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On the automatic continuity on Fréchet algebras

Mustapha Laayouni[a]

[a] *Faculty of Sciences and Technology Errachidia, UMI, Morocco*

Abstract. A topological algebra A over $\mathbb{C}$ is called functionally continuous if each complex homomorphism on A is continuous. It is well-known that each Banach algebra is functionally continuous, but it is not known whether or not each commutative Fréchet algebra has this property; This is a famous question called Michael's problem. Many mathematicians have been trying to answer this old problem in the past 70 years. To give an affirmative answer to this conjecture we use two great theorems of functional analysis, namely the closed graph theorem and the open image theorem: Let φ be complex homomorphism on a commutative Fréchet algebra $(A; (\| \|_n)_{n\geqslant 0})$. Our first result shows that there exists an integer q such that $\ker\left(\| \|_q\right) \subset \ker(\varphi)$ (Theorem 2.1). Next, we will provide A with a sequence $(\|.\|'_n)_n$ of semi-norms under which A is a Fréchet algebra (Lemma2.2) and which make the character φ continuous. The Closed graph theorem [2 , B.2, p.335] and the open image theorem [2, B.1, p.335] allow us to show that the two topologies are equivalent on A (Lemma2.3) and therefore φ is $(\| \|_n)_n$-continuous (Theorem2.3).

1. Introduction

Let A be a Banach algebra over the complex numbers. It is will known that every homomorphism $\varphi : A \to \mathbb{C}$ is automatically continuous. Michael posed the question as whether the same is true for Fréchet algebras [3]. This question is a long standing problem which remains unsolved. This problem can be described as the origin of the foundations of the locally m-convex algebras. In this era the theory of lmc algerbras was developed through the efforts of mathematecians from many countries. There were several monographs written on the subject. The purpose of this study is to give a positive answer to this question.

A Fréchet algebra A is a complete metrizable topological linear space and has a neighbourhood basis $(V_n)_n$ of zero consisting of convex sets V_n such that $V_{n+1} \subset V_n$, $V_nV_n \subset V_n$ for all $n \in \mathbb{N}$. The topology of A can be generated by an increasing sequence $(\| \|_n)_{n\geqslant 1}$ of separating seminorms

$$\|x\|_n = \inf\left\{\alpha > 0\, ; \, \alpha^{-1}x \in V_n\right\}$$

(the gauge of V_n which satisfies: $x \in V_n \Leftrightarrow \|x\|_n < 1$ [1, p.6]). If forthermor A is unital then $\| \|_n$ can be chosen such that $\|1\|_n = 1$. For each $n \in \mathbb{N}$ denote by I_n the closed ideal $I_n = \ker(\| \|_n) = \{x \in A : \|x\|_n = 0\}$,

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Email address: mustaphalaayouni@yahoo.fr (Mustapha Laayouni)

and denote by A_n the algebra A/I_n endowed with the norm $\|a+I_n\|_n = \|a\|_n$. It is well-known that $\bigcap_{n\geqslant 0} I_n = \{0\}$.

We set $\pi_n : A \longrightarrow A_n$, $x \mapsto x + I_n$. Then π_n is a continuous homomorphism.

The Fréchet algebra A with the above generating sequence of seminorms $(\|\,\|_n)_{n\geqslant 0}$ is denoted by $\left(A; (\|\,\|_n)_{n\geqslant 0}\right)$. Note that a sequence (a_k) converges to a in $\left(A; (\|\,\|_n)_{n\geqslant 0}\right)$ iff $(\|a_k - a\|_n) \to 0$ for each $n \in \mathbb{N}$, as $k \to +\infty$.

Denote by $\mathcal{S}(A)$ the set of all non zero complex-valued algebra homomorphisms and $\mathcal{M}(A)$ the set of all continuous members of $\mathcal{S}(A)$.

2. Automatic continuity

In trying to construct a discontinuous character φ on a Fréchet algebra A, that is to say construct a sequence $(a_n)_n$ with $a_n \in \ker(\|\,\|_n)$ and $(\varphi(a_n)) \to 1$ we have met the following.

