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On the existence and uniqueness of solutions of certain classes of abstract multi-term fractional differential equations

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Abstract. In this paper, we investigate the existence and uniqueness of solutions for the following abstract multi-term fractional differential equation:

$$\begin{aligned} &\mathbf{D}_t^{\alpha_n} u(t) + \sum_{i=1}^{n-1} A_i \mathbf{D}_t^{\alpha_i} u(t) = 0, \quad t > 0, \\ &u^{(k)}(0) = u_k, \quad k = 0, \cdots, \lceil \alpha_n \rceil - 1, \end{aligned} \tag{1}$$

where $n \in \mathbb{N} \setminus \{1, 2\}$, $A_1, \cdots, A_{n-1}$ are closed linear operators on a sequentially complete locally convex space X, $0 = \alpha_1 < \cdots < \alpha_n$, and $\mathbf{D}_t^{\alpha}$ denotes the Caputo fractional derivative of order α ([3]). Plenty of various examples illustrates our abstract theoretical results obtained throughout the paper.

1. Introduction and preliminaries

Fractional differential equations and fractional calculus have been attracted the attention of many authors during past three decades or so, primarily from their invaluable importance in modeling of various phenomena appearing in physics, chemistry, mathematical biology and engineering. For more details on these topics, the reader may consult the monographs by D. Baleanu, K. Diethelm, E. Scalas, J. Trujillo [2], K. Diethelm [6], K. S. Miller-B. Ross [26], I. Podlubny [29] and S. G. Samko, A. A. Kilbas, O. I. Marichev [31].

Abstract multi-term fractional differential equations have become a very active field of research (cf. [8], [14]-[15], [17] and [33]-[34] for the basic information in this direction). Without going into full details, we want to observe here that our results can be applied in the analysis of a great number of fractional PDEs describing certain real physical phenomena; for example, in the analysis of various generalizations of fractional telegraph equation, and in the analysis of the so called composite fractional relaxation-oscillation equation

$$\mathbf{D}_t^{\alpha} u(t) + B\mathbf{D}_t^{\beta} u(t) + Au(t) = f(t), \quad t \in [0, 2\pi], \tag{2}$$

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where A and B are closed linear operators defined on a complex Banach space X, $0 \leq \beta < \alpha \leq 2$ and $f \in C([0, 2\pi] : X)$ (cf. the reference [7] by R. Gorenflo and F. Mainardi for a detailed explanation of physical meaning of fractional differential equations that are special cases of (2), and the reference [8] by V. Keyantuo and C. Lizama for more details on periodic solutions of (2), in this work the authors have used the so-called Liouville-Grünwald fractional derivatives instead of Caputo's ones).

Let us also quote some other special cases of (1). The study of qualitative properties of the abstract Basset-Boussinesq-Oseen equation

$$u'(t) - A\mathbf{D}_t^{\alpha} u(t) + u(t) = f(t),\ t \geq 0,\ u(0) = 0;\ \alpha \in (0, 1), \tag{3}$$

describing the unsteady motion of a particle accelerating in a viscous fluid under the action of the gravity, has been initiated by C. Lizama and H. Prado in [24]. In 1991, S. Westerlund suggested using fractional derivatives for the description of propagation of plane electromagnetic waves in an isotropic and homogeneous material, lossy dielectric. In the abstract form, the equation suggested by S. Westerlund takes the following form (cf. [29, (10.107)]):

$$u''(t) + cA\mathbf{D}_t^{\alpha} u(t) + u(t) = f(t), \quad t \geq 0;\ u(0) = x,\ u'(0) = y, \tag{4}$$

where $c \in \mathbb{R}$, $A = \Delta$ and $1 < \alpha < 2$. Notice that Theorem 2.4 applies to various generalizations of problems (3)-(4).

