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On the Class of (n, m) Power-A-Hyponormal Operators in Semi-Hilbertian Space

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Abstract. The concept of n-power-hyponormal operator on Hilbert space defined by Messoud Guesba and Motafa Nadir see [10]. In [5] Cherifa Chellali and Abdelkader Benali gave another class of operators called class of (A, n)-power-hyponormal operator in semi-Hilbertien space.In this manuscript we introduce new class of operators on semi-Helbertian space $(\mathcal{H}, \|.\|_A)$ called (n, m) power-A-hyponormal denoted $[(n, m)H]_A$.We study some basic properties of these operators and some examples are also given .An operator $T \in \mathcal{B}_A(\mathcal{H})$ is (n, m) power-A-hyponormal for some positive operator A and for some positive integers n and m if $(T^\sharp)^m T^n - T^n (T^\sharp)^m \geq_A 0$.

1. Introduction And Premliminaries Results

A bounded linear operator T on a complex Hilbert space is n-hyponormal operator if $T^*T^n - T^nT^* \geq 0$,this class is introduced by the authors Messaoud Guesba and Mostefa Nadir see [10], they studied some proprietes for different values of the parameter n, in particular for $n = 2, n = 3$. On Hilbert space the class of p-hyponormal operator was introduced and studied by A.Aluthge [4].
The purpose of this paper is to study the class of (n, m)-power$A$-hyponormal operators in semi-Hilbertian spaces. For $m = 1$ and n any natural we obtain the work of Cherifa Chellali and Abdelkader Benali see [5]. We have also edit the results of Samir Al Mohammady, Sid Ahmed Ould Beinane and Ould Ahmed Mahmoud Sid Ahmed see [8].
The contents of the paper are the following. In Section one, we give notation and results about the concept of A-adjoint operators that will be useful in the sequel. In Section two we introduce the new concept of hyponormality of operators in semi-Hilbertian space $(\mathcal{H}, \langle.|.\rangle_A)$ called (n, m) power-A-hyponormal operator and we investigate various structural properties of this class of operators with some examples are studied .Moreover the product, direct sum, tensor product and the sum of finite numbers of these type are discussed . We start by introducing some notations.

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Along this work $\mathcal{H}$ denotes a complex Hilbert space with inner product $\langle . \rangle$, $\mathcal{B}(\mathcal{H})$ is the algebra of all bounded linear operators on $\mathcal{H}$, $\mathcal{B}(\mathcal{H})^+$ is the cone of positive operators of $\mathcal{B}(\mathcal{H})$ defined as

$$\mathcal{B}(\mathcal{H})^+ = \Big\{T \in \mathcal{B}(\mathcal{H}); \langle Tx/x\rangle \geq 0, \forall x \in \mathcal{H}\Big\}$$

For every $T \in \mathcal{B}(\mathcal{H})$, $\mathcal{N}(T)$, $\mathcal{R}(T)$ and $\overline{\mathcal{R}(T)}$ stand for respectively the null space,range and the closure of the range of T and its adjoint operator by T^* .If $\mathcal{M}$ is a closed linear subspace of $\mathcal{H}$ satisfying $T\mathcal{M} \subset \mathcal{M}$, then $\mathcal{M}$ is called invariant subspace of T. In addition if $\mathcal{M}$ also is invariant subspace of T^* , then $\mathcal{M}$ is called a reducing subspace of T . We denote the orthogonal projection onto a closed linear subspace $\mathcal{M}$ of $\mathcal{H}$ by $P_{\mathcal{M}}$.For any operator $T \in \mathcal{B}(\mathcal{H})$,$|T| = (T^*T)^{\frac{1}{2}}$ and $[T^*, T] = T^*T - TT^* = |T|^2 - |T^*|^2$
Note that for $A \in \mathcal{B}(\mathcal{H})^+$, the functional $\langle .|. \rangle_A : \mathcal{H} \times \mathcal{H} \to \mathbb{C}, \langle u|v\rangle_A = \langle Au|v\rangle$ is a semi-inner product on $\mathcal{H}$. By $\| \, . \, \|_A$ we denote the semi-norm induced by $\langle .|. \rangle_A$ i.e $\| u \|_A = \langle u|u\rangle_A^{\frac{1}{2}} = \langle Au|u\rangle^{\frac{1}{2}} = \| A^{\frac{1}{2}}u \|$ .Observe that $\| u \|_A = 0$ if and only if $u \in \mathcal{N}(\mathrm{A})$, then $\| \, . \, \|_A$ is a norm if and only if A is an injective operator and the semi-normed space $(\mathcal{H}, \| \, . \, \|_A)$ is complete if and only if $\mathcal{R}(\mathrm{A})$ is closed.
The above semi-norm induces a semi-norm on the subspace $\mathcal{B}^A(\mathcal{H})$ of $\mathcal{B}(\mathcal{H})$

$$\mathcal{B}^A(\mathcal{H}) = \Big\{T \in \mathcal{B}(\mathcal{H})/\exists c > 0 \; \| Tu \|_A \leq c \, \| u \|_A, \forall u \in \mathcal{H}\Big\}$$

Indeed, if $T \in \mathcal{B}^A(\mathcal{H})$, then $\| T \|_A = sup\Big\{\frac{\|Tu\|_A}{\|u\|_A}, u \in \overline{\mathcal{R}(A)} \wedge u \neq 0\Big\}$
Moreover $\| T \|_A = sup\Big\{|\langle Tu|v\rangle|_A : u, v \in \mathcal{H} : \| u \| \leq 1, \| v \| \leq 1\Big\}$
For $T \in \mathcal{B}(\mathcal{H})$, an operator $S \in \mathcal{B}(\mathcal{H})$ is called an A-adjoint operator of T if for every u, $v \in \mathcal{H}$, we have $\langle Tu|v\rangle_A = \langle u|Sv\rangle_A$ that is $AS = T^*A$,if T is an A-adjoint of itself, then T is called an A-selfadjoint operator $(AT = T^*A)$.
It is possible that an operator T does not have an A-adjoint, and if S is an A-adjoint of T we may find many A-adjoints , In fact in if $AR = 0$ for some $R \in \mathcal{B}(\mathcal{H})$,then $S + R$ is an A-adjoint of T. The set of all A-bounded operators which admit an A-adjoint is denoted by $\mathcal{B}_A(\mathcal{H})$. By Douglas Theorem we have that
$\mathcal{B}_A(\mathcal{H}) = \Big\{T \in \mathcal{B}(\mathcal{H})/R(T^*A) \subset R(A)\Big\}$. If $T \in \mathcal{B}_A(\mathcal{H})$, then there exists a distinguished A-adjoint operator of T, namely the reduced solution of equation $AX = T^*A$, This operator is denoted by $T^\sharp$ Therefore ,

$$T^\sharp = A^\dagger T^*A \text{ and } AT^\sharp = T^*A, \;\; R(T^\sharp) \subset \overline{R(A)} \text{ and } \mathcal{N}(T^\sharp) = \mathcal{N}(T^*A)$$

Note that in which $A^\dagger$ is the Moore-Penrose inverse of A. For more details see ([1],[2]). The classes of normal,quasinormal,isometries,hyponormal,quasihyponormal and m-isometries on Hilbert spaces have been generalized to semi-Hilbert spaces by many authors in ([1],[2],[7],[9],[11]) and other papers

Definition 1.1. *Any operators* $T \in \mathcal{B}_A(\mathcal{H})$ *is called*

1. *A-normal if* $TT^\sharp = T^\sharp T$

2. *A-isometry if* $T^\sharp T = P_{\overline{R(A)}}$

3. *A-unitary if* $T^\sharp T = TT^\sharp = P_{\overline{R(A)}}$

4. *A-hyponormal if* $T^\sharp T \geq_A TT^\sharp$

5. (A, n)*-hyponormal if* $T^\sharp T^n \geq_A T^n T^\sharp$

2. Class of (n, m) power-A-Hyponormal Operators in Semi-Hilbertian Space

In this section, we introduce the concept of (n, m) power-A-hyponormal Operators in semi-Hilbertian Space.
The following definition and results are useful for our study.