Theorem 2.1. *Let $\left(A; (\|\,\|_n)_{n\geqslant 0}\right)$ be a commutative Fréchet algebra with unit. If $\varphi \in \mathcal{S}(A)$, then there exists $q \in \mathbb{N}$ such that*

$$\ker\left(\|\,\|_q\right) \subset \ker(\varphi)$$

Proof. Suppose by way of contradiction that the theorem is not true. Then there is a sequence $(x_n)_{n\in\mathbb{N}}$ in A satisfying $x_n \in I_n$ and $x_n \notin \ker(\varphi)$ for all $n \in \mathbb{N}$. Since $\ker(\varphi)$ is a maximal ideal, it follows that $A = Ax_n + \ker(\varphi)$. Consequently, there exist $a_n \in I_n$ and $b_n \in \ker(\varphi)$ such that $1 = a_n + b_n$ for all $n \in \mathbb{N}$. It follows from [3, Corollary5.6 (a) and Lemma6.1 (a)] that for every $n \in \mathbb{N}$ there exists some $\theta_n \in \mathcal{M}(A)$ such that $\theta_n(a_n) = \varphi(a_n)$ and $\theta_n(b_n) = \varphi(b_n)$. Hence $\theta_n(b_n) = 0$ and $\theta_n(a_n) = \varphi(a_n + b_n) = \varphi(1) = 1$ for all $n \in \mathbb{N}$. Since θ_n is continuous, it follows from [2, Remarks.3.2.2 (ii), p.73]that there exists a positive integer q such that

$$|\theta_n(x)| \leqslant \|x\|_q, \ \forall x \in A$$

It is obvious that this inequality is satisfied for all $k \geqslant q$ and the integer p_n defined by

$$p_n = \min\{k \in \mathbb{N} : |\theta_n(x)| \leqslant \|x\|_k \ \forall x \in A\}$$

exists for all $n \in \mathbb{N}$.
Now define the mapping $\Gamma : A \to \mathbb{R}$ by

$$\Gamma(x) = \begin{cases} 0 & \text{if } x \in I_0 \text{ or } \theta_n(x) = 0 \text{ for some } n \in \mathbb{N} \\ \prod_{n=0}^{+\infty} \left[\frac{|\theta_n(x)|}{\|x\|_{p_n}}\right]^{\frac{|\theta_n(x)|}{2^n\|x\|_{p_n}}} & \text{otherwise} \end{cases}$$

$\prod_{n=0}^{+\infty} \left[\frac{|\theta_n(x)|}{\|x\|_{p_n}}\right]^{\frac{|\theta_n(x)|}{2^n\|x\|_{p_n}}}$ is defined to be $\lim_{k\to+\infty} \prod_{n=0}^{k} \left[\frac{|\theta_n(x)|}{\|x\|_{p_n}}\right]^{\frac{|\theta_n(x)|}{2^n\|x\|_{p_n}}}$.

Since $\theta_n(1) = 1$, and so by the above discussion $\|1\|_n = 1$ for all integer n then $\Gamma(1) = 1$.
We consider $x \in A - I_0$. Assume that there exists $n \in \mathbb{N}$ such that $\theta_n(x) = 0$ then $\Gamma(x) = 0$. Otherwise $|\theta_n(x)| > 0$ for all $n \in \mathbb{N}$ and

$$\Gamma(x) = \lim_{k\to+\infty} \exp\left[\sum_{n=0}^{k} \frac{|\theta_n(x)|}{2^n\|x\|_{p_n}} \ln\left(\frac{|\theta_n(x)|}{\|x\|_{p_n}}\right)\right]$$

is well defined. Indeed, since $I_{p_n} \subset I_0$ and $0 < |\theta_n(x)| \leqslant \|x\|_{p_n}$. Then $\sum_{n=0}^{k}\left[\frac{|\theta_n(x)|}{2^n\|x\|_{p_n}}\ln\left(\frac{|\theta_n(x)|}{\|x\|_{p_n}}\right)\right]$ becomes a decreasing sequence of negative reals. It follows from the completeness of $\mathbb{R}$ that the series $\sum \frac{|\theta_n(x)|}{2^n\|x\|_{p_n}}\ln\left(\frac{|\theta_n(x)|}{\|x\|_{p_n}}\right)$ is convergent to its infimum $\beta(x)$ or $\sum_{n=0}^{+\infty}\frac{|\theta_n(x)|}{2^n\|x\|_{p_n}}\ln\left(\frac{|\theta_n(x)|}{\|x\|_{p_n}}\right) = -\infty$. Hereby $\Gamma(x)$ is defined to be in $[0,1]$.