In [20]-[23], T. A. M. Langlands, B. I. Henry and S. L. Wearne have considered various types of fractional cable equation models describing electrodiffusion of ions in neurons for the case of anomalous subdiffusion. Notice that in some of these models abstract multi-term fractional equations with the Riemann-Liouville fractional derivatives have occurred, and that it is not clear whether these derivatives can be replaced by Caputo fractional derivatives or some combination of Caputo and Riemann-Liouville fractional derivatives, without losing some physical meaning (see e.g. the problems [20, (1.18)], [21, (22), (25)] and [23, (69)]). In this place, we wish to point out that we could not find in the existing literature any reference which treats the abstract multi-term fractional differential equations with fractional derivatives that are not of Caputo's type.

I. Podlubny [29] and K. Diethelm [6, Chapter 8] have analyzed scalar-valued multi-term Caputo fractional differential equations. We know that there will be one and only one solution $u(t)$ of the equation (1) with $A \equiv 0$, $f(t) \equiv 0$ and $A_j = c_j I$ ($c_j \in \mathbb{C}$, $j \in \mathbb{N}_{n-1}$) and that the solution $u(t)$ can be expressed in terms of the Mittag-Leffler functions and their derivatives (see [29] and [15, Example 8.1]).

The organization and main ideas of this paper can be briefly described as follows. The main purpose of the paper, as already mentioned in the abstract, is to study some existence and uniqueness theorems for the equation (1). Although we formulate our results in the setting of sequentially complete locally convex spaces, they seem to be new even in the case of abstract multi-term fractional differential equations considered in Banach spaces; it is worth noticing here that several problems occuring in the theory of abstract Volterra integro-differential equations can be analysed more effectively on locally convex spaces as on Banach spaces; see e.g. [19] for the analysis of a control problem for a one-dimensional heat equation for materials with memory (cf. [30, pp. 146-147]), which is closely connected with the problem of gluing in manufacturing polymeric materials. We continue by recalling the well-known fact from the theory of higher order abstract differential equations that the operator $-A_{n-1}$ plays a crucial role for the solvability of equation (1), and that the operators $-A_{n-2}, \cdots, -A_1$ are subordinated to $-A_{n-1}$ in some sense. F. Neubrander [27] was the first who investigated the well-posedness of problem (1) in the case that $\alpha_n = n-1$, $n \in \mathbb{N} \setminus \{1, 2\}$ and that $-A_{n-1}$ is the integral generator of a strongly continuous semigroup on a Banach space X. Concerning equations with integer order derivatives, the further contributions have been obtained by R. deLaubenfels [5] and T.-J. Xiao-J. Liang [35] (cf. also [28], [32] and [36]-[38] for more details on the subject), where the authors have analyzed the well-posedness of problem (1) in the case that there exists a number $r \in \mathbb{N}_0$ such that the operator $-A_{n-1}$ is the integral generator of an exponentially bounded r-times integrated semigroup on X. The genesis of this paper is based on the fact that the methods developed in [35] cannot be so simply modified to cover the case in which r is a non-integer number. We overcome the problem mentioned above by using an additional assumption that the operator $-A_{n-1}$ is the integral generator of an exponentially bounded C-regularized semigroup on X, for a suitable chosen injective operator $C \in L(X)$ that is practically always different from the operator $(\mu_0 - A_{n-1})^{-r}$, for some $\mu_0 \in \rho(A_{n-1})$; furthermore, we consider the case in which there exist numbers $\sigma \in [1,2]$, $r \geq 0$ and $\omega \geq 0$ such that $(\omega^\sigma, \infty) \subseteq \rho(-A_{n-1})$ and that the operator $-A_{n-1}$ generates an exponentially equicontinuous $(g_\sigma, g_{\sigma r+1})$-regularized resolvent family or an exponentially equicontinuous (g_σ, C)-regularized resolvent family (cf. Theorem 2.1). Concerning the abstract Cauchy problem $(ACP)_n$, such reasoning produces, on the concrete level, significant improvements of regularity properties of the initial data which guarantee the existence and uniqueness of solutions; cf. Example 2.3(i). On the other hand, we feel in duty bound to say that Theorem 2.1 has several disadvantages in the case that the equation under its consideration contains more than two dominating terms, the restriction in applications to real problems comes from the fact that fractional derivatives of order > 2 are allowed. This is, certainly, not the case with Theorem 2.4, whose main purpose is to transfer the assertions of [36, Theorem $(*)$] and [35, Theorem 3.4.2] to abstract multi-term fractional differential equations. The formulations of our main results, Theorem 2.1 and Theorem 2.4, are very clear and concise. Finally, we would like to observe that there exists a very large class of important multi-term problems, like the fractional analog of damped Klein-Gordon equation [14, (5.14)] or problem