Definition 2.1. *We say that $T \in \mathcal{B}(\mathcal{H})$ is an A-positive if $AT \in \mathcal{B}(\mathcal{H})^+$ or equivalently $\langle Tu|u\rangle_A \geq 0, \forall u \in \mathcal{H}$ .We note $T \geq_A 0$*

Remark 2.2. *We can define a order relation by $T \geq_A S \Leftrightarrow T - S \geq_A 0$*

Remark 2.3. *Inequality de Cauchy-Schwarz for A-positive operator. If $T \in \mathcal{B}(\mathcal{H})$ is an A-positive , then$|\langle Tu|v\rangle_A|^2 \leq \langle Tu|u\rangle_A \langle Tv|v\rangle_A, \forall u \in \mathcal{H}$*

Lemma 2.4. *([3]) Let $T, S \in \mathcal{B}(\mathcal{H})$ such that $T \geq_A S$ and let $R \in \mathcal{B}_A(\mathcal{H})$,then the following properties hold*

1. $R^\sharp TR \geq_A R^\sharp SR$
2. $RTR^\sharp \geq_A RSR^\sharp$
3. *If R is A-selfadjoint then $RTR \geq_A RSR$*

Remark 2.5. *from lemma (2.4) we deduce the following results*

1. $R^{\sharp n}TR^n \geq_A R^{\sharp n}SR^n$ *and* $R^nTR^{\sharp n} \geq_A R^nSR^{\sharp n}$ *for all positive integer n*
2. *if $m \geq n$ i.e ($\exists k \geq 0 : m = n + k$) and $TR^{\sharp K} \geq_A SR^{\sharp K}$, then $R^nTR^{\sharp m} \geq_A R^nSR^{\sharp m}$*
3. *If $m \geq n$ i.e($\exists k \geq 0 : m = n + k$) and $R^{\sharp K}T \geq_A AR^{\sharp K}S$,then $R^{\sharp m}TR^n \geq_A R^{\sharp m}SR^n$*

Lemma 2.6. *Let $T, S \in \mathcal{B}(\mathcal{H})$ are A-positive operators, if T commutes with S then TS is A-positive*

Proof. Let $T, S \in \mathcal{B}(\mathcal{H})$ are an A-positive . Since T is A-positive then exists only one operator R is A-positive such that $R^2 = T$ i.e $R = T^{\frac{1}{2}}$ and commutes with S
Then for all $u \in \mathcal{H}$ we have

$$\begin{aligned} \langle TSu|u\rangle &= \langle R^2Su|u\rangle \\ &= \langle RSu|Ru\rangle \\ &= \langle S(Ru)|Ru\rangle \\ &\geq_A 0 \end{aligned}$$

Hence $TS \geq_A 0$ □

Definition 2.7. *An operator $T \in \mathcal{B}_A(H)$ is said to be (n, m) power-A-hyponormal operator for a positive integers n, m if $(T^\sharp)^mT^n - T^n(T^\sharp)^m \geq_A 0$.Or equivalently*

$$\langle \Big((T^\sharp)^mT^n - T^n(T^\sharp)^m\Big)u|u\rangle \geq_A 0 \textit{ for all } u \in \mathcal{H}$$

We denote the set of all (n, m) power-A-hyponormal operators by $[(n, m)H]_A$

Remark 2.8. *We make the following observations:*

1. *If $n = m = 1$ then $(1, 1)$-A-Hyponormal is A Hyponormal*
2. *every (n, m) power-A-normal is (n, m) power-A-hyponormal i.e $[(n, m)N]_A \subset [(n, m)H]_A$*

In the following example we give an operator that is (n, m)-A-Hyponormal for some positive integers n and m but is not a A -hyponormal

Example 2.9.

$$\text{Let } T = \begin{pmatrix} -3 & 2 \\ 0 & 3 \end{pmatrix} \text{ and } A = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

be operators acting on two dimensional Hilbert space $\mathbb{C}^2$. *A simple calculation shows that*

$$T^* = \begin{pmatrix} -3 & 0 \\ 2 & 3 \end{pmatrix} \text{ and } T^\sharp = \begin{pmatrix} -3 & 0 \\ \frac{2}{3} & 3 \end{pmatrix}.$$

A direct calculation show that T is of class $[(3,2)H]_A$ *but T is not in* $[H]_A$

In the following theorem we give the characterization of (n, m) power-A-hyponormal

Theorem 2.10. *Let* $T \in \mathcal{B}_A(\mathcal{H})$.*Then T is* (n, m) *power-A-hyponormal operator for some positive integers n and m if and only if T satisfying the following condition*

$$\langle T^n u / T^m u\rangle_A \geq \langle (T^\sharp)^m u / (T^\sharp)^n u\rangle_A, \forall u \in \mathcal{H}$$

Proof. Assume that T is a (n, m) power-A-hyponormal operator then

$$\begin{aligned} (T^\sharp)^m T^n - T^n (T^\sharp)^m \geq_A 0 \quad &\Leftrightarrow \quad \langle ((T^\sharp)^m T^n - T^n (T^\sharp)^m) u / u\rangle_A \geq 0 \text{ for all } u \in H \\ &\Leftrightarrow \quad \langle A (T^\sharp)^m T^n u / u\rangle - \langle A T^n (T^\sharp)^m u / u\rangle \geq 0 \\ &\Leftrightarrow \quad \langle A (T^\sharp)^m T^n u / u\rangle - \langle ((T^\sharp)^m) u / (T^*)^n A u\rangle \geq 0 \\ &\Leftrightarrow \quad \langle (T^*)^m A T^n u / u\rangle - \langle (T^\sharp)^m u / A (T^\sharp)^n u\rangle \geq 0 \\ &\Leftrightarrow \quad \langle T^n u / T^m u\rangle_A - \langle (T^\sharp)^m u / (T^\sharp)^n u\rangle_A \geq \\ &\Leftrightarrow \quad \langle T^n u / T^m u\rangle_A \geq \langle (T^\sharp)^m u / (T^\sharp)^n u\rangle_A \end{aligned}$$

The proof is complete □

Remark 2.11. *If* $n = m$ *then*

$$\begin{aligned} T \in [(n,n)H]_A &\Leftrightarrow \langle T^n u / T^n u\rangle_A \geq \langle (T^\sharp)^n u / (T^\sharp)^n u\rangle_A \\ &\Leftrightarrow \| T^n u \|_A \geq \| T^{\sharp n} u \|_A \end{aligned}$$

