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# On common fixed point theorems in partial metric spaces using $C$-class function

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**Abstract.** The purpose of this paper is to prove some common fixed point theorems in the set up of partial metric spaces with the help of $C$-class function and auxiliary functions and give some consequences of the established results. Also we give some examples in support of the result. Our results extend and generalize several results in the existing literature regarding rational type contraction mappings and partial metric spaces.

## 1. Introduction

The classical Banach contraction principle is one of the most celebrated and useful results in fixed point theory. In a metric space setting it can be briefly stated as follows.

**Theorem 1.1.** *([7]) Let $(\mathcal{Y}, d)$ be a complete metric space and $\mathcal{R}\colon \mathcal{Y} \to \mathcal{Y}$ be a map satisfying*

$$d(\mathcal{R}(y), \mathcal{R}(z)) \le s\, d(y, z), \quad \text{for all } y, z \in \mathcal{Y}, \tag{1}$$

*where $0 < s < 1$ is a constant. Then*

(1) *$\mathcal{R}$ has a unique fixed point $x$ in $\mathcal{Y}$.*

(2) *The Picard iteration $\{u_n\}_{n=0}^{\infty}$ defined by*

$$u_{n+1} = \mathcal{R}u_n, \quad n = 0, 1, 2, \dots \tag{2}$$

*converges to $x$, for any $u_0 \in \mathcal{Y}$.*

**Remark 1.2.** *(i) A self-map satisfying* (1) *and* (2) *is said to be a Picard operator (see, [25, 26]).*

*(ii) Inequality (1) also implies the continuity of $\mathcal{R}$.*

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There are many generalizations of this principle. These generalizations are made either by using different contractive conditions or by imposing some additional condition on the ambient spaces. On the other hand, a number of generalizations of metric spaces have been done and one of such generalization is partial metric space introduced in 1992 by Matthews [21, 22]. It is widely recognized that partial metric spaces play an important role in constructing models in the theory of computation. In partial metric spaces the distance of a point in the self may not be zero. Introducing partial metric space, Matthews proved the partial metric version of Banach fixed point theorem ([7]). Then, many authors gave some generalizations of the result of Matthews and proved some fixed point theorems in this space (see, i.e., [1], [2], [3], [15], [16], [17], [18], [24], [33]-[37], [38] and many others).

Das and Gupta [13] in 1975, proved the following fixed point theorem using contractive condition involving rational expressions.

**Theorem 1.3.** *([13]) Let $(\mathcal{Y}, d)$ be a complete metric space and let $\mathcal{R}\colon \mathcal{Y} \to \mathcal{Y}$ be a mapping such that there exists $\alpha, \beta > 0$ with $\alpha + \beta < 1$ satisfying*

$$d(\mathcal{R}(y), \mathcal{R}(z)) \leq \alpha\, d(y, z) + \beta\, \frac{d(z, \mathcal{R}(z))[1 + d(y, \mathcal{R}(y))]}{1 + d(y, z)}, \tag{3}$$

*for all $y, z \in \mathcal{Y}$. Then $\mathcal{R}$ has a unique fixed point.*

Recently, many authors have proved fixed point and common fixed point theorems via contractive condition using rational expressions or rational type expressions in various ambient spaces (see, e.g., [8–11, 18, 27–32] and many others).

Quite recently, Kumar et al. [19] have proved some existence and uniqueness of fixed point theorems for contractive condition using auxiliary function in the framework of partial metric spaces.

The purpose of this work is to prove some common fixed point theorems for contractive condition involving rational expression with $C$-class function and some auxiliary functions in the set up of partial metric spaces.

## 2. Preliminaries

Now, we recall some basic definitions, properties and auxiliary results of partial metric spaces.

**Definition 2.1.** *([22]) Let $\mathcal{Y}$ be a nonempty set and $p\colon \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}^+$ be such that for all $u, v, w \in \mathcal{Y}$ the followings are satisfied:*

$(P1)$ $u = v \Leftrightarrow p(u, u) = p(u, v) = p(v, v)$,

$(P2)$ $p(u, u) \leq p(u, v)$,

$(P3)$ $p(u, v) = p(v, u)$,

$(P4)$ $p(u, v) \leq p(u, w) + p(w, v) - p(w, w)$.

*Then $p$ is called partial metric on $\mathcal{Y}$ and the pair $(\mathcal{Y}, p)$ is called partial metric space (in short PMS).*

**Remark 2.2.** *It is clear that if $p(u, v) = 0$, then $u = v$. But, on the contrary $p(u, u)$ need not be zero.*

**Example 2.3.** *([6]) Let $\mathcal{Y} = \mathbb{R}^+$ and $p\colon \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}^+$ given by $p(u, v) = \max\{u, v\}$ for all $u, v \in \mathbb{R}^+$. Then $(\mathbb{R}^+, p)$ is a partial metric space.*

**Example 2.4.** *([6]) Let $\mathcal{Y} = \{[a, b] : a, b \in \mathbb{R}, a \leq b\}$. Then $p\Big([a, b], [c, d]\Big) = \max\{b, d\} - \min\{a, c\}$ defines a partial metric $p$ on $\mathcal{Y}$.*

Various applications of this space has been extensively investigated by many authors (see [20], [38] for details).

**Remark 2.5.** *([16]) Let $(\mathcal{Y}, p)$ be a partial metric space.*

(1) *The function $d_p \colon \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}^+$ defined as $d_p(u, v) = 2p(u, v) - p(u, u) - p(v, v)$ is a (usual) metric on $\mathcal{Y}$ and $(\mathcal{Y}, d_p)$ is a (usual) metric space.*

(2) *The function $d_s \colon \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}^+$ defined as $d_s(u, v) = \max\{p(u, v) - p(u, u), p(u, v) - p(v, v)\}$ is a (usual) metric on $\mathcal{Y}$ and $(\mathcal{Y}, d_s)$ is a (usual) metric space.*

Note also that each partial metric $p$ on $\mathcal{Y}$ generates a $T_0$ topology $\tau_p$ on $\mathcal{Y}$, whose base is a family of open $p$-balls $\{B_p(u, \varepsilon) : u \in \mathcal{Y}, \varepsilon > 0\}$ where $B_p(u, \varepsilon) = \{v \in \mathcal{Y} : p(u, v) < p(u, u) + \varepsilon\}$ for all $u \in \mathcal{Y}$ and $\varepsilon > 0$.

On a partial metric space the notions of convergence, the Cauchy sequence, completeness and continuity are defined as follows [21].

**Definition 2.6.** *([21]) Let $(\mathcal{Y}, p)$ be a partial metric space. Then*

$(a1)$ *a sequence $\{q_n\}$ in $(\mathcal{Y}, p)$ is said to be convergent to a point $q \in \mathcal{Y}$ if and only if $p(q, q) = \lim_{n\to\infty} p(q_n, q)$,*

$(a2)$ *a sequence $\{q_n\}$ is called a Cauchy sequence if $\lim_{m,n\to\infty} p(q_m, q_n)$ exists and finite,*

$(a3)$ *$(\mathcal{Y}, p)$ is said to be complete if every Cauchy sequence $\{q_n\}$ in $\mathcal{Y}$ converges to a point $q \in \mathcal{Y}$ with respect to $\tau_p$. Furthermore,*

$$\lim_{m,n\to\infty} p(q_m, q_n) = \lim_{n\to\infty} p(q_n, q) = p(q, q).$$

$(a4)$ *A mapping $g \colon \mathcal{Y} \to \mathcal{Y}$ is said to be continuous at $r_0 \in \mathcal{Y}$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $g\big(B_p(r_0, \delta)\big) \subset B_p\big(g(r_0), \varepsilon\big)$.*

**Definition 2.7.** *([23]) Let $(\mathcal{Y}, p)$ be a partial metric space. Then*

$(a5)$ *a sequence $\{q_n\}$ in $(\mathcal{Y}, p)$ is called 0-Cauchy if $\lim_{m,n\to\infty} p(q_m, q_n) = 0$,*