We claim tha Γ is continuous in the identity. Indeed, 1 belong to the open subset $A - I_0$. We must show that $|\Gamma(x) - 1| \to 0$ if $\|x-1\|_q \to 0$ for some $q \in \mathbb{N}$. To this end, we will show that, $|\beta(x)| \to 0$ when $\|x-1\|_q \to 0$.

We must find $q \in \mathbb{N}$ such that for all $\varepsilon > 0$, there exists $\eta > 0$ satisfying for every $x \in A - I_0$

$$\|x-1\|_q < \eta \quad \Rightarrow \quad -\varepsilon < \beta(x) \leqslant 0 \tag{2.1}$$

Consider an element $x \in A - I_0$. The desired assertion (2.1) is trivial in the case where $\theta_n(x) = 0$ for some n (note that, according to the definition of the function Γ, this may happen for some $x \in A - I_0$). Assume now that $\theta_n(x) \neq 0$ holds for all n. Pik $\varepsilon > 0$

Let $u \in \mathbb{R}$ such that $0 < u < 1$. Since for each $n \in \mathbb{N}$ we have

$$(1-u)\left(1+u+u^2+\ldots+u^n\right) = 1-u^{n+1}$$

then

$$\frac{1}{1-u} = \sum_{n=0}^{+\infty} u^n \text{ and } \ln(1-u) = -\sum_{n=0}^{+\infty}\frac{1}{n+1}u^{n+1}$$

Applying to $t = 1-u$ the last equality, we obtain

$$\begin{aligned}
\ln(t) &= -\sum_{k=1}^{+\infty}\left(\frac{1}{k}(1-t)^k\right)\\
&\geqslant -\sum_{k=1}^{+\infty}(1-t)^k\\
&\geqslant -(1-t)\sum_{k=0}^{+\infty}(1-t)^k\\
&\geqslant -(1-t)\frac{1}{t}
\end{aligned} \tag{2.2}$$

we conclude that

$$\begin{aligned}
t\ln(t) &\geqslant t\frac{t-1}{t}\\
&\geqslant t-1\\
&> -1
\end{aligned} \tag{2.3}$$

holds for every $0 < t < 1$. Obviously 1 satisfies (2.3). Then for every $n \in \mathbb{N}$ and for $0 < t = \frac{|\theta_n(x)|}{\|x\|_{p_n}} \leqslant 1$ we have

$$\frac{|\theta_n(x)|}{\|x\|_{p_n}}\ln\left(\frac{|\theta_n(x)|}{\|x\|_{p_n}}\right) > -1$$

and so

$$\frac{|\theta_n(x)|}{2^n\|x\|_{p_n}}\ln\left(\frac{|\theta_n(x)|}{\|x\|_{p_n}}\right) > -\frac{1}{2^n} \tag{2.4}$$

By the convergence of the series $\sum \frac{1}{2^n}$, there exists $s \in \mathbb{N}$ with

$$-\sum_{k=s}^{+\infty}\frac{1}{2^n} \geqslant -\frac{\varepsilon}{2}$$

and then

$$\sum_{n=s}^{+\infty}\frac{|\theta_n(x)|}{2^n\|x\|_{p_n}}\ln\left(\frac{|\theta_n(x)|}{\|x\|_{p_n}}\right) > -\frac{\varepsilon}{2} \tag{2.5}$$