Proposition 2.12. *Let* $T \in \mathcal{B}_A(\mathcal{H})$ *is* (n, n) *power-A-hyponormal operator ,then* $\| T^n u \|_A = \| T^{\sharp n} u \|_A$ *if and only if* $\left((T^\sharp)^n T^n - T^n (T^\sharp)^n\right) \in \mathcal{N}(A)$

Proof. Suppose that $\| T^n u \|_A = \| T^{\sharp n} u \|_A$,then for all $u \in \mathcal{H}$ we have

$$\begin{aligned} \| T^n u \|_A^2 = \| T^{\sharp n} u \|^2 &\Rightarrow \langle T^n u / T^n u\rangle_A = \langle (T^\sharp)^n u / (T^\sharp)^n u\rangle_A \\ &\Rightarrow \langle A T^n u / T^n u\rangle - \langle (T^\sharp)^n u / A (T^\sharp)^n u\rangle = 0 \\ &\Rightarrow \langle (T^*)^n A T^n u / u\rangle - \langle (T^\sharp)^n u / (T^*)^n A u\rangle = 0 \\ &\Rightarrow \langle A (T^\sharp)^n T^n u / u\rangle - \langle A T^n (T^\sharp)^n u / u\rangle = 0 \\ &\Rightarrow \langle (T^\sharp)^n T^n u / u\rangle_A - \langle T^n (T^\sharp)^n u / u\rangle_A = 0 \\ &\Rightarrow \langle (T^{\sharp n} T^n - T^n T^{\sharp n}) u / u\rangle_A = 0 \end{aligned}$$

Since $(T^{\sharp n}T^n - T^nT^{\sharp n})$ is A-positive we have for all $u \in \mathcal{H}$ by the A-Cauchy Schwarz inequality

$$|\langle (T^{\sharp n}T^n - T^nT^{\sharp n})u/v\rangle_A|^2 \leq \langle (T^{\sharp n}T^n - T^nT^{\sharp n})u/u\rangle_A \langle (T^{\sharp n}T^n - T^nT^{\sharp n})v/v\rangle_A = 0$$

Hence $\langle (T^{\sharp n}T^n - T^nT^{\sharp n})u/v\rangle_A = 0$ for all $v \in \mathcal{H}$ and this implies that

$$A(T^{\sharp n}T^n - T^nT^{\sharp n})u = 0$$

Then $\left((T^\sharp)^nT^n - T^n(T^\sharp)^n\right) \in \mathcal{N}(A)$
Conversely if $A(T^{\sharp n}T^n - T^nT^{\sharp n})u = 0$ it is clear that $\langle (T^{\sharp n}T^n - T^nT^{\sharp n})u/u\rangle_A = 0$
And hence $\| T^nu \|_A = \| T^{\sharp n}u \|_A$ □

The idea of the following proposition is inspired from ([3] proposition (2.3))

Proposition 2.13. *Let $T \in \mathcal{B}_A(\mathcal{H})$,then T is (n, n) power-A-hyponormal operator if and only if*

$$T^{\sharp n}T^n + 2\lambda T^nT^{\sharp n} + \lambda^2 T^{\sharp n}T^n \geq_A 0 \textit{for all } \lambda \in \mathbb{R}$$

Proof. Assume$T$ is (n, n) power-A-hyponormal operator, for all$u \in \mathcal{H}$ and $\lambda \in \mathbb{R}$ we have

$$\begin{aligned}
\| T^nu \|_A \geq \| T^{\sharp n}u \|_A &\Leftrightarrow \| T^{\sharp n}u \|_A^4 - \| T^nu \|_A^4 \leq 0\\
&\Leftrightarrow \| T^nu \|_A^2 + 2\lambda \| T^{\sharp n}u \|^2 + \lambda^2 \| T^nu \|_A^2 \geq 0\\
&\Leftrightarrow \langle AT^{\sharp n}T^nu|u\rangle + 2\lambda\langle AT^nT^{\sharp n}u|u\rangle + \lambda^2\langle AT^{\sharp n}T^nu|u\rangle \geq 0\\
&\Leftrightarrow \langle (T^{\sharp n}T^n + 2\lambda T^nT^{\sharp n} + \lambda^2 T^{\sharp n}T^n)u|u\rangle_A \geq 0\\
&\Leftrightarrow (T^{\sharp n}T^n + 2\lambda T^nT^{\sharp n} + \lambda^2 T^{\sharp n}T^n) \geq_A 0
\end{aligned}$$

□

The folloing proposition discuss the relation between (n, m) power-A-hyponormal operator and (m, n) power-A-hyponormal operator

Proposition 2.14. *Let $T \in \mathcal{B}_A(\mathcal{H})$ such that $\mathcal{N}(A)$ is invariant subspace of T.The following statements are equivalent*

1. *T is (n, m) power-A-hyponormal operator*

2. *T is (m, n) power-A-hyponormal operator*

Proof. from the condition we have $P_{\overline{\mathbb{R}(\mathbb{A})}}T = TP_{\overline{\mathbb{R}(\mathbb{A})}}$ and $P_{\overline{\mathbb{R}(\mathbb{A})}}A = AP_{\overline{\mathbb{R}(\mathbb{A})}} = A$
(1)implies (2).Assume that T is a (n, m) power-A-hyponormal operator it follow that

$$\begin{aligned}
(T^\sharp)^mT^n \geq_A T^n(T^\sharp)^m \quad &\Rightarrow \quad (T^\sharp)^n((T^\sharp)^\sharp)^m \geq_A ((T^\sharp)^\sharp)^m(T^\sharp)^n\\
&\Rightarrow \quad (T^\sharp)^n(P_{\overline{\mathbb{R}(\mathbb{A})}}TP_{\overline{\mathbb{R}(\mathbb{A})}})^m \geq_A (P_{\overline{\mathbb{R}(\mathbb{A})}}TP_{\overline{\mathbb{R}(\mathbb{A})}})^m(T^\sharp)^n\\
&\Rightarrow \quad (T^\sharp)^n(P_{\overline{\mathbb{R}(\mathbb{A})}}T^mP_{\overline{\mathbb{R}(\mathbb{A})}}) \geq_A (P_{\overline{\mathbb{R}(\mathbb{A})}}T^mP_{\overline{\mathbb{R}(\mathbb{A})}})(T^\sharp)^n\\
&\Rightarrow \quad P_{\overline{\mathbb{R}(\mathbb{A})}}((T^\sharp)^nT^m - T^m(T^\sharp)^n) \geq_A 0\\
&\Rightarrow \quad (T^\sharp)^nT^m - T^m(T^\sharp)^n \geq_A 0
\end{aligned}$$

Then T is a (m, n) power-A-hyponormal operator
(2) implies (1).By same way hence we get it. □

This result is extented of the result of Theorem (2.5) in [3]

Theorem 2.15. *Let $T \in \mathcal{B}_A(\mathcal{H})$ such that $\mathcal{N}(A)$ is invariant subspace of T.The following statements hold*