$(a6)$ *$(\mathcal{Y}, p)$ is said to be 0-complete if every 0-Cauchy sequence $\{q_n\}$ in $\mathcal{Y}$ converges to a point $q \in \mathcal{Y}$, such that $p(q, q) = 0$.*

**Lemma 2.8.** *([21, 22]) Let $(\mathcal{Y}, p)$ be a partial metric space. Then*

*(a7) a sequence $\{q_n\}$ in $(\mathcal{Y}, p)$ is a Cauchy sequence if and only if it is a Cauchy sequence in the metric space $(\mathcal{Y}, d_p)$,*

*(a8) $(\mathcal{Y}, p)$ is complete if and only if the metric space $(\mathcal{Y}, d_p)$ is complete,*

*(a9) a subset $E$ of a partial metric space $(\mathcal{Y}, p)$ is closed if a sequence $\{q_n\}$ in $E$ such that $\{q_n\}$ converges to some $q \in \mathcal{Y}$, then $q \in E$.*

**Lemma 2.9.** *([2]) Assume that $q_n \to q$ as $n \to \infty$ in a partial metric space $(\mathcal{Y}, p)$ such that $p(q, q) = 0$. Then $\lim_{n\to\infty} p(q_n, u) = p(q, u)$ for every $u \in \mathcal{Y}$.*

**Lemma 2.10.** *([12]) Let $(\mathcal{Y}, p)$ be a partial metric space and let $\{y_n\}$ be a sequence in $(\mathcal{Y}, p)$ such that $\lim_{n\to\infty} p(y_{n+1}, y_n) = 0$.*

*If the sequence $\{y_{2n}\}$ is not a Cauchy sequence in $(\mathcal{Y}, p)$, then there exists $\varepsilon > 0$ and two subsequences $\{y_{2m(k)}\}$ and $\{y_{2n(k)}\}$ of positive integers with $n(k) > m(k) > k$ such that the four sequences*

$$p(y_{2m(k)}, y_{2n(k)+1}),\ p(y_{2m(k)}, y_{2n(k)}),\ p(y_{2m(k)-1}, y_{2n(k)+1}),\ p(y_{2m(k)-1}, y_{2n(k)})$$

*tend to $\varepsilon > 0$ when $k \to \infty$.*

**Remark 2.11.** *(see [16]) Let $(\mathcal{Y}, p)$ be a PMS. Therefore, for all $u, v \in \mathcal{Y}$*
*(i) if $p(u, v) = 0$, then $u = v$;*
*(ii) if $u \neq v$, then $p(u, v) > 0$.*

In 2014, Ansari [4] was introduced the notion of $C$-class function (see Definition 2.12) which actually covers a large class of contractive conditions.

**Definition 2.12.** *([4]) A mapping $F\colon [0,\infty)\times[0,\infty) \to R$ is called a C-class function if it is continuous and satisfies the following axioms:*
(1) $F(s,t) \leq s$,
(2) $F(s,t) = s$ *implies that either* $s = 0$ *or* $t = 0$, *for all* $s, t \in [0,\infty)$.

Note for some $F$ we have $F(0,0) = 0$.
We denote the set of all $C$-class functions by letter $\mathcal{C}$.

**Example 2.13.** *([4]) The following functions $F\colon [0,\infty)\times[0,\infty) \to R$ are elements of $\mathcal{C}$, for all $s, t \in [0,\infty)$, we have:*
(1) $F(s,t) = s - t$, $F(s,t) = s \Rightarrow t = 0$;
(2) $F(s,t) = ms, 0 < m < 1$, $F(s,t) = s \Rightarrow s = 0$;
(3) $F(s,t) = \frac{s}{(1+t)^r}$, $r \in (0,\infty)$, $F(s,t) = s \Rightarrow s = 0$ *or* $t = 0$;
(4) $F(s,t) = \frac{log(t+a^s)}{1+t}$, $a > 1$, $F(s,t) = s \Rightarrow s = 0$ *or* $t = 0$;
(5) $F(s,t) = \frac{ln(1+a^s)}{2}$, $a > e$, $F(s,1) = s \Rightarrow s = 0$;
(6) $F(s,t) = (s+l)^{(1/(1+t)^r)} - l$, $l > 1$, $r \in (0,\infty)$, $F(s,t) = s \Rightarrow t = 0$;
(7) $F(s,t) = slog_{t+a}a$, $a > 1$, $F(s,t) = s \Rightarrow s = 0$ *or* $t = 0$;
(8) $F(s,t) = s - \left(\frac{1+s}{2+s}\right)\left(\frac{t}{1+t}\right)$, $F(s,t) = s \Rightarrow t = 0$;
(9) $F(s,t) = s\beta(s)$, *where* $\beta\colon [0,\infty) \to [0,1)$ *and is continuous*, $F(s,t) = s \Rightarrow s = 0$;
(10) $F(s,t) = s - \left(\frac{t}{k+t}\right)$, $F(s,t) = s \Rightarrow t = 0$;
(11) $F(s,t) = s - \varphi(s)$, $F(s,t) = s \Rightarrow s = 0$, *here* $\varphi\colon [0,\infty) \to [0,\infty)$ *is a continuous function such that* $\varphi(t) = 0$ *if and only if* $t = 0$;
(12) $F(s,t) = sh(s,t)$, $F(s,t) = s \Rightarrow s = 0$, *here* $h\colon [0,\infty)\times[0,\infty) \to [0,\infty)$ *is a continuous function such that* $h(s,t) < 1$ *for all* $t, s > 0$;
(13) $F(s,t) = s - \left(\frac{2+t}{1+t}t\right)$, $F(s,t) = s \Rightarrow t = 0$;
(14) $F(s,t) = \sqrt[n]{ln(1+s^n)}$, $F(s,t) = s \Rightarrow s = 0$;
(15) $F(s,t) = \phi(s)$, $F(s,t) = s \Rightarrow s = 0$, *here* $\phi\colon [0,\infty) \to [0,\infty)$ *is a upper semi-continuous function such that* $\phi(0) = 0$ *and* $\phi(t) < t$ *for all* $t > 0$;
(16) $F(s,t) = \frac{s}{(1+s)^r}$, $r \in (0,\infty)$, $F(s,t) = s \Rightarrow s = 0$;
(17) $F(s,t) = \frac{s}{\Gamma(1/2)}\int_0^\infty \frac{e^{-x}}{\sqrt{x}+t}dx$, *where* $\Gamma$ *is the Euler Gamma function.*

**Remark 2.14.** *Number (1), (2), (9) and (15) from Example 2.13 are pivotal results in fixed point theory ([4]). Also see [5] and [14].*

**Definition 2.15.** *([4]) A function $\psi\colon [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:*
($\psi_1$) *$\psi$ is non-decreasing and continuous function,*
($\psi_2$) *$\psi(t) = 0$ if and only if $t = 0$.*

**Remark 2.16.** *([4]) We denote $\Psi$ the class of all altering distance functions.*

**Definition 2.17.** *([4]) A function $\varphi\colon [0,\infty) \to [0,\infty)$ is said to be an ultra altering distance function, if it is continuous, non-decreasing such that $\varphi(t) > 0$ for $t > 0$ and $\varphi(0) \geq 0$.*

**Remark 2.18.** *([4]) We denote $\Phi_u$ the class of all ultra altering distance functions.*

## 3. Main Results

In this section, we shall prove some unique common fixed point theorems in the set up of partial metric spaces with the help of $C$-class function and auxiliary functions.