Put $q = \max\{p_n : n \leqslant s\}$, we can assume without loss of generality that $q \geqslant s$. Since $\|\,\|_{p_n} \leqslant \|\,\|_q$ then $\|x-1\|_q \to 0$ implies that $\|x-1\|_{p_n} \to 0$ and $|\theta_n(x)| \to 1$ for $n = 0, 1, ..., s$. On the other hand, the real-valued function $t \to t\ln t$ satisfies $\lim_{t\to 1}(t\ln t) = 0$. Then, for each $n = 0, 1, ..., s$, there exists a real $\eta_n > 0$ such that $\|x-1\|_{p_n} < \eta_n$ implies $\frac{|\theta_n(x)|}{\|x\|_{p_n}}\ln\left(\frac{|\theta_n(x)|}{\|x\|_{p_n}}\right) > -\frac{\varepsilon}{4}$. In this way, the real $\eta = \min\{\eta_n : n = 0, 1, 2, ..., s\}$ satisfies $\eta > 0$ and for every $n = 0, 1, ..., s$

$$\|x-1\|_q < \eta \quad \Rightarrow \quad \frac{|\theta_n(x)|}{\|x\|_{p_n}}\ln\left(\frac{|\theta_n(x)|}{\|x\|_{p_n}}\right) > -\frac{\varepsilon}{4} \tag{2.6}$$

we conclude from (2.5) and (2.6) that

$$\begin{aligned}
0 > \beta(x) &= \sum_{n=0}^{+\infty}\frac{|\theta_n(x)|}{2^n\|x\|_{p_n}}\ln\left(\frac{|\theta_n(x)|}{\|x\|_{p_n}}\right)\\
&= \sum_{n=0}^{s}\frac{|\theta_n(x)|}{2^n\|x\|_{p_n}}\ln\left(\frac{|\theta_n(x)|}{\|x\|_{p_n}}\right) + \sum_{n=s}^{+\infty}\frac{|\theta_n(x)|}{2^n\|x\|_{p_n}}\ln\left(\frac{|\theta_n(x)|}{\|x\|_{p_n}}\right)\\
&> -\frac{\varepsilon}{2} + \sum_{n=0}^{s}\frac{|\theta_n(x)|}{2^n\|x\|_{p_n}}\ln\left(\frac{|\theta_n(x)|}{\|x\|_{p_n}}\right)\\
&> -\frac{\varepsilon}{2} - \frac{\varepsilon}{4}\sum_{n=0}^{s}\frac{1}{2^n}\\
&> -\frac{\varepsilon}{2} - \frac{\varepsilon}{4}\sum_{n=0}^{+\infty}\frac{1}{2^n}\\
&> -\varepsilon
\end{aligned} \tag{2.7}$$

remain true for every $x \in A - I_0$ satisfying $\|x-1\|_q < \eta$ given by (2.6). Then β is continuous in the identity as well as $\Gamma = \exp\circ\beta$ as desired.

On the other hand, it is easily realized that $\lim_{n\to\infty} a_n = 0$. Indeed, for all $m, n \in \mathbb{N}$.

$$a_{n+m} \in I_{n+m} \subset V_{n+m} \subset V_n$$

So,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}(1 - a_n) = 1$$

with $\Gamma(b_n) = 0$ fot all $n \in \mathbb{N}$ and $\Gamma(1) = 1$. Which is a contradiction. This completes the proof.
□

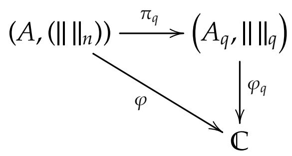
Denote by q_φ the positive integer defined by

$$q_\varphi = \min\left\{q \in \mathbb{N}; \ker\left(\|\,\|_q \subset \ker(\varphi)\right)\right\}$$

For every integer $q \geqslant q_\varphi$, consider the homomorphism φ_q from A_q to $\mathbb{C}$ such that

$$\varphi_q\left(x + \ker\left(\|\,\|_q\right)\right) = \varphi(x)$$

in such a way that the following diagram is commutatif.



According to Theorem2.1, we have the following.