1. *T and $T^{\sharp}$ are (n,n) power-A-hyponormal if and only if $\| T^n u \|_A = \| T^{\sharp n} u \|_A$*

2. *If A is injective then T and $T^{\sharp}$ are (n,n) power-A-hyponormal operators if and only if T is (n,n) power-A-normal*

Proof. 1 Assume T and $T^{\sharp}$ are (n,m) power-A-hyponormal operators that is

$$(T^{\sharp n}T^n - T^nT^{\sharp n}) \geq_A 0 \text{ and } ((T^{\sharp})^{\sharp n}T^{\sharp n} - T^{\sharp n}(T^{\sharp})^{\sharp n}) \geq_A 0$$

Since $\mathcal{N}(A)$ in invariant subspace for T we have

$$TP_{\overline{\mathcal{R}(\mathcal{A})}} = P_{\overline{\mathcal{R}(\mathcal{A})}}T \text{ and } P_{\overline{\mathcal{R}(\mathcal{A})}}A = AP_{\overline{\mathcal{R}(\mathcal{A})}} = A$$

Therefore in view of the fact that $(T^{\sharp})^{\sharp} = P_{\overline{\mathcal{R}(\mathcal{A})}}TP_{\overline{\mathcal{R}(\mathcal{A})}}$ we have for all$u \in \mathcal{H}$

$$\begin{aligned}
\langle(((T^{\sharp})^{\sharp n}T^{\sharp n} - T^{\sharp n}(T^{\sharp})^{\sharp n})u|u\rangle_A \geq 0 \quad &\Leftrightarrow \quad \langle(P_{\overline{\mathcal{R}(\mathcal{A})}}T^nP_{\overline{\mathcal{R}(\mathcal{A})}}T^{\sharp n} - T^{\sharp n}P_{\overline{\mathcal{R}(\mathcal{A})}}T^nP_{\overline{\mathcal{R}(\mathcal{A})}})u|u\rangle_A \geq 0\\
&\Leftrightarrow \quad \langle(P_{\overline{\mathcal{R}(\mathcal{A})}}T^nT^{\sharp n} - T^{\sharp n}T^nP_{\overline{\mathcal{R}(\mathcal{A})}})u|u\rangle_A \geq 0\\
&\Leftrightarrow \quad \langle(AP_{\overline{\mathcal{R}(\mathcal{A})}}T^nT^{\sharp n} - AT^{\sharp n}T^nP_{\overline{\mathcal{R}(\mathcal{A})}})u|u\rangle \geq 0\\
&\Leftrightarrow \quad \langle(AT^nT^{\sharp n} - AT^{\sharp n}T^n)u|u\rangle \geq 0\\
&\Leftrightarrow \quad \langle(T^nT^{\sharp n} - T^{\sharp n}T^n)u|u\rangle_A \geq 0\\
&\Leftrightarrow \quad T^nT^{\sharp n} \geq_A T^{\sharp n}T^n
\end{aligned}$$

It follows that $T^nT^{\sharp n} \geq_A T^{\sharp n}T^n \geq_A T^nT^{\sharp n}$ and hence

$$\| T^{\sharp n}u \|_A \geq \| T^n u \|_A \geq \| T^{\sharp n}u \|_A$$

Therefore $\| T^n u \|_A = \| T^{\sharp n} u \|_A$
Conversely, assume that $\| T^n u \|_A = \| T^{\sharp n} u \|_A$ $\forall u \in \mathcal{H}$ .From which it is clear that T is (n,n) A-hyponormal and $\langle(T^{\sharp n}T^n - T^nT^{\sharp n})u|u\rangle_A = 0$ $\forall u \in \mathcal{H}$. Now we have

$$\begin{aligned}
\langle(T^{\sharp n}T^n - T^nT^{\sharp n})u|u\rangle_A = 0 \quad &\Leftrightarrow \quad \langle A(T^{\sharp n}T^n - T^nT^{\sharp n})u|u\rangle = 0\\
&\Leftrightarrow \quad \langle(T^{\sharp n}P_{\overline{\mathcal{R}(\mathcal{A})}}T^nP_{\overline{\mathcal{R}(\mathcal{A})}} - P_{\overline{\mathcal{R}(\mathcal{A})}}T^nP_{\overline{\mathcal{R}(\mathcal{A})}}T^{\sharp n})u|u\rangle_A = 0\\
&\Leftrightarrow \quad \langle(T^{\sharp n}(T^{\sharp})^{\sharp n} - (T^{\sharp})^{\sharp n}T^{\sharp n})u|u\rangle_A = 0\\
&\Leftrightarrow \quad \| T^{\sharp n}u \|_A^2 - \| (T^{\sharp})^{\sharp n}u \|_A^2 = 0
\end{aligned}$$

Thus, $T^{\sharp}$ is (n,n) A-hyponormal.
2. If we assume that T and $T^{\sharp}$ are (n,n) power-A-hyponormal,from statement (1) it follows that $\| T^n u \|_A = \| T^{\sharp n} u \|_A$ $\forall u \in \mathcal{H}$.
Applying Proposition (2.12) and taking into account A is injective we see that
$T^{\sharp n}T^n - T^nT^{\sharp n} = 0$ We clearly have T is (n,n) power-A-normal
Conversely, if T is(n,n) power-A-normal, we have $T^{\sharp}$ is (n,n) power-A-normal ([11] corollary 3.2) then T and $T^{\sharp}$ are (n,n) power-A-hyponormal .The proof is complete. □

Proposition 2.16. *Let $T \in \mathcal{B}_A(\mathcal{H})$,$X = T^n + (T^{\sharp})^m$, $Y = T^n - (T^{\sharp})^m$ and $Z = T^n.(T^{\sharp})^m$.Then the followig statements hold*

1. *T is (n,m) power-A-hyponormal operator if and only if $[X,Y] \geq_A 0$*

2. *If T is of class $[(n,m)H]_A$ such that $T^n \geq_A 0$ and T^n commutes with $(T^{\sharp m}T^n - T^nT^{\sharp m})$ then $[Z, X+Y]$ is A-positive*

3. *If T is of class $[(n,m)H]_A$ such that $T^n \geq_A 0$,$T^{\sharp m} \geq_A 0$ and $(T^{\sharp m}T^n - T^nT^{\sharp m})$ commutes with both T^n and $T^{\sharp m}$ then $[Z,Y]$ is A-positive*

4. *T is of class* $[(n,m)H]_A$ *if and only if* $[X,T^n]\geq_A 0$ *and* $[T^n,Y]\geq_A 0$

5. *T is of class* $[(n,m)H]_A$ *if and only if* $[T^{\sharp m},X]\geq_A 0$ *and* $[Y,T^{\sharp m}]\geq_A 0$

Proof. 1.

$$\begin{aligned}
[X,Y] &= XY-YX\geq_A 0\\
&\Leftrightarrow (T^n+T^{\sharp m})(T^n-T^{\sharp m})-(T^n-T^{\sharp m})(T^n+T^{\sharp m})\geq_A 0\\
&\Leftrightarrow (T^{2n}-T^nT^{\sharp m}+T^{\sharp m}T^n-T^{\sharp 2m})-(T^{2n}+T^nT^{\sharp m}-T^{\sharp m}T^n-T^{\sharp 2m})\geq_A 0\\
&\Leftrightarrow T^{\sharp m}T^n-T^nT^{\sharp m}\geq_A 0\\
&\Leftrightarrow T\in[(n,m)H]_A
\end{aligned}$$