$$\mathbf{D}_t^{4/3}u(t) + \big(I - \Delta\big)^{1/2}\mathbf{D}_t^{1/2}u(t) + (I - \Delta)u(t) = 0,$$

to which both Theorem 2.1 and Theorem 2.4 cannot be applied.

We use the standard notation throughout the paper. A Hausdorff sequentially complete locally convex space over the field of complex numbers, SCLCS for short, will be denoted by X. The abbreviations $\circledast$ and $L(X)$ stand for the fundamental system of seminorms which defines the topology of X, and the space of all continuous linear mappings from X into X, respectively. Let $\mathcal{B}$ be the family of bounded subsets of X and let $p_B(T) := \sup_{x \in B} p(Tx)$, $p \in \circledast$, $B \in \mathcal{B}$, $T \in L(X)$. Then $p_B(\cdot)$ is a seminorm on $L(X)$ and the system $(p_B)_{(p,B) \in \circledast \times \mathcal{B}}$ induces the Hausdorff locally convex topology on $L(X)$. Henceforth $C \in L(X)$ is an injective operator, and the convolution like mapping $*$ is given by $f * g(t) := \int_0^t f(t-s)g(s)\,ds$. The domain, resolvent set and range of a closed linear operator A acting on X are denoted by $D(A)$, $\rho(A)$ and $R(A)$, respectively. In the case that X is a Banach space, then we denote by $\|\cdot\|$ the norm on X. Recall that the C-resolvent set of A, denoted by $\rho_C(A)$, is defined by $\rho_C(A) := \{\lambda \in \mathbb{C} \ ; \ \lambda - A \text{ is injective and } (\lambda - A)^{-1}C \in L(X)\}$. By I we denote the identity operator on X.

Given $s \in \mathbb{R}$ in advance, set $\lfloor s \rfloor := \sup\{l \in \mathbb{Z} : s \geq l\}$ and $\lceil s \rceil := \inf\{l \in \mathbb{Z} : s \leq l\}$. The Gamma function is denoted by $\Gamma(\cdot)$ and the principal branch is always used to take the powers. Set $\mathbb{C}_+ := \{z \in \mathbb{C} : \Re z > 0\}$, $\mathbb{N}_l := \{1, \cdot\cdot\cdot, l\}$, $\mathbb{N}_l^0 := \{0, 1, \cdot\cdot\cdot, l\}$, $0^\zeta := 0$, $g_\zeta(t) := t^{\zeta-1}/\Gamma(\zeta)$ $(\zeta > 0,\ t > 0)$ and $g_0 :=$ the Dirac δ-distribution; the symbol δ_{kl} denotes the Kronecker delta. If $\omega \geq 0$, then we say that a function $f : (\omega, \infty) \to X$ belongs to the class $LT - X$, if there exists a function $h(\cdot) \in C([0,\infty) : X)$ such that, for every $p \in \circledast$, there exists $M_p > 0$ satisfying $p(h(t)) \leq M_p e^{\omega t}$, $t \geq 0$ and $f(t) = \int_0^\infty e^{-\lambda t} h(t)\,dt$, $\lambda > \omega$. In the sequel, we shall always assume that $A_1, \cdot\cdot\cdot, A_{n-1}$ are closed linear operators on X as well as that $0 = \alpha_1 < \cdots < \alpha_n$; notice that the assumption $\alpha_1 = 0$ is not restrictive since we can always add the additional term $A_0 u(t) \equiv 0u(t)$ on the left hand side of (1). Set $m_j := \lceil \alpha_j \rceil$, $1 \leq j \leq n$, $D_i := \{j \in \mathbb{N}_{n-1} : m_j - 1 \geq i\}$ $(i \in \mathbb{N}^0_{m_n - 1})$, and

$$P_\lambda := \lambda^{\alpha_n} + \sum_{j=1}^{n-1} \lambda^{\alpha_j} A_j, \quad \lambda \in \mathbb{C} \setminus \{0\}.$$