Theorem 2.2. *Let $\left(A; (\|\,\|_n)_{n\geqslant 0}\right)$ be a commutative Fréchet algebra, and let φ be a Character of A. Then φ is continuous if and only if there exists an integer $q \geqslant q_\varphi$ such that φ_q is continuous.*

Proof. First assume that A has a unit. In view of Theorem2.1, it is enough to show that, $x + \ker\left(\|\,\|_q\right) \to \varphi(x)$ becomes an homomorphism φ_q from A_q onto $\mathbb{C}$ such that $\varphi_q \circ \pi_q = \varphi$. It is well known that φ is continuous if and only if φ_q is too.

Now, assume that A does not have a unit. Set $A' = A \oplus \mathbb{C}$ the algebra obtained by adjoining an identity to A endowed with the topology wich is generated by seminorms

$$\|x + \alpha\|_{n,1} = \|x\|_n + |\alpha|$$

It is easy to show that A' becomes a Fréchet algebra and $0_A + 1$ is an identity. On the other hand, the mapping $\widehat{\varphi} : A' \to \mathbb{C}$ defined by

$$\widehat{\varphi}(x + \alpha) = \varphi(x) + \alpha$$

is an algebra homomorphism. From the conclusion of the unitary case, $\widehat{\varphi}$ is continous, and so is φ.
□

It is easy to see that without loss of generality we can assume that $I_q = \ker\left(\|\,\|_q\right) \subset \ker(\varphi)$ holds for all q. Denote $\overline{A_q}$, the completion of $A_q = A/I_q$ with respect to the norm $x + I_q \mapsto \|x\|_q$. We also denote by $\|\,\|_q$ the norm of $\overline{A_q}$.

We now provide A with an interesting Fréchet topology which we call, as in the Banach case [1, p.20], the Fréchet topology of the graph.

$$\begin{array}{rcl} \|\,\|'_q : A & \to & \mathbb{R}_+ \\ x & \mapsto & \|x\|'_q \end{array}$$

by $\|x\|'_q = \|x\|_q + |\varphi(x)|$. It is easy to see that $\|\,\|'_q$ is a submultiplicative seminorm satisfying

$$\|x\|'_q \leqslant \|x\|'_{q+1} \text{ for all } x \in A \text{ and } q \geqslant 1 \tag{2.8}$$

Lemma 2.3. *With this notations we have:* $\ker\left(\|\,\|'_q\right) = \ker\left(\|\,\|_q\right)$.

Proof. If an element x of A is in $\ker\left(\|\,\|_q\right)$ then Theorem2.1 implies x is in $\ker(\varphi)$. Thus x belongs to $\ker\left(\|\,\|'_q\right)$. The converse inclusion follows easily from the inequality $\|\,\|_q \leqslant \|\,\|'_q$.
□

Clearly, $\pi_q(x) = x + I_q \mapsto \|x\|'_q$ makes A_q a normed algebra. $\widetilde{A_q}$ denotes the completion of A_q with respect to this norm. We keep the notation $\|\,\|'_q$ for the norm of $\widetilde{A_q}$. So from the relation $|\varphi_q\left(\pi_q(x)\right)| = |\varphi(x)| \leqslant \|\pi_q(x)\|'_q$, we easily see that φ_q is $\|\,\|'_q$-continuous and hence, extends to a continuous morphism $\widetilde{\varphi_q} : \widetilde{A_q} \to \mathbb{C}$.

Lemma 2.4. *A equipped with the seminorms* $\left(\|\,\|'_q\right)_n$ *is a commutative Fréchet algebra.*

Proof. Clearly, $\left(\|\,\|'_q\right)_n$ are submultiplicative. On the other hand, it follows from Lemma2.1 that

$$\bigcap_q \ker\left(\|\,\|'_q\right) = \bigcap_q \ker\left(\|\,\|_q\right) = \{0\}$$