2. Assume that $T\in[(n,m)H]_A$ .Since $T^n\geq_A 0$ and commutes with $(T^{\sharp m}T^n-T^nT^{\sharp m})$ by lemma (2.6) we have $T^n(T^{\sharp m}T^n-T^nT^{\sharp m})\geq_A 0$ i.e $T^nT^{\sharp m}T^n\geq_A T^{2n}T^{\sharp m}$ .Then

$$\begin{aligned}
[Z,X+Y] &= Z(X+Y)-(X+Y)Z\\
&= T^n.(T^\sharp)^m.(2T^n)-(2T^n).T^n.(T^\sharp)^m\\
&= 2T^n.(T^\sharp)^mT^n-2T^{2n}.(T^\sharp)^m\\
&\geq_A T^{2n}T^{\sharp m}-T^{2n}T^{\sharp m}\\
&= 0
\end{aligned}$$

in the same we proved The statements 3,4,and 5 □

Proposition 2.17. *Let* $T,S\in\mathcal{B}_A(\mathcal{H})$ *such that* $\mathcal{N}(A)$ *is a reducing subspace for both T and S .If T is an* (n,m) *power-A-hyponormal operator and S is an A-isometry ,then* $STS^\sharp$ *is an* (n,m) *power-A-hyponormal operator.*

Proof. Note first that since S is an A-isometry then $S^\sharp S=P_{\overline{\mathcal{R}(\mathcal{A})}}$.Moreover from the fact that $\mathcal{N}(A)$ is reducing subspace for both T and S we have

$$P_{\overline{\mathcal{R}(\mathcal{A})}}T=TP_{\overline{\mathcal{R}(\mathcal{A})}},P_{\overline{\mathcal{R}(\mathcal{A})}}T^\sharp=T^\sharp P_{\overline{\mathcal{R}(\mathcal{A})}},P_{\overline{\mathcal{R}(\mathcal{A})}}S=SP_{\overline{\mathcal{R}(\mathcal{A})}}\text{ and }P_{\overline{\mathcal{R}(\mathcal{A})}}S^\sharp=S^\sharp P_{\overline{\mathcal{R}(\mathcal{A})}}$$

It easly to check that

$$\begin{aligned}
(STS^\sharp)^n &= \underbrace{(STS^\sharp)(STS^\sharp)...(STS^\sharp)}_{n\text{ times}}\\
&= \underbrace{(STP_{\overline{\mathcal{R}(\mathcal{A})}}TS^\sharp)...(STS^\sharp)}_{n-1\text{ times}}\\
&= (P_{\overline{\mathcal{R}(\mathcal{A})}}ST^2S^\sharp)...(STS^\sharp)\\
&= \vdots\\
&= P_{\overline{\mathcal{R}(\mathcal{A})}}ST^nS^\sharp
\end{aligned}$$

And

$$\begin{aligned}
(STS^\sharp)^{\sharp n} &= (STS^\sharp)^\sharp(STS^\sharp)^\sharp...(STS^\sharp)^\sharp)\\
&= \vdots\\
&= (P_{\overline{\mathcal{R}(\mathcal{A})}}ST^{\sharp n}S^\sharp)
\end{aligned}$$

From the above calculation we deduce that

$$\begin{aligned}\langle((STS^\sharp)^\sharp)^m u/((STS^\sharp)^\sharp)^n u\rangle_A &= \langle P_{\overline{\mathcal{R}(\mathcal{A})}} ST^{\sharp m} S^\sharp u/P_{\overline{\mathcal{R}(\mathcal{A})}} ST^{\sharp n} S^\sharp u\rangle_A (\text{S is A-isometry }) \\ &= \langle T^{\sharp m} S^\sharp u/T^{\sharp n} S^\sharp u\rangle_A \end{aligned} \quad (1)$$

And

$$\begin{aligned}\langle(STS^\sharp)^n u/(STS^\sharp)^m u\rangle_A &= \langle P_{\overline{\mathcal{R}(\mathcal{A})}} ST^n S^\sharp u/P_{\overline{\mathcal{R}(\mathcal{A})}} ST^m S^\sharp u\rangle_A \\ &= \langle T^n S^\sharp u/T^m S^\sharp u\rangle_A \end{aligned} \quad (2)$$

Since T is an (n, m) power-A-hyponormal operator we have by combining (1) and (2)

$$\langle(STS^\sharp)^n u/(STS^\sharp)^m u\rangle_A \geq \langle((STS^\sharp)^\sharp)^m u/((STS^\sharp)^\sharp)^n u\rangle_A$$

Then $(STS^\sharp)$ is an (n, m) power-A-hyponormal operator □

Proposition 2.18. *Let $T \in \mathcal{B}_A(\mathcal{H})$ is (n, m) power-A-hyponormal such that $\mathcal{N}(A)$ is a reducing subspace of T, if S is A-unitary equivalent of T ,then S is (n, m) power-A-hyponormal operator*

Proof. Let T be an (n, m) power A-hyponormal operator , since S is A-unitary equivalent of T then there exists A-unitary operator U such that $S = UTU^\sharp$, it is easly to chek that $S^n = UT^nU^\sharp$ and $S^\sharp = UT^\sharp U^\sharp$ .We have

$$\begin{aligned} S^n(S^\sharp)^m &= (UT^nU^\sharp)(U(T^\sharp)^m U^\sharp) \\ &= UT^nU^\sharp U(T^\sharp)^m U^\sharp \\ &= UT^n P_{\overline{\mathcal{R}(\mathcal{A})}} (T^\sharp)^m U^\sharp \\ &= UT^n(T^\sharp)^m U^\sharp \\ &\leq_A UT^{\sharp m} T^n U^\sharp (\text{ lemma (2.4) }) \\ &= (UT^{\sharp m} U^\sharp)(UT^n U^\sharp) \\ &= (S^\sharp)^m S^n \end{aligned}$$

Hence ,$S^n(S^\sharp)^m \leq_A (S^\sharp)^m S^n$ then $S \in [(n, m)H]_A$ □

The following discusses the conditions for product and sum of two (n, m) power-A-hyponormal operators to be (n, m) power-A-hyponormal

Proposition 2.19. *Let $T \in \mathcal{B}_A(\mathcal{H})$ and $S \in \mathcal{B}_A(\mathcal{H})$ such that $[T, S] = [S, T^\sharp] = 0$,if T is (n, n) power-A-hyponormal,then the following statements hold*

