; \tag{2.5}$$

thus if $n \otimes m$ is the coset generated by $(n, m) \in N \times M$ then there is equality

$$(\alpha n + \alpha' n') \otimes m = \alpha n \otimes m + \alpha' n' \otimes m , \tag{2.6}$$

and also

$$n \otimes (\alpha m + \alpha' m') = \alpha n \otimes m + \alpha' n \otimes m' . \tag{2.7}$$

Here we find it convenient to muddle the meaning of "linear", which refers to real or complex vector spaces, with "additive", which applies to abelian groups. Thus bilinear mappings

$$T : N \times M \to P \tag{2.8}$$

from the cartesian product to an abelian group convert to linear, or additive mappings

$$T : N \otimes M \to P . \tag{2.9}$$

Now the mixed identity takes the form

$$T(n \otimes m) = T(m)(n) . \tag{2.10}$$

3. Ring modules

Suppose now that A is a ring, with identity 1, and that M and N are respectively (unital) left and right A modules, then the tensor product

$$N \otimes_A M \tag{3.1}$$

is obtained by quotienting out from the linear product the subgroup of all cosets of the form

$$na \otimes m - n \otimes am . \tag{3.2}$$

Thus when $A = \mathbb{Z}$ is the integers then we are looking at abelian groups, while if $A = \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is either the real or the complex numbers we are looking at vector spaces. Now the "mixed identity" is the isomorphism

$$Hom(N \otimes_A M, P) \cong Hom_A(M, Hom(N, P)) \tag{3.3}$$

of abelian groups. More generally if A and B are unital rings and if M, N and P are respectively a left (A, B) bimodule, a (left B, right A) bimodule and a left B module, then

$$Hom_B(N \otimes_A M, P) \cong Hom_A(M, Hom_B(N, P)) . \tag{3.4}$$

When $B = \mathbb{Z}$ is either the integers or the usual scalars we are back in the "linear" case.

Generally if a ring A is commutative then the distinction between left and right modules collapses: every left A module can be treated as a (left A, right A) bimodule. This of course applies to abelian groups, and to real or complex vector spaces.

4. Normed spaces and algebras

For normed left (or right) A modules M and N there is a range of norms on the direct sum $N \oplus M$

$$\|n \oplus m\|_\infty \leq \|n \oplus m\|_p \leq \|n \oplus m\|_1 \tag{4.1}$$

with $1 \leq p \leq \infty$, and embeddings

$$N \oplus_1 M \subseteq N \oplus_p M \subseteq N \oplus_\infty M \tag{4.2}$$

and then

$$BL_A(N \oplus_p M, P) \cong BL_A(N, P) \oplus_q BL(M, P) \tag{4.3}$$

with

$$1/p + 1/q = 1 .$$

When M and N are normed spaces and A a normed algebra then $N \otimes_A M$ is again a normed space, and the linear space

$$N \otimes_A M \tag{4.4}$$

can be given various interesting crossnorms, for which

$$\|n \otimes m\| = \|m\| \, \|n\|. \tag{4.5}$$

For the analogue of the mixed identity we select the "projective" or *greatest crossnorm*, for which for arbitrary $w \in N \otimes M$,

$$\|w\| = \inf \left\{ \sum_{j \in J} \|n_j\| \, \|m_j\| : w = \sum_{j \in J} n_j \otimes m_j . \right\} \tag{4.6}$$

When A is a real or a complex normed algebra, we impose the induced quotient norm on the normed space $N \otimes_A M$. If we need to consider Banach spaces, we pass to the usual *completion*. The mixed identity now takes the form, for arbitrary P,

$$BL(N \otimes_A M, P) \cong BL_A(M, BL(N, P)) . \tag{4.7}$$

5. Duality

For a normed algebra A, the obvious "dual" of a normed left A module M would seem to be

$$BL_A(M, A) \tag{5.1}$$

and indeed Bonsall and Goldie [2] and Nachbin [14] have flirted wih this; it seems that it is necessary for this to work that the algebra A in some sense "represents its own linear functionals". If however we write

$$M^* = BL_{\mathbb{K}}(M, \mathbb{K}) \tag{5.2}$$

for the usual normed space dual of a real or complex normed space M, then the mixed identity tells us

$$(N \otimes_A M)^* \cong BL_A(M, N^*) . \tag{5.3}$$

Since

$$N = A \Longrightarrow N \otimes_A M \cong M \cong BL_A(N, M) \tag{5.4}$$

it follows

$$M^* \cong BL_A(M, A^*) ; \tag{5.5}$$

thus the true module dual coincides with the normed linear space dual. For example the assunption

$$A \cong A^* \tag{5.6}$$

is certainly sufficient, as Bonsall/Goldie and Nachbin believed, for $BL_A(M, A)$ to behave as a satisfactory dual for M .