**Theorem 3.1.** *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two self-maps on a complete partial metric space $(\mathcal{Y}, p)$ satisfying the condition:*

$$\psi\Big(p(\mathcal{G}_1 y, \mathcal{G}_2 z)\Big) \leq F\Big(\psi\Big(\mathcal{M}_1^p(y,z)\Big), \varphi\Big(\mathcal{M}_2^p(y,z)\Big)\Big), \tag{4}$$

*for all $y, z \in \mathcal{Y}$, where*

$$\mathcal{M}_1^p(y,z) = \max\Big\{p(y,z), p(y,\mathcal{G}_1 y), p(z,\mathcal{G}_2 z), p(z,\mathcal{G}_2 z)\frac{1+p(y,\mathcal{G}_1 y)}{1+p(y,z)}\Big\}, \tag{5}$$

*and*

$$\begin{aligned}\mathcal{M}_2^p(y,z) = \max\Big\{&p(y,z), \frac{1}{2}[p(z,\mathcal{G}_1 y)+p(y,\mathcal{G}_2 z)], \frac{p(y,\mathcal{G}_1 y)p(z,\mathcal{G}_2 z)}{1+p(y,z)},\\ &\frac{p(y,\mathcal{G}_1 y)p(z,\mathcal{G}_2 z)}{1+p(\mathcal{G}_1 y,\mathcal{G}_2 z)}\Big\},\end{aligned} \tag{6}$$

*for all $\psi \in \Psi$, $\varphi \in \Phi_u$ and $F \in \mathcal{C}$. Then $\mathcal{G}_1$ and $\mathcal{G}_2$ have a unique common fixed point in $\mathcal{Y}$.*

*Proof.* For each $y_0 \in \mathcal{Y}$. Let $y_{2n+1} = \mathcal{G}_1 y_{2n}$ and $y_{2n+2} = \mathcal{G}_2 y_{2n+1}$ for $n = 0, 1, 2, \ldots$. We prove that $\{y_n\}$ is a Cauchy sequence in $(\mathcal{Y}, p)$. It follows from (4) for $y = y_{2n}$ and $z = y_{2n-1}$ that

$$\begin{aligned}\psi\Big(p(y_{2n+1}, y_{2n})\Big) &= \psi\Big(p(\mathcal{G}_1 y_{2n}, \mathcal{G}_2 y_{2n-1})\Big)\\ &\leq F\Big(\psi\Big(\mathcal{M}_1^p(y_{2n}, y_{2n-1})\Big), \varphi\Big(\mathcal{M}_2^p(y_{2n}, y_{2n-1})\Big)\Big),\end{aligned} \tag{7}$$

where

$$\begin{aligned}\mathcal{M}_1^p(y_{2n}, y_{2n-1}) &= \max\Big\{p(y_{2n}, y_{2n-1}), p(y_{2n}, \mathcal{G}_1 y_{2n}), p(y_{2n-1}, \mathcal{G}_2 y_{2n-1}),\\ &\qquad p(y_{2n-1}, \mathcal{G}_2 y_{2n-1})\frac{1+p(y_{2n}, \mathcal{G}_1 y_{2n})}{1+p(y_{2n}, y_{2n-1})}\Big\}\\ &= \max\Big\{p(y_{2n}, y_{2n-1}), p(y_{2n}, y_{2n+1}), p(y_{2n-1}, y_{2n}),\\ &\qquad p(y_{2n-1}, y_{2n})\frac{1+p(y_{2n}, y_{2n+1})}{1+p(y_{2n}, y_{2n-1})}\Big\}\\ &= \max\Big\{p(y_{2n-1}, y_{2n}), p(y_{2n+1}, y_{2n}), p(y_{2n-1}, y_{2n}),\\ &\qquad p(y_{2n-1}, y_{2n})\frac{1+p(y_{2n+1}, y_{2n})}{1+p(y_{2n-1}, y_{2n})}\Big\} \text{ (by (P3))}\\ &= \max\Big\{p(y_{2n-1}, y_{2n}), p(y_{2n+1}, y_{2n})\Big\},\end{aligned} \tag{8}$$

and

$$
\begin{aligned}
\mathcal{M}_2^p(y_{2n}, y_{2n-1}) &= \max\Big\{p(y_{2n}, y_{2n-1}), \frac{1}{2}[p(y_{2n-1}, \mathcal{G}_1 y_{2n}) + p(y_{2n}, \mathcal{G}_2 y_{2n-1})], \\
&\qquad \frac{p(y_{2n}, \mathcal{G}_1 y_{2n})p(y_{2n-1}, \mathcal{G}_2 y_{2n-1})}{1 + p(y_{2n}, y_{2n-1})}, \frac{p(y_{2n}, \mathcal{G}_1 y_{2n})p(y_{2n-1}, \mathcal{G}_2 y_{2n-1})}{1 + p(\mathcal{G}_1 y_{2n}, \mathcal{G}_2 y_{2n-1})}\Big\} \\
&= \max\Big\{p(y_{2n}, y_{2n-1}), \frac{1}{2}[p(y_{2n-1}, y_{2n+1}) + p(y_{2n}, y_{2n})], \\
&\qquad \frac{p(y_{2n}, y_{2n+1})p(y_{2n-1}, y_{2n})}{1 + p(y_{2n}, y_{2n-1})}, \frac{p(y_{2n}, y_{2n+1})p(y_{2n-1}, y_{2n})}{1 + p(y_{2n+1}, y_{2n})}\Big\} \\
&\leq \max\Big\{p(y_{2n-1}, y_{2n}), \frac{1}{2}[p(y_{2n-1}, y_{2n}) + p(y_{2n+1}, y_{2n})], \\
&\qquad \frac{p(y_{2n+1}, y_{2n})p(y_{2n-1}, y_{2n})}{1 + p(y_{2n-1}, y_{2n})}, \frac{p(y_{2n+1}, y_{2n})p(y_{2n-1}, y_{2n})}{1 + p(y_{2n+1}, y_{2n})}\Big\} \\
&\qquad \text{(by (P3) and (P4))} \\
&= \max\Big\{p(y_{2n-1}, y_{2n}), p(y_{2n+1}, y_{2n})\Big\}. \qquad (9)
\end{aligned}
$$

From equations (7), (8) and (9), we have

$$
\begin{aligned}
\psi\Big(p(y_{2n+1}, y_{2n})\Big) &\leq F\Big(\psi\Big(\max\Big\{p(y_{2n-1}, y_{2n}), p(y_{2n+1}, y_{2n})\Big\}\Big), \\
&\qquad \varphi\Big(\max\Big\{p(y_{2n-1}, y_{2n}), p(y_{2n+1}, y_{2n})\Big\}\Big)\Big). \qquad (10)
\end{aligned}
$$

If $p(y_{2n+1}, y_{2n}) > p(y_{2n-1}, y_{2n})$, then from equation (10) and by using the property of $F$, we get

$$
\begin{aligned}
\psi\Big(p(y_{2n+1}, y_{2n})\Big) &\leq F\Big(\psi\Big(p(y_{2n+1}, y_{2n})\Big), \varphi\Big(p(y_{2n+1}, y_{2n})\Big)\Big) \\
&\leq \psi\Big(p(y_{2n+1}, y_{2n})\Big), \qquad (11)
\end{aligned}
$$

and since $\psi \in \Psi$, we deduce that

$$
p(y_{2n+1}, y_{2n}) \leq p(y_{2n+1}, y_{2n}), \qquad (12)
$$

which is a contradiction since $p(y_{2n+1}, y_{2n}) > 0$ (by Remark 2.14(ii)). So, we have $p(y_{2n+1}, y_{2n}) \leq p(y_{2n-1}, y_{2n})$, that is, $\{p(y_{2n+1}, y_{2n})\}$ is a non-increasing sequence of positive real numbers. Thus there exists $c \geq 0$ such that

$$
p(y_{2n+1}, y_{2n}) = c. \qquad (13)
$$

Suppose that $c > 0$. Taking the limit in (11) as $n \to \infty$ and using (13) and the properties of $\psi$, $\varphi$, we have

$$
\psi(c) \leq \psi(c) - \varphi(c) < \psi(c),
$$

which is a contradiction. Therefore,

$$
\lim_{n\to\infty} p(y_{2n+1}, y_{2n}) = 0,
$$

which implies

$$
\lim_{n\to\infty} p(y_{n+1}, y_n) = 0. \qquad (14)
$$

Now, we shall show that $\{y_{2n}\}$ is a Cauchy sequence in $(\mathcal{Y}, p)$. On the contrary, assume that $\{y_{2n}\}$ is not a Cauchy sequence in $(\mathcal{Y}, p)$, then by Lemma 2.10, there exists $\varepsilon > 0$ and two subsequences $\{y_{2m(k)}\}$ and $\{y_{2n(k)}\}$ of $\{y_{2n}\}$ with $n(k) > m(k) > k$ such that the sequences

$$
p(y_{2m(k)}, y_{2n(k)+1}),\ p(y_{2m(k)}, y_{2n(k)}),\ p(y_{2m(k)-1}, y_{2n(k)+1}),\ p(y_{2m(k)-1}, y_{2n(k)})
$$

tend to $\varepsilon > 0$ when $k \to \infty$.