1. *If S is A-self adjoint,then TS is (n, n) power-A-hyponormal operator*

2. *If S is A-normal operator,then TS is (n, n) power-A-hyponormal operator*

Proof. 1. Let S is A-self adjoint then $AS = S^*A$,we have

$$\begin{aligned} \langle(TS)^{\sharp n} u/(TS)^{\sharp n} u\rangle_A &= \langle S^{\sharp n} T^{\sharp n} u/S^{\sharp n} T^{\sharp n} u\rangle_A \\ &= \langle AS^{\sharp n} T^{\sharp n} u/S^{\sharp n} T^{\sharp n} u\rangle \\ &= \langle S^{*n} AT^{\sharp n} u/S^{\sharp n} T^{\sharp n} u\rangle \\ &= \langle AS^n T^{\sharp n} u/S^{\sharp n} T^{\sharp n} u\rangle \\ &= \langle S^n T^{\sharp n} u/AS^n T^{\sharp n} u\rangle \\ &= \langle T^{\sharp n} S^n u/T^{\sharp n} S^n u\rangle_A \\ &\leq \langle T^n S^n u/T^n S^n u\rangle_A (\text{ since } T \in [(n, n)H]_A) \\ &= \langle(TS)^n u/(TS)^n u\rangle_A \end{aligned}$$

Then $TS \in [(n,n)H]_A$
2. The same steps can prove that TS is (n,n) power A-hyponormal operator □

Proposition 2.20. *If $T,S \in \mathcal{B}_A(\mathcal{H})$ are commuting (n,m) power A-hyponormal operators such that $ST^\sharp = T^\sharp S$ and $TS^\sharp = S^\sharp T$,if $m \geq n$ i.e $(m = n+k)$,$(S^{\sharp m}S^n)T^{\sharp k} \geq_A (S^nS^{\sharp m})T^{\sharp k}$ and $S^{\sharp k}(T^{\sharp m}T^n) \geq_A S^{\sharp k}(T^nT^{\sharp m})$ then TS is (n,m) power A-hyponormal operator*

Proof. Let $TS = ST$,$ST^\sharp = T^\sharp S$ and $TS^\sharp = S^\sharp T$, since $(S^{\sharp m}S^n)T^{\sharp k} \geq_A (S^nS^{\sharp m})T^{\sharp k}$ and $S^{\sharp k}(T^{\sharp m}T^n) \geq_A S^{\sharp k}(T^nT^{\sharp m})$ from remark (2.5) we get $T^nS^{\sharp m}S^nT^{\sharp m} \geq_A T^nS^nS^{\sharp m}T^{\sharp m}$ and $S^{\sharp m}T^{\sharp m}T^nS^n \geq_A S^{\sharp m}T^nT^{\sharp m}S^n$,then

$$\begin{aligned} (TS)^n.(TS)^{\sharp m} &= T^nS^nS^{\sharp m}T^{\sharp m} \\ &\leq_A T^nS^{\sharp m}S^nT^{\sharp m} \\ &= S^{\sharp m}T^nT^{\sharp m}S^n \\ &\leq_A S^{\sharp m}T^{\sharp m}T^nS^n \\ &= (TS)^{\sharp m}.(TS)^n \end{aligned}$$

Then TS is (n,m) power A-hyponormal □

Proposition 2.21. *Let $T,S \in \mathcal{B}_A(\mathcal{H})$ are (n,m) power A-hyponormal operators,if $S^n(T^nT^{\sharp m}) = (T^nT^{\sharp m})S^n$, $T^n(S^{\sharp m}S^n) = (S^{\sharp m}S^n)T^n$ and$(TS)^j = T^jS^j$ then TS and ST are (n,m) power A-hyponormal operators*

Proof. We have $S^nS^{\sharp m} \leq_A S^{\sharp m}S^n$ and $T^nT^{\sharp m} \leq_A T^{\sharp m}T^n$ then

$$\begin{aligned} (TS)^n.(TS)^{\sharp m} &= T^nS^nS^{\sharp m}T^{\sharp m} \\ &\leq_A T^nS^{\sharp m}S^nT^{\sharp m}(\text{ remark (2.5)}) \\ &= S^{\sharp m}S^nT^nT^{\sharp m} \\ &= S^{\sharp m}T^nT^{\sharp m}S^n \\ &\leq_A S^{\sharp m}T^{\sharp m}T^nS^n(\text{ remark (2.5)}) \\ &= (TS)^{\sharp m}.(TS)^n \end{aligned}$$

Then TS is (n,m) power A-hyponormal operator
The same steps can prove that ST is (n,m) power A-hyponormal operator □

Proposition 2.22. *Let $T,S \in \mathcal{B}_A(\mathcal{H})$ are (n,m) power A-hyponormal operators for some positive integers n and m such that $TS^\sharp = S^\sharp T$,$ST^\sharp = T^\sharp S$ and $TS = ST = 0$.Then $(S+T)$ are (n,m) power A-hyponormal operators for some positive integers n and m*

Proof. Under assumption we have

$$\begin{aligned} (S+T)^{\sharp m}(S+T)^n &= (S^{\sharp m} + T^{\sharp m})(S^n + T^n) \\ &= S^{\sharp m}S^n + S^{\sharp m}T^n + T^{\sharp m}S^n + T^{\sharp m}T^n \\ &= S^{\sharp m}S^n + T^nS^{\sharp m} + S^nT^{\sharp m} + T^{\sharp m}T^n \\ &\geq_A S^nS^{\sharp m} + T^nS^{\sharp m} + S^nT^{\sharp m} + T^nT^{\sharp m} \\ &= (S^n + T^n)(S^{\sharp m} + T^{\sharp m}) \\ &= (S+T)^n(S+T)^{\sharp m} \end{aligned}$$

Then $(S+T) \in [(n,m)H]_A$ and the proof is complete □

The following example show that the conditions of proposition (2.22)are necessary but not sufficient

Example 2.23.

$$\textit{Let } T = \begin{pmatrix} -3 & 2 \\ 0 & 3 \end{pmatrix}, S = \begin{pmatrix} -2 & 0 \\ -1 & 2 \end{pmatrix} \textit{ and } A = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

be operators acting on two dimensional Hilbert space $\mathbb{C}^2$*. It is easy to see that* $T \in [(2,3)H]_A$ *,*$S \in [(2,3)H]_A$ *and* $(S+T) \in [(2,3)H]_A$ *but* $TS \neq ST \neq 0$ *and* $TS^\sharp \neq S^\sharp T$

Proposition 2.24. *Let* $T, S \in \mathcal{B}_A(\mathcal{H})$ *are commuting* $(n,1)$ *power-A-hyponormal operators, such that* $TS^\sharp = S^\sharp T$*,*$ST^\sharp = T^\sharp T$ *and* $(T+S)^\sharp$ *is commutes with* $\sum_{1\leq p\leq n-1}\binom{n}{p}\Big(T^pS^{n-p}\Big)$ *Then* $(T+S)$ *is an* $(n,1)$ *power -A-hyponormal*

Proof.

$$\begin{aligned}
\Big(T+S\Big)^n\Big(T+S\Big)^\sharp &= \left[\sum_{0\leq p\leq n}\binom{n}{p}\Big(T^pS^{n-p}\Big)\right]\Big(T+S\Big)^\sharp \\
&= \left[S^n + T^n + \sum_{1\leq p\leq n-1}\binom{n}{p}\Big(T^pS^{n-p}\Big)\right]\Big(T+S\Big)^\sharp \\
&= S^nS^\sharp + T^nS^\sharp + S^nT^\sharp + T^nT^\sharp + \sum_{1\leq p\leq n-1}\binom{n}{p}\Big(T^pS^{n-p}\Big)\Big(T+S\Big)^\sharp \\
&\leq_A S^\sharp S^n + S^\sharp T^n + T^\sharp S^n + T^\sharp T^n + \Big(T+S\Big)^\sharp\sum_{1\leq p\leq n-1}\binom{n}{p}\Big(T^pS^{n-p}\Big) \\
&= \Big(T+S\Big)^\sharp\Big(S^n+T^n\Big) + \Big(T+S\Big)^\sharp\sum_{1\leq p\leq n-1}\binom{n}{p}\Big(T^pS^{n-p}\Big) \\
&= \Big(T+S\Big)^\sharp\Big(T+S\Big)^n.
\end{aligned}$$