The most immediate fallout is that the real and the complex duals of a complex normed space coincide, which historically took some years to emerge. It also clear that the real and "quaternion" duals coincide for normed modules over the normed algebra of quaternions. In another direction, the real dual of a finite dimensional normed space also coincides with its *Pontrjagin* dual, its character group of continuous group homomorphims into the circle $\mathbb{S} \subseteq \mathbb{C}$.

This duality offers a partial extension to reflexive Banach spaces of the von Neumann double commutant theorem, which says, for a Hilbert space X, that a *subalgebra $A \subseteq B(X)$ is a dual space if and only if it is a commutant:

$$A = \text{comm}(B) \subseteq B(X) \Longrightarrow (X^* \otimes_B X)^* \cong BL_B(X, X) = A . \tag{5.7}$$

Our mixed identity also explains the *Arens product*, which makes the second dual of a Banach algebra again an algebra:

$$A^{**} \cong BL_A(A^*, A^*) . \tag{5.8}$$

Indeed there are in general two possibly distinct such products; we get the second by treating the dual A^* as a right A module.

Generally if a subalgebra $A \subseteq B(X)$ includes all the finite rank operators then its commutant reduces to the scalars:

$$X^* \odot X \subseteq A \subseteq B(X) \Longrightarrow X^* \otimes_A X \cong \mathbb{C} \cong BL_A(X, X) . \tag{5.9}$$

When A is a C* algebra, then left A modules can be interpreted as bimodules: for if $m \to m^* : M \to M$ is an involution then we can define, for $m \in M$ and $a \in A$,

$$ma = (a^* m^*)^*. \tag{5.10}$$

This works with $m^* \equiv m$, but we might sometimes hope for more interesting involutions. For a Hilbert A module [10],[11],[12], the norm is generated by an A-valued inner product.

6. Integration spaces

The *integration spaces* $L_p(\Omega)$ over a measure space Ω are, under pointwise multiplication, normed modules over the Banach algebra $A = L_\infty(\Omega)$:

$$L_p(\Omega) \cdot L_q(\Omega) = L_r(\Omega) , \tag{6.1}$$

where, if $1 \le p, q < \infty$ (Holder's inequality)

$$1/p + 1/q = 1/r. \tag{6.2}$$

We now [6] have

$$A = L_\infty(\Omega) \Longrightarrow L_p(\Omega) \otimes_A L_q(\Omega) \cong L_p(\Omega) \cdot L_q(\Omega) , \tag{6.3}$$

and hence the mixed identity tells us, in particular, that L_∞ acts on each L_p as a "maximal abelian" algebra of operators.

When in particular Ω carries the discrete *counting measure* we write ℓ_p in place of L_p. We can now extend this discrete version to the *Schatten* ideals on a Hilbert, or more general Banach space; for example ℓ_1 gives the *trace class* or "nuclear" operators, and ℓ_2 the *Hilbert Schmidt*. Now ℓ_∞ can be the full algebra of bounded operators, or the compact operators (with the scalars thrown in).

When $\Omega = G$ is derived from a locally compact abelian group G with Haar measure the convolution algebra $L_1(G)$ need not have an identity, but does have an "approximate identity". Here we take [5] its "multiplier algebra" $M(G)$: we observe

$$A = M(G) \Longrightarrow L_p(G) \otimes_A L_q(G) \cong L_p(G) * L_q(G) = L_r(G)], \tag{6.4}$$

where now

$$1/p + 1/q = 1 + 1/r. \tag{6.5}$$

7. Clarification

Back in 1967 the first author submitted a version of the present paper to Proceedings LMS, which was returned to him for rewriting; in the words of the anonymous referee, "Harte's account is muddled and unclear". While he was addressing this problem, the paper [16] of Rieffel appeared, which rather set him back on his heels. The first author is profoundly grateful to the second for encouraging him to pick up again the threads of this work. As the first author is himself becoming generally more "muddled and unclear", this has been something of a race against time.

The authors are also seriously grateful to Michael Mackey for scanning the fragile document [6], and also [15], into the form of a pdf .

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