Now, using the given contractive condition (4) for $y = y_{2m(k)}$ and $z = y_{2n(k)+1}$, we have

$$\begin{aligned} \psi\Big(p(y_{2m(k)}, y_{2n(k)+1})\Big) &= \psi\Big(p(\mathcal{G}_1 y_{2m(k)-1}, \mathcal{G}_2 y_{2n(k)}\Big) \\ &\leq F\Big(\psi\Big(\mathcal{M}_1^p(y_{2m(k)-1}, y_{2n(k)})\Big), \varphi\Big(\mathcal{M}_2^p(y_{2m(k)-1}, y_{2n(k)})\Big)\Big), \end{aligned} \tag{15}$$

where

$$\begin{aligned} \mathcal{M}_1^p(y_{2m(k)-1}, y_{2n(k)}) &= \max\Big\{p(y_{2m(k)-1}, y_{2n(k)}), p(y_{2m(k)-1}, \mathcal{G}_1 y_{2m(k)-1}), p(y_{2n(k)}, \mathcal{G}_2 y_{2n(k)}), \\ &\qquad p(y_{2n(k)}, \mathcal{G}_2 y_{2n(k)}) \frac{1 + p(y_{2m(k)-1}, \mathcal{G}_1 y_{2m(k)-1}}{1 + p(y_{2m(k)-1}, y_{2n(k)})}\Big\} \\ &= \max\Big\{p(y_{2m(k)-1}, y_{2n(k)}), p(y_{2m(k)-1}, y_{2m(k)}), p(y_{2n(k)}, y_{2n(k)+1}), \\ &\qquad p(y_{2n(k)}, y_{2n(k)+1}) \frac{1 + p(y_{2m(k)-1}, y_{2m(k)}}{1 + p(y_{2m(k)-1}, y_{2n(k)})}\Big\}. \end{aligned} \tag{16}$$

Taking the limit as $k \to \infty$ and using (P3) and (14) in (16), we get

$$\mathcal{M}_1^p(y_{2m(k)-1}, y_{2n(k)}) \to \max\{\varepsilon, 0, 0, \varepsilon\} = \varepsilon, \tag{17}$$

and

$$\begin{aligned} \mathcal{M}_2^p(y_{2m(k)-1}, y_{2n(k)}) &= \max\Big\{p(y_{2m(k)-1}, y_{2n(k)}), \\ &\qquad \frac{1}{2}[p(y_{2n(k)}, \mathcal{G}_1 y_{2m(k)-1}) + p(y_{2m(k)-1}, \mathcal{G}_2 y_{2n(k)})], \\ &\qquad \frac{p(y_{2m(k)-1}, \mathcal{G}_1 y_{2m(k)-1}) p(y_{2n(k)}, \mathcal{G}_2 y_{2n(k)})}{1 + p(y_{2m(k)-1}, y_{2n(k)})}, \\ &\qquad \frac{p(y_{2m(k)-1}, \mathcal{G}_1 y_{2m(k)-1}) p(y_{2n(k)}, \mathcal{G}_2 y_{2n(k)})}{1 + p(\mathcal{G}_1 y_{2m(k)-1}, \mathcal{G}_2 y_{2n(k)})}\Big\} \\ &= \max\Big\{p(y_{2m(k)-1}, y_{2n(k)}), \\ &\qquad \frac{1}{2}[p(y_{2n(k)}, y_{2m(k)}) + p(y_{2m(k)-1}, y_{2n(k)+1})], \\ &\qquad \frac{p(y_{2m(k)-1}, y_{2m(k)}) p(y_{2n(k)}, y_{2n(k)+1})}{1 + p(y_{2m(k)-1}, y_{2n(k)})}, \\ &\qquad \frac{p(y_{2m(k)-1}, y_{2m(k)}) p(y_{2n(k)}, y_{2n(k)+1})}{1 + p(y_{2m(k)}, y_{2n(k)+1})}\Big\}. \end{aligned} \tag{18}$$

Taking the limit as $k \to \infty$ and using (P3) and (14) in (18), we get

$$\mathcal{M}_2^p(y_{2m(k)-1}, y_{2n(k)}) \to \max\{\varepsilon, \varepsilon, 0, 0\} = \varepsilon. \tag{19}$$

Thus, by (15) for any $k \to \infty$ and using (17), (18), we have

$$\psi(\varepsilon) \leq F\Big(\psi(\varepsilon), \varphi(\varepsilon)\Big), \tag{20}$$

so, $\psi(\varepsilon) = 0$ or $\varphi(\varepsilon) = 0$, we deduce that $\varepsilon = 0$, which is a contradiction since $\varepsilon > 0$. Hence, we have

$$\lim_{n,m\to\infty} p(y_n, y_m) = 0. \tag{21}$$

Since $\lim_{n,m\to\infty} p(y_n, y_m)$ exists and is finite, we conclude that $\{y_n\}$ is a Cauchy sequence in $(\mathcal{Y}, p)$.

On the other hand from Remark 2.5(1), since

$$d_p(y_n, y_m) \leq 2p(y_n, y_m).$$

Therefore, taking the limit as $n, m \to \infty$ and using (21), we have

$$\lim_{n,m\to\infty} d_p(y_n, y_m) = 0. \tag{22}$$

This shows that $\{y_n\}$ is also a Cauchy sequence in the metric space $(\mathcal{Y}, d_p)$. Since $(\mathcal{Y}, p)$ is complete, then from Lemma 2.8, the sequence $\{y_n\}$ converges in the metric space $(\mathcal{Y}, d_p)$, say to a point $a \in \mathcal{Y}$ and $\lim_{n\to\infty} d_p(y_n, a) = 0$. Again from Lemma 2.8, we have

$$p(a, a) = \lim_{n\to\infty} p(y_n, a) = \lim_{n,m\to\infty} p(y_n, y_m). \tag{23}$$

Hence from (21) and (23), we get

$$p(a, a) = \lim_{n\to\infty} p(y_n, a) = \lim_{n,m\to\infty} p(y_n, y_m) = 0. \tag{24}$$

Now, we shall show that $a$ is a common fixed point of $\mathcal{G}_1$ and $\mathcal{G}_2$. Notice that due to (24), we have $p(a, a) = 0$. By (4) with $y = y_{2n}$ and $z = a$ and using (24), we have

$$\begin{aligned} \psi\Big(p(y_{2n+1}, \mathcal{G}_2 a)\Big) &= \psi\Big(p(\mathcal{G}_1 y_{2n}, \mathcal{G}_2 a)\Big) \\ &\le F\Big(\psi\Big(\mathcal{M}_1^p(y_{2n}, a)\Big), \phi\Big(\mathcal{M}_2^p(y_{2n}, a)\Big)\Big), \end{aligned} \tag{25}$$

where

$$\begin{aligned} \mathcal{M}_1^p(y_{2n}, a) &= \max\Big\{p(y_{2n}, a), p(y_{2n}, \mathcal{G}_1 y_{2n}), p(a, \mathcal{G}_2 a), p(a, \mathcal{G}_2 a)\frac{1 + p(y_{2n}, \mathcal{G}_1 y_{2n})}{1 + p(y_{2n}, a)}\Big\} \\ &= \max\Big\{p(y_{2n}, a), p(y_{2n}, y_{2n+1}), p(a, \mathcal{G}_2 a), p(a, \mathcal{G}_2 a)\frac{1 + p(y_{2n}, y_{2n+1})}{1 + p(y_{2n}, a)}\Big\}. \end{aligned}$$