If $\alpha > 0$ and $\beta > 0$, then we define the Mittag-Leffler function $E_{\alpha,\beta}(z)$ by $E_{\alpha,\beta}(z) := \sum_{n=0}^\infty z^n/\Gamma(\alpha n + \beta)$, $z \in \mathbb{C}$; set, for short, $E_\alpha(z) := E_{\alpha,1}(z)$, $z \in \mathbb{C}$ (cf. [3, Section 1.3] for more details about the Mittag-Leffler functions).

We need the following definition from [11].

Definition 1.1. *Let $k \in C([0,\infty))$, $k \neq 0$, and let $a \in L^1_{loc}([0,\infty))$, $a \neq 0$. A strongly continuous operator family $(R(t))_{t \geq 0} \subseteq L(X)$ is called an (a,k)-regularized C-resolvent family having A as a subgenerator iff the following holds:*

(i) $R(t)A \subseteq AR(t),\ t \geq 0,\ R(0) = k(0)C$ and $CA \subseteq AC$,

(ii) $R(t)C = CR(t),\ t \geq 0$ *and*

(iii) $R(t)x = k(t)Cx + \int_0^t a(t-s)AR(s)x\,ds,\ t \geq 0,\ x \in D(A).$

$(R(t))_{t\geq 0}$ *is said to be non-degenerate if the condition* $R(t)x = 0,\ t \geq 0$ *implies* $x = 0$, *and* $(R(t))_{t\geq 0}$ *is said to be exponentially equicontinuous (equicontinuous) if there exists* $\omega \in \mathbb{R}$ *(*$\omega = 0$*) such that the family* $\{e^{-\omega t}R(t) : t \geq 0\}$ *is equicontinuous.*

In the case that $k(t) \equiv 1$, then it is also said that $(R(t))_{t\geq 0}$ is an (a, C)-regularized resolvent family with subgenerator A. Henceforth, any considered operator family will be non-degenerate, and the functions $a(t)$, $k(t)$ will be scalar-valued kernels. Then we are in a position to define the integral generator $\hat{A}$ of $(R(t))_{t\geq 0}$ by setting

$$\hat{A} := \left\{ (x, y) \in X \times X : R(t)x - k(t)Cx = \int_0^t a(t-s)R(s)y\,ds \text{ for all } t \geq 0 \right\}.$$

The integral generator $\hat{A}$ of $(R(t))_{t\geq 0}$ is a linear operator on X which extends any subgenerator of $(R(t))_{t\geq 0}$ and satisfies $C^{-1}\hat{A}C = \hat{A}$. The exponential equicontinuity of $(R(t))_{t\geq 0}$ guarantees that $\hat{A}$ is a closed linear operator on X; if, additionally,

$$A \int_0^t a(t-s)R(s)x\,ds = R(t)x - k(t)Cx,\ t \geq 0,\ x \in X, \tag{5}$$

then $R(t)R(s) = R(s)R(t),\ t,\ s \geq 0$, $\hat{A}$ itself is a subgenerator of $(R(t))_{t\geq 0}$ and $\hat{A} = C^{-1}AC$. For further information on subgenerators of (a, k)-regularized C-resolvent families, we refer the reader to [10]-[11]; in the sequel, we shall always assume that the functional equation (5) holds for any considered (a, k)-regularized C-resolvent family. By $a^{*,l}(t)$ we denote the l-th convolution power of $a(t)$.

We need the following condition.