Then $(S+T) \in [(n,1)H]_A$ □

Proposition 2.25. *Let* $T \in \mathcal{B}_A(\mathcal{H})$ *such that* $T \in [(n,m)N]_A \wedge [(n+1,m)H]_A$ *,if T is A-positive and commutes with* $(T^{\sharp m}T^{n+1} - T^{n+1}T^{\sharp m})$*,then* $T \in [(n+2,m)H]_A$ *for some positive integers n and m.*

Proof. Let $T \in [(n,m)N]_A \wedge [(n+1,m)H]_A$ .Since $T \geq_A 0$ and T commutes with $(T^{\sharp m}T^{n+1} - T^{n+1}T^{\sharp m})$ by lemma (2.6) we deduce that

$$T^{\sharp m}T^{n+1}T \geq_A T^{n+1}T^{\sharp m}T \text{ and } TT^{\sharp m}T^{n+1} \geq_A TT^{n+1}T^{\sharp m}. \text{ So}$$

$$\begin{aligned}
T^{\sharp m}T^{n+2} &= T^{\sharp m}T^{n+1}T \\
&\geq_A T^{n+1}T^{\sharp m}T \\
&= TT^nT^{\sharp m}T \\
&= TT^{\sharp m}T^{n+1} \\
&\geq_A TT^{n+1}T^{\sharp m} \\
&= T^{n+2}T^{\sharp m}
\end{aligned}$$

Hence $T \in [(n+2,m)H]_A$. □

Proposition 2.26. *Let* $T, \in \mathcal{B}_A(\mathcal{H})$ *such that* $T \in ([(n,m)N]_A \cap [(n,m+1)H]_A)$ *,if* $T^\sharp \geq_A 0$ *and commutes with* $(T^{\sharp m+1}T^n - T^nT^{\sharp m+1})$ *then* $T \in [(n,m+2)H]_A$ *for some positive integers n and m.*

Proof. .A similar argument as in the proof of proposition (2.25) gives the desired results □

Theorem 2.27. *Let $T, \in \mathcal{B}_A(\mathcal{H})$ be an (n,m) power A-hyponormal operator for some positive integers n and m.The following statements hold*

1. *If $n \geq m$ and $T^m(T^\sharp)^m T^m = T^m$,such that $T^m \geq_A 0$ and T^m commutes with $(T^{\sharp m}T^n - T^n T^{\sharp m})$ then $T \in [(n+m,m)H]_A$.*

2. *If $m \geq n$ and $(T^\sharp)^n T^n (T^\sharp)^n = (T^\sharp)^n$,such that $T^{\sharp n} \geq_A 0$ and $T^{\sharp n}$ commutes with $(T^{\sharp m}T^n - T^n T^{\sharp m})$ then $T \in [(n,n+m)H]_A$.*

Proof. 1 we have $T^m(T^\sharp)^m T^m = T^m$.It follows that $T^n(T^\sharp)^m T^m = T^n$ and $T^m(T^\sharp)^m T^n = T^n$ for $n \geq m$ wich means that

$$T^n(T^\sharp)^m T^m = T^m(T^\sharp)^m T^n \tag{3}$$

Since $T \in [(n,m)H]_A$ and T^m is A-positive commutes with $(T^{\sharp m}T^n - T^n T^{\sharp m})$ we get

$$T^n(T^\sharp)^m T^m \leq_A (T^\sharp)^m T^{n+m} \tag{4}$$

and

$$T^{n+m}(T^\sharp)^m \leq_A T^m(T^\sharp)^m T^n \tag{5}$$

By combining (3),(4) and (5) we get

$$T^{n+m}(T^\sharp)^m \leq_A (T^\sharp)^m T^{n+m}$$

Then $T \in [(n+m,m)H]_A$.
2. in the same way we get $T \in [(n,n+m)H]_A$. □

In the following theorem we will prove the stability of the class of (n,m) power$A$-hyponormal operators under the direct sum and tensor product.

Theorem 2.28. *Let $T_1, T_2, ..., T_k$ be (n,m) power -A-hyponormal operators in $\mathcal{B}_A(\mathcal{H})$,such that $T_d^n T_d^{\sharp m} \geq_A 0$ for all $d \in \{1, 2, \dots k\}$,then*

1. *$(T_1 \oplus T_2 \oplus .... \oplus T_k)$ is (n,m) power $(A \oplus A \oplus ... \oplus A)$-hyponormal operator*

2. *$(T_1 \otimes T_2 \otimes .... \otimes T_k)$ is (n,m) power $(A \otimes A \otimes ... \otimes A)$-hyponormal operator.*

Proof. 1.Assume $T_1, T_2, ..., T_k$ are (n,m) power -A-hyponormal operators,we have

$$\begin{aligned}
(T_1 \oplus T_2 \oplus ... \oplus T_k)^n (T_1 \oplus T_2 \oplus ... \oplus T_k)^{\sharp m} &= \left(T_1^n \oplus T_2^n \oplus ... \oplus T_k^n\right)\left(T_1^{\sharp m} \oplus T_2^{\sharp m} \oplus ... \oplus T_k^{\sharp m}\right) \\
&= T_1^n T_1^{\sharp m} \oplus T_2^n T_2^{\sharp m} \oplus .... \oplus T_k^n T_k^{\sharp m} \\
&\leq_{A\oplus A ... \oplus A} T_1^{\sharp m} T_1^n \oplus T_2^{\sharp m} T_2^n \oplus ... \oplus T_k^{\sharp m} T_k^n \\
&= \left(T_1^{\sharp m} \oplus T_2^{\sharp m} \oplus ... \oplus T_k^{\sharp m}\right)\left(T_1^n \oplus T_2^n \oplus ... \oplus T_k^n\right) \\
&= (T_1 \oplus T_2 \oplus ... \oplus T_k)^{\sharp m} (T_1 \oplus T_2 \oplus ... \oplus T_k)^n .
\end{aligned}$$