Passing to the limit as $n \to \infty$ and using (24) in the above inequality, we obtain

$$\mathcal{M}_1^p(y_{2n}, a) \to \max\Big\{0, 0, p(a, \mathcal{G}_2 a), p(a, \mathcal{G}_2 a)\Big\} = p(a, \mathcal{G}_2 a), \tag{26}$$

and

$$\begin{aligned} \mathcal{M}_2^p(y_{2n}, a) &= \max\Big\{p(y_{2n}, a), \frac{1}{2}[p(a, \mathcal{G}_1 y_{2n}) + p(y_{2n}, \mathcal{G}_2 a)], \\ &\qquad \frac{p(y_{2n}, \mathcal{G}_1 y_{2n}) p(a, \mathcal{G}_2 a)}{1 + p(y_{2n}, a)}, \frac{p(y_{2n}, \mathcal{G}_1 y_{2n}) p(a, \mathcal{G}_2 a)}{1 + p(\mathcal{G}_1 y_{2n}, \mathcal{G}_2 a)}\Big\} \\ &= \max\Big\{p(y_{2n}, a), \frac{1}{2}[p(a, y_{2n+1}) + p(y_{2n}, \mathcal{G}_2 a)], \\ &\qquad \frac{p(y_{2n}, y_{2n+1}) p(a, \mathcal{G}_2 a)}{1 + p(y_{2n}, a)}, \frac{p(y_{2n}, y_{2n+1}) p(a, \mathcal{G}_2 a)}{1 + p(y_{2n+1}, \mathcal{G}_2 a)}\Big\}. \end{aligned}$$

Passing to the limit as $n \to \infty$ and using (24) in the above inequality, we obtain

$$\mathcal{M}_2^p(y_{2n}, a) \to \max\Big\{0, \frac{p(a, \mathcal{G}_2 a)}{2}, 0, 0\Big\} = \frac{p(a, \mathcal{G}_2 a)}{2} < p(a, \mathcal{G}_2 a). \tag{27}$$

Now, from (25), (26) and (27), we have

$$\psi\Big(p(y_{2n+1}, \mathcal{G}_2 a)\Big) \le F\Big(\psi\Big(p(a, \mathcal{G}_2 a)\Big), \varphi\Big(p(a, \mathcal{G}_2 a)\Big)\Big). \tag{28}$$

Passing to the limit as $n \to \infty$ in the above inequality and using the property of $\psi$, $\varphi$, we obtain

$$\psi\Big(p(a, \mathcal{G}_2 a)\Big) \le F\Big(\psi\Big(p(a, \mathcal{G}_2 a)\Big), \varphi\Big(p(a, \mathcal{G}_2 a)\Big)\Big). \tag{29}$$

So, $\psi\Big(p(a, \mathcal{G}_2 a)\Big) = 0$ or $\varphi\Big(p(a, \mathcal{G}_2 a)\Big) = 0$, which implies that $p(a, \mathcal{G}_2 a) = 0$, that is, $a = \mathcal{G}_2 a$. This shows that $a$ is a fixed point of $\mathcal{G}_2$. By similar fashion we can show that $a = \mathcal{G}_1 a$. Hence, $a$ is a common fixed point of $\mathcal{G}_1$ and $\mathcal{G}_2$.

Finally, we prove the uniqueness of common fixed point. Suppose $a'$ is another common fixed point of $\mathcal{G}_1$ and $\mathcal{G}_2$ such that $\mathcal{G}_1 a' = a' = \mathcal{G}_2 a'$ with $a \neq a'$. From (4) and (24), we have

$$\begin{aligned} \psi\big(p(a,a')\big) &= \psi\big(p(\mathcal{G}_1 a, \mathcal{G}_2 a')\big) \\ &\leq F\Big(\psi\big(\mathcal{M}_1^p(a,a')\big), \phi\big(\mathcal{M}_2^p(a,a')\big)\Big), \end{aligned} \tag{30}$$

where

$$\begin{aligned} \mathcal{M}_1^p(a,a') &= \max\Big\{p(a,a'), p(a,\mathcal{G}_1 a), p(a',\mathcal{G}_2 a'), p(a',\mathcal{G}_2 a')\frac{1+p(a,\mathcal{G}_1 a)}{1+p(a,a')}\Big\} \\ &= \max\Big\{p(a,a'), p(a,a), p(a',a'), p(a',a')\frac{1+p(a,a)}{1+p(a,a')}\Big\} \\ &= \max\Big\{p(a,a'), 0, 0, 0\Big\} = p(a,a'), \end{aligned} \tag{31}$$

and

$$\begin{aligned} \mathcal{M}_2^p(a,a') &= \max\Big\{p(a,a'), \frac{1}{2}[p(a',\mathcal{G}_1 a) + p(a,\mathcal{G}_2 a')], \\ &\qquad \frac{p(a,\mathcal{G}_1 a)p(a',\mathcal{G}_2 a')}{1+p(a,a')}, \frac{p(a,\mathcal{G}_1 a)p(a',\mathcal{G}_2 a')}{1+p(\mathcal{G}_1 a,\mathcal{G}_2 a')}\Big\} \\ &= \max\Big\{p(a,a'), \frac{1}{2}[p(a',a) + p(a,a')], \\ &\qquad \frac{p(a,a)p(a',a')}{1+p(a,a')}, \frac{p(a,a)p(a',a')}{1+p(a,a')}\Big\} \\ &= \max\Big\{p(a,a'), p(a,a'), 0, 0\Big\} = p(a,a'). \end{aligned} \tag{32}$$

Now, from equations (30)-(32) and using the property of $\psi$, $\varphi$, we obtain

$$\psi\big(p(a,a')\big) \leq F\Big(\psi\big(p(a,a')\big), \varphi\big(p(a,a')\big)\Big).$$

So, $\psi\big(p(a,a')\big) = 0$ or $\varphi\big(p(a,a')\big) = 0$, we deduce that $p(a,a') = 0$, that is, $a = a'$. This shows that the common fixed point of $\mathcal{G}_1$ and $\mathcal{G}_2$ is unique. This completes the proof. □

**Theorem 3.2.** *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two continuous self-maps on a complete partial metric space $(\mathcal{Y}, p)$ satisfying the condition:*

$$\psi\big(p(\mathcal{F}_1^m y, \mathcal{F}_2^n z)\big) \leq F\Big(\psi\big(\mathcal{N}_1^p(y,z)\big), \varphi\big(\mathcal{N}_2^p(y,z)\big)\Big), \tag{33}$$

*for all $y, z \in \mathcal{Y}$, where $m$ and $n$ are some positive integers,*

$$\begin{aligned} \mathcal{N}_1^p(y,z) &= \max\Big\{p(y,z), p(y,\mathcal{F}_1^m y), p(z,\mathcal{F}_2^n z), \\ &\qquad p(z,\mathcal{F}_2^n z)\frac{1+p(y,\mathcal{F}_1^m y)}{1+p(y,z)}\Big\}, \end{aligned} \tag{34}$$

*and*

$$\begin{aligned} \mathcal{N}_2^p(y,z) &= \max\Big\{p(y,z), \frac{1}{2}[p(z,\mathcal{F}_1^m y) + p(y,\mathcal{F}_2^n z)], \\ &\qquad \frac{p(y,\mathcal{F}_1^m y)p(z,\mathcal{F}_2^n z)}{1+p(y,z)}, \frac{p(y,\mathcal{F}_1^m y)p(z,\mathcal{F}_2^n z)}{1+p(\mathcal{F}_1^m y,\mathcal{F}_2^n z)}\Big\}, \end{aligned} \tag{35}$$

*for all $\psi \in \Psi$, $\varphi \in \Phi_u$ and $F \in \mathcal{C}$. Then $\mathcal{F}_1$ and $\mathcal{F}_2$ have a unique common fixed point in $\mathcal{Y}$.*

*Proof.* Since $\mathcal{F}_1^m$ and $\mathcal{F}_2^n$ satisfy the conditions of the Theorem 3.1. So $\mathcal{F}_1^m$ and $\mathcal{F}_2^n$ have a unique common fixed point. Let $b$ be the common fixed point. Then we have

$$\mathcal{F}_1^m b = b \Rightarrow \mathcal{F}_1(\mathcal{F}_1^m b) = \mathcal{F}_1 b$$
$$\Rightarrow \mathcal{F}_1^m(\mathcal{F}_1 b) = \mathcal{F}_1 b.$$

If $\mathcal{F}_1 b = b_0$, then $\mathcal{F}_1^m b_0 = b_0$. So $\mathcal{F}_1 b$ is a fixed point of $\mathcal{F}_1^m$. Similarly, $\mathcal{F}_2^n(\mathcal{F}_2 b) = \mathcal{F}_2 b$, that is, $\mathcal{F}_2 b$ is a fixed point of $\mathcal{F}_2^n$.