(P1): $k(t)$ is Laplace transformable, i.e., it is locally integrable on $[0, \infty)$ and there exists $\beta \in \mathbb{R}$ such that $\tilde{k}(\lambda) := \mathcal{L}(k)(\lambda) := \lim_{b\to\infty} \int_0^b e^{-\lambda t}k(t)\,dt := \int_0^\infty e^{-\lambda t}k(t)\,dt$ exists for all $\lambda \in \mathbb{C}$ with $\Re\lambda > \beta$. Put $abs(k)$:=inf$\{\Re\lambda : \tilde{k}(\lambda)$ exists$\}$, $\tilde{\delta}(\lambda) := 1$ and denote by $\mathcal{L}^{-1}$ the inverse Laplace transform.

Let $\sigma > 0$ and $l \in \mathbb{N}$. Set, for any X-valued function $f(t)$ satisfying (P1),

$$F_{\sigma,f}(z) := \int_0^\infty e^{-z^{1/\sigma}t} f(t)\,dt, \quad z > \max(abs(f), 0)^\sigma.$$

Then there exist uniquely determined real numbers $(c_{l_0,l,\sigma})_{1\leq l_0\leq l}$, independent of X and $f(t)$, such that:

$$\frac{d^l}{dz^l}F_{\sigma,f}(z) = \sum_{l_0=1}^{l} c_{l_0,l,\sigma} z^{\frac{l_0}{\sigma}-l} \int_0^\infty e^{-z^{1/\sigma}t} t^{l_0} f(t)\,dt,\ z > \max(abs(f), 0)^\sigma. \tag{6}$$

Notice that $c_{l,l,\sigma} = (-1)^l\sigma^{-l}$, $l \geq 1$ and that [13, Lemma 3] implies that there exists a number $\zeta \geq 1$ such that

$$\sum_{l_0=1}^{l} l_0!|c_{l_0,l,\sigma}| \leq \zeta^l l! \text{ for all } l \in \mathbb{N}. \tag{7}$$

Following [14, Definition 2.1], it will be said that a function $u \in C^{m_n-1}([0, \infty) : X)$ is a (strong) solution of (1) iff $A_i\mathbf{D}_t^{\alpha_i}u \in C([0, \infty) : X)$ for $1 \leq i \leq n-1$, $g_{m_n-\alpha_n} * (u - \sum_{k=0}^{m_n-1} u_k g_{k+1}) \in C^{m_n}([0, \infty) : X)$

and (1) holds. Recall that the Caputo fractional derivative $\mathbf{D}_t^{\alpha_n}u(t)$ is defined for those functions $u \in C^{m_n-1}([0,\infty) : X)$ for which $g_{m_n-\alpha_n} * (u - \sum_{k=0}^{m_n-1} u_k g_{k+1}) \in C^{m_n}([0,\infty) : X)$, by

$$\mathbf{D}_t^{\alpha_n}u(t) = \frac{d^{m_n}}{dt^{m_n}}\left[g_{m_n-\alpha_n} * \left(u - \sum_{k=0}^{m_n-1} u_k g_{k+1}\right)\right].$$

We need the following recent result [17] on the existence and uniqueness of strong solutions of (1); notice only that we always have in formulations of our results that $\mathbb{N}_{n-1} \setminus D_k \neq \emptyset$, $k \in \mathbb{N}^0_{m_n-1}$.

Lemma 1.2. *(i) Suppose $A_1, \cdots, A_{n-1}$ are closed linear operators on X, $\omega \geq 0$, $C \in L(X)$ is injective and $u_0, \cdots, u_{m_n-1} \in X$. Let the following conditions hold:*

(a) The operator P_λ is injective for $\lambda > \omega$ and $D(P_\lambda^{-1}C) = X$, $\lambda > \omega$.