Then $(T_1 \oplus T_2 \oplus ... \oplus T_k)$ is (n,m) power $(A \oplus A \oplus ... \oplus A)$-hyponormal operator.
2.Let $x_1, x_2, x_3 \dots x_k \in \mathcal{H}$,then

$$\left(T_1 \otimes T_2 \otimes ... \otimes T_k\right)^{\sharp m}\left(T_1 \otimes T_2 \otimes ... \otimes T_k\right)^n (x_1 \otimes x_2 \otimes \dots \otimes x_k)$$

$$\begin{aligned}
&= \Big(T_1^{\sharp m} \otimes T_2^{\sharp m} \otimes ... \otimes T_k^{\sharp m}\Big)\Big(T_1^n \otimes T_2^n \otimes ... \otimes T_k^n\Big)(x_1 \otimes x_2 \otimes \ldots \otimes x_k)\\
&= T_1^{\sharp m}T_1^n x_1 \otimes T_2^{\sharp m}T_2^n x_2 \otimes .... \otimes T_k^{\sharp m}T_k^n x_k\\
&\geq_{A\otimes A ...\otimes A} T_1^n T_1^{\sharp m} x_1 \otimes T_2^n T_2^{\sharp m} x_2 \otimes .... \otimes T_k^n T_k^{\sharp m} x_k\\
&= \Big(T_1^n \otimes T_2^n \otimes ... \oplus T_k^n\Big)\Big(T_1^{\sharp m} \otimes T_2^{\sharp m} \otimes ... \otimes T_k^{\sharp m}\Big)(x_1 \otimes x_2 \otimes \ldots \otimes x_k)\\
&= \Big(T_1 \otimes T_2 \otimes ... \otimes T_k\Big)^n\Big(T_1 \otimes T_2 \otimes ... \otimes T_k\Big)^{\sharp m}(x_1 \otimes x_2 \otimes \ldots \otimes x_k).
\end{aligned}$$

Then $(T_1 \otimes T_2 \otimes ... \otimes T_K)$ is (n, m)-power$(A \otimes A \otimes ... \otimes A)$-hyponormal operator. □

Proposition 2.29. *Let $T, S \in \mathcal{B}_A(\mathcal{H})$ are (n, m) power A-hyponormal such that $T^nT^{\sharp m} \geq_A 0$,$S^nS^{\sharp m} \geq_A 0$ and $(TS)^j = T^jS^j$, then the following statements hold*

1. *If $S^n(T^nT^{\sharp m}) = (T^nT^{\sharp m})S^n$ and $T^n(S^{\sharp m}S^n) = (S^{\sharp m}S^n)T^n$ then $TS \otimes T$ and $TS \otimes S \in \mathcal{B}_{A\otimes A}(\mathcal{H}\overline{\otimes}\mathcal{H})$ are (n, m) power $(A \otimes A)$-hyponormal*

2. *If $T^n(S^nS^{\sharp m}) = (S^nS^{\sharp m})T^n$ and $S^n(T^{\sharp m}T^n) = (T^{\sharp m}T^n)S^n$ then $ST \otimes T$ and $ST \otimes S \in \mathcal{B}_{A\otimes A}(\mathcal{H}\overline{\otimes}\mathcal{H})$ are (n, m) power $(A \otimes A)$-hyponormal*

Proof. 1.Assume that the conditions (1) are hold , T and S are (n, m) power A-hyponormal,we have

$$\begin{aligned}
(TS \otimes T)^{\sharp m}(TS \otimes T)^n &= \Big((TS)^{\sharp m} \otimes T^{\sharp m}\Big)\Big((TS)^n \otimes T^n\Big)\\
&= \Big(S^{\sharp m}T^{\sharp m}T^nS^n \otimes T^{\sharp m}T^n\Big)
\end{aligned}$$

Since $T^{\sharp m}T^n \geq_A T^nT^{\sharp m} \geq_A 0$ and $S^{\sharp m}S^n \geq_A S^nS^{\sharp m} \geq_A 0$,from lemma (2.4)and remak (2.5) we have

$$\begin{aligned}
S^{\sharp m}T^{\sharp m}T^nS^n &\geq_A S^{\sharp m}T^nT^{\sharp m}S^n\\
&= S^{\sharp m}S^nT^nT^{\sharp m}\\
&= T^nS^{\sharp m}S^nT^{\sharp m}\\
&\geq_A T^nS^nS^{\sharp m}T^{\sharp m}\\
&= (TS)^n(TS)^{\sharp m}
\end{aligned}$$

Thus

$$\begin{cases} S^{\sharp m}T^{\sharp m}T^nS^n \geq_A (TS)^n(TS)^{\sharp} \geq_A 0\\ \text{and}\\ T^{\sharp m}T^n \geq_A T^nT^{\sharp m} \geq_A 0\end{cases}$$

Lemma (3.1) in [3] implies that

$$\Big(S^{\sharp m}T^{\sharp m}T^nS^n \otimes T^{\sharp m}T^n\Big) \geq_A \Big((TS)^n(TS)^{\sharp m} \otimes T^nT^{\sharp m}\Big)$$

Then, $(TS \otimes T)^{\sharp m}(TS \otimes T)^n \geq_A (TS \otimes T)^n(TS \otimes T)^{\sharp m}$
Hence $(TS \otimes T)$ is (n, m) power $(A \otimes A)$-hyponormal. In the same way, we may deduce the (n, m) power $(A \otimes A)$-hyponormality of $TS \otimes S$,$(ST \otimes T)$ and $(ST \otimes S)$ □

References

[1] M.L. Arias, G. Corach, M.C. Gonzalez, *Partial isometries in semi- Hilbertian spaces*, Linear Algebra Appl, 428, (7), (2008), 1460-1475. DOI:10.1016/j.laa.2007.09.031.

[2] M.L. Arias, G. Corach, M.C. Gonzalez, *Metric properties of projections in semi- Hilbertian spaces*, Integral Equations Operator Theory , 62 (1),(2008), 11-28.

[3] Ould Ahmed Mahmoud Sid Ahmed and Abdelkader Benali, *Hyponormal And k-Quasi-hyponormal Operators On Semi-Hilbertian Spaces*, The Australian Journal of Mathematical Analysis and Applications (2016). doi:10.1155/2012/975745.

[4] A.Aluthge, *On p-hyponormal Operators for* $0 < p < 1$, Integral Equations Operators Theory , 13(1990), 307-315.

[5] Cherifa Chelalli and Abdelkader Benali , *Class of* (A, n)*-power-Hyponormal and Quasihyponormal operators in semi-Hilbertian space*, Functional Analysis Approximation And Computation 11(2)(2019), 13-21.

[6] B. P. Duggal, *Tensor products of operators-strong stability and p-hyponormality*, Glasgow Math journal,42(2000), 371-381.

[7] M. C. Gonzalez, *Operator norm inequalities in semi-Hilbertian spaces*, Linear Algebra Appl. 434 (2011), pp. 370-378.

[8] Samir Al Mohammady, Sid Ahmed Ould Beinane and Ould Ahmed Mahmoud Sid Ahmed, *On* (n, m)*-A-power-normal and* (n, m)*-A-power quasi-normal operators in semi-Hilbertian space*, Mathematica Bohemica (February 2020)

[9] O. A. Mahmoud Sid Ahmed and A. Saddi, *A-m-Isomertic operators in semi-Hilbertian spaces*, Linear Algebra and its Applications, 436 (2012), pp. 3930-3942.

[10] Messaoud Guesba and Mostefa Nadir, *On n-power-hyponormal operators*, Global Journal of Pure and Applied Mathematics. ISSN 0973-1768 Volume 12, Number 1 (2016), pp. 473-479

[11] A. Saddi, *A- Normal operators in Semi-Hilbertian spaces*, The Australian Journal of Mathematical Analysis and Applications Volume 9, issue 1,Article 5,(2012), 1-12.