Now, using equations (33) and (24), we have

$$\begin{aligned}\psi\Big(p(b,\mathcal{F}_1 b)\Big) &= \psi\Big(p(\mathcal{F}_1^m b,\mathcal{F}_1^m(\mathcal{F}_1 b))\Big)\\ &\leq F\Big(\psi\Big(\mathcal{N}_1^p(b,\mathcal{F}_1 b)\Big),\varphi\Big(\mathcal{N}_2^p(b,\mathcal{F}_1 b)\Big)\Big),\end{aligned} \tag{36}$$

where

$$\begin{aligned}\mathcal{N}_1^p(b,\mathcal{F}_1 b) &= \max\Big\{p(b,\mathcal{F}_1 b), p(b,\mathcal{F}_1^m b), p(\mathcal{F}_1 b,\mathcal{F}_2^n(\mathcal{F}_1 b)), p(\mathcal{F}_1 b,\mathcal{F}_2^n(\mathcal{F}_1 b))\frac{1+p(b,\mathcal{F}_1^m b)}{1+p(b,\mathcal{F}_1 b)}\Big\}\\ &= \max\Big\{p(b,\mathcal{F}_1 b), p(b,b), p(\mathcal{F}_1 b,\mathcal{F}_1 b), p(\mathcal{F}_1 b,\mathcal{F}_1 b)\frac{1+p(b,b)}{1+p(b,b)}\Big\}\\ &= \max\Big\{p(b,\mathcal{F}_1 b),0,0,0\Big\} = p(b,\mathcal{F}_1 b),\end{aligned} \tag{37}$$

and

$$\begin{aligned}\mathcal{N}_1^p(b,\mathcal{F}_1 b) &= \max\Big\{p(b,\mathcal{F}_1 b), \frac{1}{2}[p(\mathcal{F}_1 b,\mathcal{F}_1^m b) + p(b,\mathcal{F}_2^n(\mathcal{F}_1 b))],\\ &\qquad \frac{p(b,\mathcal{F}_1^m b)p(\mathcal{F}_1 b,\mathcal{F}_2^n(\mathcal{F}_1 b))}{1+p(b,\mathcal{F}_1 b)}, \frac{p(b,\mathcal{F}_1^m b)p(\mathcal{F}_1 b,\mathcal{F}_2^n(\mathcal{F}_1 b))}{1+p(\mathcal{F}_1^m b,\mathcal{F}_2^n(\mathcal{F}_1 b))}\Big\}\\ &= \max\Big\{p(b,\mathcal{F}_1 b), \frac{1}{2}[p(\mathcal{F}_1 b,b) + p(b,\mathcal{F}_1 b)],\\ &\qquad \frac{p(b,b)p(\mathcal{F}_1 b,\mathcal{F}_1 b)}{1+p(b,\mathcal{F}_1 b)}, \frac{p(b,b)p(\mathcal{F}_1 b,\mathcal{F}_1 b)}{1+p(b,\mathcal{F}_1 b)}\Big\}\\ &= \max\Big\{p(b,\mathcal{F}_1 b), p(b,\mathcal{F}_1 b),0,0\Big\} = p(b,\mathcal{F}_1 b). \text{ (using (P3))}\end{aligned} \tag{38}$$

From equations (36)-(38) and using the property of $\psi$, $\phi$, we obtain

$$\psi\Big(p(b,\mathcal{F}_1 b)\Big) \leq F\Big(\psi\Big(p(b,\mathcal{F}_1 b)\Big),\varphi\Big(p(b,\mathcal{F}_1 b)\Big)\Big).$$

So, $\psi\Big(p(b,\mathcal{F}_1 b)\Big) = 0$ or $\varphi\Big(p(b,\mathcal{F}_1 b)\Big) = 0$, hence we deduce that $p(b,\mathcal{F}_1 b) = 0$, that is, $b = \mathcal{F}_1 b$ for all $b \in \mathcal{Y}$. Similarly, we can show that $b = \mathcal{F}_2 b$. This shows that $b$ is a common fixed point of $\mathcal{F}_1$ and $\mathcal{F}_2$. For the uniqueness of $b$, let $b' \neq b$ be another common fixed point of $\mathcal{F}_1$ and $\mathcal{F}_2$. Then clearly $b'$ is also a common fixed point of $\mathcal{F}_1^m$ and $\mathcal{F}_2^n$ which implies $b = b'$. Thus $\mathcal{F}_1$ and $\mathcal{F}_2$ have a unique common fixed point in $\mathcal{Y}$. This completes the proof. □

If we take $\mathcal{G}_1 = \mathcal{G}_2 = \mathcal{T}$ in Theorem 3.1, then we have the following result as corollaries.

**Corollary 3.3.** *Let $\mathcal{T}$ be a self-map on a complete partial metric space $(\mathcal{Y}, p)$ satisfying the condition:*

$$\psi\Big(p(\mathcal{T}y,\mathcal{T}z)\Big) \leq F\Big(\psi\Big(\mathcal{M}_a^p(y,z)\Big),\varphi\Big(\mathcal{M}_b^p(y,z)\Big)\Big),$$

*for all $y,z \in \mathcal{Y}$, where*

$$\mathcal{M}_a^p(y,z) = \max\Big\{p(y,z), p(y,\mathcal{T}y), p(z,\mathcal{T}z), p(z,\mathcal{T}z)\frac{1+p(y,\mathcal{T}y)}{1+p(y,z)}\Big\},$$

*and*

$$\mathcal{M}_b^p(y,z) = \max\Big\{p(y,z), \frac{1}{2}[p(z,\mathcal{T}y)+p(y,\mathcal{T}z)], \frac{p(y,\mathcal{T}y)p(z,\mathcal{T}z)}{1+p(y,z)}, \frac{p(y,\mathcal{T}y)p(z,\mathcal{T}z)}{1+p(\mathcal{T}y,\mathcal{T}z)}\Big\},$$

*for all $\psi \in \Psi$, $\varphi \in \Phi_u$ and $F \in \mathcal{C}$. Then $\mathcal{T}$ has a unique fixed point in $\mathcal{Y}$.*

**Corollary 3.4.** *Let $\mathcal{T}$ be a self-map on a complete partial metric space $(\mathcal{Y}, p)$ satisfying the condition:*

$$\psi\big(p(\mathcal{T}y,\mathcal{T}z)\big) \le F\Big(\psi\big(\mathcal{M}_a^p(y,z)\big), \varphi\big(\mathcal{M}_a^p(y,z)\big)\Big),$$

*for all $y, z \in \mathcal{Y}$, where $\mathcal{M}_a^p(y,z)$, $\psi$, $\varphi$ and $F$ are as in Corollary 3.3. Then $\mathcal{T}$ has a unique fixed point in $\mathcal{Y}$.*

**Corollary 3.5.** *Let $\mathcal{T}$ be a self-map on a complete partial metric space $(\mathcal{Y}, p)$ satisfying the condition:*

$$\psi\big(p(\mathcal{T}y,\mathcal{T}z)\big) \le F\Big(\psi\big(\mathcal{M}_b^p(y,z)\big), \varphi\big(\mathcal{M}_b^p(y,z)\big)\Big),$$

*for all $y, z \in \mathcal{Y}$, where $\mathcal{M}_b^p(y,z)$, $\psi$, $\varphi$ and $F$ are as in Corollary 3.3. Then $\mathcal{T}$ has a unique fixed point in $\mathcal{Y}$.*

If we take $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{S}$ in Theorem 3.2, then we have the following result as corollaries.