To complete the proof, it suffices to show that the seminorms $\left(\|\,\|'_q\right)_n$ generate on A a complete topology. To this end, let $(x_n)_n$ be a sequence in A satisfying $\lim_{m,n\to\infty} \|x_m - x_n\|'_q = 0$ for each q. Since $\|x_m - x_n\|_q \leqslant \|x_m - x_n\|'_q$ then $(x_n)_n$ is a $\left(\|\,\|_q\right)_q$-Cauchy sequence and so $\left(\|\,\|_q\right)_q$-converges to $x \in A$. Fix q, the graph $\mathcal{G}\left(\widetilde{\varphi_q}\right)$ of the continuous mapping $\widetilde{\varphi_q}$ is closed in the Banach space $\widetilde{A_q} \times \mathbb{C}$ then $\mathcal{G}\left(\widetilde{\varphi_q}\right)$ is itself complete. So, $\left(\widetilde{\pi_q}(x_n), \widetilde{\varphi_q}\left(\widetilde{\pi_q}(x_n)\right)\right) \to \left(y, \widetilde{\varphi_q}(y)\right)$ for some $y \in \widetilde{A_q}$. So, for each $n \geqslant 1$

$$\begin{aligned} |\varphi(x_n) - \widetilde{\varphi_q}(y)| &= |\varphi_q\left(\pi_q(x_n)\right) - \widetilde{\varphi_q}(y)| \\ &= |\widetilde{\varphi_q}\left(\widetilde{\pi_q}(x_n)\right) - \widetilde{\varphi_q}(y)| \\ &\leqslant \|\widetilde{\pi_q}(x_n) - y\|_q + |\widetilde{\varphi_q}\left(\widetilde{\pi_q}(x_n)\right) - \widetilde{\varphi_q}(y)| \\ &= \|\widetilde{\pi_q}(x_n) - y\|'_q \\ &\to 0 \text{ whenever } n \to \infty \end{aligned} \tag{2.9}$$

$$\begin{aligned} \|\widetilde{\pi_q}(x_n) - y\|_q &\leqslant \|\widetilde{\pi_q}(x_n) - y\|_q + |\widetilde{\varphi_q}\left(\widetilde{\pi_q}(x_n)\right) - \widetilde{\varphi_q}(y)| \\ &= \|\widetilde{\pi_q}(x_n) - y\|'_q \\ &\to 0 \text{ whenever } n \to \infty \end{aligned} \tag{2.10}$$

Furthermore

$$\begin{aligned} \|\widetilde{\pi_q}(x_n) - \widetilde{\pi_q}(x)\|_q &= \|x_n - x\|_q \\ &\to 0 \text{ whenever } n \to \infty \end{aligned} \tag{2.11}$$

Combining (2.10) with (2.11), we get $y = \widetilde{\pi_q}(x)$ and $\varphi(x) = \widetilde{\varphi_q}\left(\widetilde{\pi_q}(x)\right) = \widetilde{\varphi_q}(y)$. Which gives in addition to the identity (2.9)

$$\begin{aligned} \|x_n - x\|'_q &= \|\widetilde{\pi_q}(x_n) - \widetilde{\pi_q}(x)\|'_q \\ &= \|\widetilde{\pi_q}(x_n) - y\|'_q \\ &\to 0 \text{ whenever } n \to \infty \end{aligned}$$

Since q is arbitrary, it follows that A is complete under the topology generated by the seminorms $\left(\|\,\|'_q\right)_q$ as desired. □

This lemma means that $\left(A, \left(\|.\|'_p\right)_p\right)$ is a Fréchet algebra. Clearly, the homomorphism $\varphi : \left(A, \left(\|.\|'_p\right)_p\right) \to \mathbb{C}$ is continuous. To return to the initial topology of A we have the following lemma.

Lemma 2.5. *The two topologies* $(\|.\|'_n)_n$ *and* $(\|.\|_n)_n$ *of* A *are equivalent.*

Proof. Obviously, the identity $\left(A, \left(\|.\|'_p\right)_p\right) \to \left(A, \left(\|.\|_p\right)_p\right)$ is continuous, and so by the open mapping theorem, we get the conclusion of the lemma. □

Thus we obtain the following main result.

Theorem 2.6. *If* A *is a commutative Fréchet algebra, then every caracter of* A *is automatically continuous.*

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