(b) If $1 \leq j \leq n-1$, $1 \leq l \leq n-1$, $0 \leq k \leq m_n - 1$, $k > m_l - 1$ and $\lambda > \omega$, then $Cu_k \in D(P_\lambda^{-1}A_l)$,

$$A_j\left\{\lambda^{\alpha_j}\left[\lambda^{-k-1}Cu_k - \sum_{l \in \mathbb{N}_{n-1}\setminus D_k} \lambda^{\alpha_l-k-1}P_\lambda^{-1}A_lCu_k\right] - \sum_{l=0}^{m_j-1}\delta_{kl}\lambda^{\alpha_j-1-l}Cu_k\right\} \in LT - X \tag{8}$$

and

$$\lambda^{\alpha_n}\left[\lambda^{-k-1}Cu_k - \sum_{l \in \mathbb{N}_{n-1}\setminus D_k} \lambda^{\alpha_l-k-1}P_\lambda^{-1}A_lCu_k\right] - \lambda^{\alpha_n-1-k}Cu_k \in LT - X. \tag{9}$$

Then the abstract Cauchy problem (1) has a strong solution, with u_k replaced by Cu_k ($0 \leq k \leq m_n-1$). Furthermore, $u(t) = \sum_{k=0}^{m_n-1} u_k(t)$, $t \geq 0$, where

$$\int_0^\infty e^{-\lambda t}u_k(t)\,dt = \lambda^{-k-1}Cu_k - \sum_{l \in \mathbb{N}_{n-1}\setminus D_k} \lambda^{\alpha_l-k-1}P_\lambda^{-1}A_lCu_k, \tag{10}$$

for any $k \in \mathbb{N}^0_{m_n-1}$.

(ii) Let $\lambda > 0$, let $C \in L(X)$ be injective, and let $D(P_{n\lambda}^{-1}C) = X$, $n \in \mathbb{N}$. Suppose that, for every positive real number $\sigma > 0$ and for every null sequence $(x_n)_{n\in\mathbb{N}}$ in X, one has:

$$\lim_{n\to\infty} e^{-n\lambda\sigma}P_{n\lambda}^{-1}Cx_n = 0.$$

Then, for every $u_0, \cdots, u_{m_n-1} \in X$, the abstract Cauchy problem (1) has at most one strong solution.

2. Applications of certain subclasses of (a, k)-regularized C-resolvent families in the analysis of the abstract Cauchy problem (1)

We start this section by stating the following important result.

Theorem 2.1. *Suppose $n \in \mathbb{N} \setminus \{1, 2\}$, $\sigma \in [1, 2]$, $r \geq 0$, $\alpha_n - \alpha_{n-1} = \sigma$, $\alpha_{n-1} - \alpha_{n-2} \geq \sigma$, $\omega \geq 0$, $D(A_{n-1}) \subseteq \bigcap_{i=0}^{n-2} D(A_i)$ and $(\omega^\sigma, \infty) \subseteq \rho(-A_{n-1})$. Put $\breve{A}_i(\lambda) := \lambda^{\alpha_i-\alpha_{n-1}}A_i(\lambda^\sigma + A_{n-1})^{-1}$, $\lambda > \omega$, $i \in \mathbb{N}_{n-2}$ and suppose that the following conditions hold:*

(i) $A_iA_jx = A_jA_ix,\ 1 \le i,\ j \le n-1,\ x \in D(A_{n-1}^2)$ *and* $CA_j \subseteq A_jC,\ j \in \mathbb{N}_{n-2}$.

(ii) *There exists* $\omega_0 \ge \omega$ *such that, for every* $p \in \circledast$, *there exists* $c_p \in (0, 1/(n-2))$ *satisfying*

$$p\big(\breve{A}_i(\lambda)x\big) \le c_p p(x),\ \lambda > \omega_0,\ x \in X,\ i \in \mathbb{N}_{n-2}. \tag{11}$$