**Corollary 3.6.** *Let $\mathcal{S}$ be a continuous self-map on a complete partial metric space $(\mathcal{Y}, p)$ satisfying the condition:*

$$\psi\big(p(\mathcal{S}^m y,\mathcal{S}^n z)\big) \le F\Big(\psi\big(\mathcal{N}_a^p(y,z)\big), \varphi\big(\mathcal{N}_b^p(y,z)\big)\Big),$$

*for all $y, z \in \mathcal{Y}$, where $m$ and $n$ are some positive integers,*

$$\mathcal{N}_a^p(y,z) = \max\Big\{p(y,z), p(y,\mathcal{S}^m y), p(z,\mathcal{S}^n z), p(z,\mathcal{S}^n z)\frac{1+p(y,\mathcal{S}^m y)}{1+p(y,z)}\Big\},$$

*and*

$$\mathcal{N}_b^p(y,z) = \max\Big\{p(y,z), \frac{1}{2}[p(z,\mathcal{S}^m y)+p(y,\mathcal{S}^n z)], \frac{p(y,\mathcal{S}^m y)p(z,\mathcal{S}^n z)}{1+p(y,z)}, \frac{p(y,\mathcal{S}^m y)p(z,\mathcal{S}^n z)}{1+p(\mathcal{S}^m y,\mathcal{S}^n z)}\Big\},$$

*for all $\psi \in \Psi$, $\varphi \in \Phi_u$ and $F \in \mathcal{C}$. Then $\mathcal{S}$ has a unique fixed point in $\mathcal{Y}$.*

**Corollary 3.7.** *Let $\mathcal{S}$ be a continuous self-map on a complete partial metric space $(\mathcal{Y}, p)$ satisfying the condition:*

$$\psi\big(p(\mathcal{S}^m y,\mathcal{S}^n z)\big) \le F\Big(\psi\big(\mathcal{N}_a^p(y,z)\big), \varphi\big(\mathcal{N}_a^p(y,z)\big)\Big),$$

*for all $y, z \in \mathcal{Y}$, where $m$ and $n$ are some positive integers and $\mathcal{N}_a^p(y,z)$, $\psi$, $\varphi$, $F$ are as in Corollary 3.6. Then $\mathcal{S}$ has a unique fixed point in $\mathcal{Y}$.*

**Corollary 3.8.** *Let $\mathcal{S}$ be a continuous self-map on a complete partial metric space $(\mathcal{Y}, p)$ satisfying the condition:*

$$\psi\big(p(\mathcal{S}^m y,\mathcal{S}^n z)\big) \le F\Big(\psi\big(\mathcal{N}_b^p(y,z)\big), \varphi\big(\mathcal{N}_b^p(y,z)\big)\Big),$$

*for all $y, z \in \mathcal{Y}$, where $m$ and $n$ are some positive integers and $\mathcal{N}_b^p(y,z)$, $\psi$, $\varphi$, $F$ are as in Corollary 3.6. Then $\mathcal{S}$ has a unique fixed point in $\mathcal{Y}$.*

Other consequences of Theorem 3.1 and 3.2 are as follows.

If we take $F(s,t) = ks$ where $0 < k < 1$ and $\psi(t) = t$ for all $t \ge 0$ in Theorem 3.1 and 3.2, then we obtain the following results.

**Corollary 3.9.** *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two self-maps on a complete partial metric space $(\mathcal{Y}, p)$ satisfying the condition:*

$$p(\mathcal{G}_1 y, \mathcal{G}_2 z) \leq k\, \mathcal{M}_1^p(y,z),$$

*for all $y, z \in \mathcal{Y}$, where $0 < k < 1$ is a constant and $\mathcal{M}_1^p(y,z)$ is as in Theorem 3.1. Then $\mathcal{G}_1$ and $\mathcal{G}_2$ have a unique common fixed point in $\mathcal{Y}$.*

**Corollary 3.10.** *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two continuous self-maps on a complete partial metric space $(\mathcal{Y}, p)$ satisfying the condition:*

$$p(\mathcal{F}_1^m y, \mathcal{F}_2^n z) \leq k\, \mathcal{N}_1^p(y,z),$$

*for all $y, z \in \mathcal{Y}$, where $0 < k < 1$ is a constant, $m$ and $n$ are some positive integers and $\mathcal{N}_1^p(y,z)$ is as in Theorem 3.2. Then $\mathcal{F}_1$ and $\mathcal{F}_2$ have a unique common fixed point in $\mathcal{Y}$.*

If we take $F(s,t) = s - t$ in Theorem 3.1 and 3.2, then we obtain the following results.

**Corollary 3.11.** *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two self-maps on a complete partial metric space $(\mathcal{Y}, p)$ satisfying the condition:*

$$\psi\Big(p(\mathcal{G}_1 y, \mathcal{G}_2 z)\Big) \leq \psi\Big(\mathcal{M}_1^p(y,z)\Big) - \varphi\Big(\mathcal{M}_2^p(y,z)\Big),$$

*for all $y, z \in \mathcal{Y}$, where $\mathcal{M}_1^p(y,z)$, $\mathcal{M}_2^p(y,z)$, $\psi$ and $\varphi$ are as in Theorem 3.1. Then $\mathcal{G}_1$ and $\mathcal{G}_2$ have a unique common fixed point in $\mathcal{Y}$.*

**Corollary 3.12.** *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two continuous self-maps on a complete partial metric space $(\mathcal{Y}, p)$ satisfying the condition:*

$$\psi\Big(p(\mathcal{F}_1^m y, \mathcal{F}_2^n z)\Big) \leq \psi\Big(\mathcal{N}_1^p(y,z)\Big) - \varphi\Big(\mathcal{N}_2^p(y,z)\Big),$$

*for all $y, z \in \mathcal{Y}$, where $m$ and $n$ are some positive integers and $\mathcal{N}_1^p(y,z)$, $\mathcal{N}_2^p(y,z)$, $\psi$, $\varphi$ are as in Theorem 3.2. Then $\mathcal{F}_1$ and $\mathcal{F}_2$ have a unique common fixed point in $\mathcal{Y}$.*

If we take $F(s,t) = m\,s$, $\psi(t) = t$ for all $t \geq 0$ and $\mathcal{M}_a^p(y,z) = p(y,z)$ in Corollary 3.3, then we have the following result.

**Corollary 3.13.** *([22]) Let $\mathcal{T}$ be a self-map on a complete partial metric space $(\mathcal{Y}, p)$ satisfying the condition:*

$$p(\mathcal{T} y, \mathcal{T} z) \leq m\, p(y,z),$$

*for all $y, z \in \mathcal{Y}$, where $m \in [0,1)$ is a constant. Then $\mathcal{T}$ has a unique fixed point in $\mathcal{Y}$.*

If we take $F(s,t) = s - t$ in Corollary 3.3, then we have the following result.

**Corollary 3.14.** *Let $\mathcal{T}$ be a self-map on a complete partial metric space $(\mathcal{Y}, p)$ satisfying the condition:*

$$\psi\Big(p(\mathcal{T} y, \mathcal{T} z)\Big) \leq \psi\Big(\mathcal{M}_a^p(y,z)\Big) - \varphi\Big(\mathcal{M}_b^p(y,z)\Big),$$

*for all $y, z \in \mathcal{Y}$, where $\mathcal{M}_a^p(y,z)$, $\mathcal{M}_b^p(y,z)$, $\psi$ and $\varphi$ are as in Corollary 3.3. Then $\mathcal{T}$ has a unique fixed point in $\mathcal{Y}$.*

Now, we give some examples in support of the results.

**Example 3.15.** *Let $\mathcal{Y} = \{1,2,3,4\}$ and $p \colon \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}$ be defined by*

$$p(y,z) = \begin{cases} |y - z| + \max\{y,z\}, & \text{if } y \neq z, \\ y, & \text{if } y = z \neq 1, \\ 0, & \text{if } y = z = 1, \end{cases}$$

*for all $y, z \in \mathcal{Y}$. Then $(\mathcal{Y}, p)$ is a complete partial metric space.*

*Define the mapping $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ by*

$$\mathcal{T}(1) = 1,\ \mathcal{T}(2) = 1,\ \mathcal{T}(3) = 2,\ \mathcal{T}(4) = 2.$$

*Now, we have*

$$p(\mathcal{T}(1), \mathcal{T}(2)) = p(1,1) = 0 \le \frac{3}{4}.3 = \frac{3}{4}p(1,2),$$

$$p(\mathcal{T}(1), \mathcal{T}(3)) = p(1,2) = 3 \le \frac{3}{4}.5 = \frac{3}{4}p(1,3),$$

$$p(\mathcal{T}(1), \mathcal{T}(4)) = p(1,2) = 3 \le \frac{3}{4}.7 = \frac{3}{4}p(1,4),$$

$$p(\mathcal{T}(2), \mathcal{T}(3)) = p(1,2) = 3 \le \frac{3}{4}.4 = \frac{3}{4}p(2,3),$$

$$p(\mathcal{T}(2), \mathcal{T}(4)) = p(1,2) = 3 \le \frac{3}{4}.6 = \frac{3}{4}p(2,4),$$

$$p(\mathcal{T}(3), \mathcal{T}(4)) = p(2,2) = 2 \le \frac{3}{4}.5 = \frac{3}{4}p(3,4).$$

*Thus, $\mathcal{T}$ satisfies all the conditions of Corollary 3.13 with $m = \frac{3}{4} < 1$. Now by applying Corollary 3.13, $\mathcal{T}$ has a unique fixed point. Indeed 1 is the required unique fixed point in this case.*

**Example 3.16.** *Let $\mathcal{Y} = [0, \infty)$ and $p \colon \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}$ be defined by $p(y, z) = \max\{y, z\}$. Then $(\mathcal{Y}, p)$ is a complete partial metric space. Consider the mappings $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ defined by*

$$\mathcal{T}(y) = \begin{cases} 0, & \text{if } 0 \le y < 1, \\ \frac{y^2}{1+y}, & \text{if } y \ge 1, \end{cases}$$

*and $\psi, \varphi \colon [0, \infty) \to [0, \infty)$ are defined by $\psi(t) = t$ and $\varphi(t) = \frac{t}{1+t}$.*

*We have the following cases.*

**Case (i)** *If $y, z \in [0, 1)$ and assume that $y \ge z$, then we have*

$$p(\mathcal{T}(y), \mathcal{T}(z)) = 0,$$

*and*

$$\begin{aligned} \mathcal{M}_a^p(y,z) &= \max\Big\{p(y,z), p(y,\mathcal{T}y), p(z,\mathcal{T}z), p(z,\mathcal{T}z)\frac{1 + p(y,\mathcal{T}y)}{1 + p(y,z)}\Big\} \\ &= \max\Big\{y, y, z, \frac{z(1+y)}{1+y}\Big\} = \max\{y, z\} = y. \end{aligned}$$

*On the similar fashion*

$$\mathcal{M}_b^p(y,z) = y.$$

*Therefore,*

$$\psi\Big(p(\mathcal{T}(y), \mathcal{T}(z))\Big) = 0, \tag{39}$$

*and*

$$\psi\Big(\mathcal{M}_a^p(y,z)\Big) - \varphi\Big(\mathcal{M}_b^p(y,z)\Big) = y - \frac{y}{1+y} = \frac{y^2}{1+y}. \tag{40}$$

*From equations (39) and (40), we have*

$$\psi\Big(p(\mathcal{T}(y),\mathcal{T}(z))\Big) \leq \psi\Big(\mathcal{M}_a^p(y,z)\Big) - \varphi\Big(\mathcal{M}_b^p(y,z)\Big).$$

**Case (ii)** *If* $z \in [0,1)$*,* $y \geq 1$ *and assume that* $y \geq z$*, then we have*

$$p(\mathcal{T}(y),\mathcal{T}(z)) = \max\Big\{\frac{y^2}{1+y}, 0\Big\} = \frac{y^2}{1+y},$$

*and*

$$\begin{aligned}\mathcal{M}_a^p(y,z) &= \max\Big\{p(y,z), p(y,\mathcal{T}y), p(z,\mathcal{T}z), p(z,\mathcal{T}z)\frac{1+p(y,\mathcal{T}y)}{1+p(y,z)}\Big\} \\ &= \max\Big\{y, y, z, \frac{z(1+y)}{1+y}\Big\} = \max\{y,z\} = y.\end{aligned}$$

*On the similar fashion*

$$\mathcal{M}_b^p(y,z) = y.$$

*Therefore,*

$$\psi\Big(p(\mathcal{T}(y),\mathcal{T}(z))\Big) = \frac{y^2}{1+y}, \tag{41}$$

*and*

$$\psi\Big(\mathcal{M}_a^p(y,z)\Big) - \varphi\Big(\mathcal{M}_b^p(y,z)\Big) = y - \frac{y}{1+y} = \frac{y^2}{1+y}. \tag{42}$$

*From equations (41) and (42), we have*

$$\psi\Big(p(\mathcal{T}(y),\mathcal{T}(z))\Big) = \psi\Big(\mathcal{M}_a^p(y,z)\Big) - \varphi\Big(\mathcal{M}_b^p(y,z)\Big).$$

**Case (iii)** *If* $y \geq z \geq 1$ *and assume that* $y \geq z$*, then we have*

$$p(\mathcal{T}(y),\mathcal{T}(z)) = \max\Big\{\frac{y^2}{1+y}, \frac{z^2}{1+z}\Big\} = \frac{y^2}{1+y},$$

*and*

$$\begin{aligned}\mathcal{M}_a^p(y,z) &= \max\Big\{p(y,z), p(y,\mathcal{T}y), p(z,\mathcal{T}z), p(z,\mathcal{T}z)\frac{1+p(y,\mathcal{T}y)}{1+p(y,z)}\Big\} \\ &= \max\Big\{y, y, z, \frac{z(1+y)}{1+y}\Big\} = \max\{y,z\} = y.\end{aligned}$$

*On the similar fashion*

$$\mathcal{M}_b^p(y,z) = y.$$

*Therefore,*

$$\psi\Big(p(\mathcal{T}(y),\mathcal{T}(z))\Big) = \frac{y^2}{1+y}, \tag{43}$$

*and*

$$\psi\Big(\mathcal{M}_a^p(y,z)\Big) - \varphi\Big(\mathcal{M}_b^p(y,z)\Big) = y - \frac{y}{1+y} = \frac{y^2}{1+y}. \tag{44}$$

*From equations (43) and (44), we have*

$$\psi\Big(p(\mathcal{T}(y),\mathcal{T}(z))\Big) = \psi\Big(\mathcal{M}_a^p(y,z)\Big) - \varphi\Big(\mathcal{M}_b^p(y,z)\Big).$$

*Thus, in all the above cases* $\mathcal{T}$ *satisfies all the conditions of Corollary 3.14. Hence* $\mathcal{T}$ *has a unique fixed point in* $\mathcal{Y}$*, indeed,* $y = 0$ *is the required point.*

## 4. Conclusion

In this paper, we establish some common fixed point theorems in the set up of partial metric spaces with the help of $C$-class function and some auxiliary functions and give some consequences of the established results. We also give some examples in support of the results. The presented results in this paper extend and generalize several results from the existing literature regarding partial metric spaces and other supported functions.

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