If

(a) *The operator* $-A_{n-1}$ *is the integral generator of a* $(g_\sigma, g_{\sigma r+1})$*-regularized resolvent family* $(S_{\sigma,r}(t))_{t\ge0}$ *on* X, *the family* $\{e^{-\omega t}S_{\sigma,r}(t) : t \ge 0\}$ *is equicontinuous, and* $A_lu_k \in D\big(A_{n-1}^{\max(\lceil \frac{1}{\sigma}(\sigma r+\alpha_l-k)\rceil,0)}\big)$, *provided* $0 \le k \le m_n - 1$ *and* $l \in \mathbb{N}_{n-1} \setminus D_k$,

or

(b) *The operator* $-A_{n-1}$ *is the integral generator of a* (g_σ, C)*-regularized resolvent family* $(T_\sigma(t))_{t\ge0}$ *on* X, *the family* $\{e^{-\omega t}T_\sigma(t) : t \ge 0\}$ *is equicontinuous, and* $u_k \in C\big(\bigcap_{l\in\mathbb{N}_{n-1}\setminus D_k} D(A_l)\big)$ *for* $0 \le k \le m_n - 1$,

then the abstract Cauchy problem (1) has a unique strong solution.

Proof. Let $\mu_0 < -\omega_0^\sigma$. By (ii), it follows that, for every $p \in \circledast$ and $l \in \mathbb{N}$, one has $p(\breve{A}_i(\lambda)^l x) \le c_p^l p(x)$, $\lambda > \omega_0$, $x \in X$, $i \in \mathbb{N}_{n-2}$. Using this inequality and the polynomial formula, we obtain that, for every $p \in \circledast$,

$$p\left(\left[\sum_{i=1}^{n-2} \breve{A}_i(\lambda)\right]^k x\right) \le c_p^k(n-2)^k p(x), \quad \lambda > \omega_0,\ k \in \mathbb{N}_0,\ x \in X.$$

Since $c_p(n-2) < 1$, $p \in \circledast$, the above implies that, for every $x \in X$ and $\lambda > \omega_0$, the series

$$B_\lambda x \equiv \sum_{k=0}^{\infty} \big(\lambda^\sigma + A_{n-1}\big)^{-1} \left[-\sum_{i=1}^{n-2} \breve{A}_i(\lambda)\right]^k x \tag{12}$$

is convergent. Put $\tilde{A}_i := A_i(\mu_0 - A_{n-1})^{-1}$, $i \in \mathbb{N}_{n-2}$. Then (11) implies $\tilde{A}_i \in L(X)$, $i \in \mathbb{N}_{n-2}$. Using the polynomial formula again, we get that, for every $\lambda > \omega$, $k \in \mathbb{N}_0$ and $j \in \mathbb{N}_0$,

$$\left[\sum_{i=1}^{n-2} \lambda^{\alpha_i-\alpha_{n-1}+\sigma}\tilde{A}_i\right]^k \Big(1 + \frac{\mu_0}{\lambda^\sigma}\Big)^j = \sum_{\substack{(l_1,\cdots,l_{n-2})\in\mathbb{N}_0^{n-2}\\ l_1+\cdots+l_{n-2}=k}} \sum_{s=0}^{j} \frac{k!}{l_1!\cdots l_{n-2}!}$$
$$\times \big(\lambda^{\alpha_1-\alpha_{n-1}+\sigma}\tilde{A}_1\big)^{l_1} \cdots \big(\lambda^{\alpha_{n-2}-\alpha_{n-1}+\sigma}\tilde{A}_{n-2}\big)^{l_{n-2}} \binom{j}{s} \frac{\mu_0^s}{\lambda^{\sigma s}}. \tag{13}$$

Since $\alpha_{n-1} - \alpha_{n-2} \ge \sigma$, (13) yields that, for every $k \in \mathbb{N}_0$ and $j \in \mathbb{N}_0$ with $0 \le j \le k$, there exist numbers $l_{kj} \in \mathbb{N}$, $\beta_0, \cdots, \beta_{l_{kj}} \in (-\infty, 0]$ and operators $A_{kjm} \in L(X)$ $(0 \le m \le l_{kj})$ such that $\beta_0 > \cdots > \beta_{l_{kj}}$ and

$$\left[\sum_{i=1}^{n-2} \lambda^{\alpha_i-\alpha_{n-1}+\sigma}\tilde{A}_i\right]^k \Big(1 + \frac{\mu_0}{\lambda^\sigma}\Big)^j = \sum_{m=0}^{l_{kj}} \lambda^{\beta_m} A_{kjm}, \quad \lambda > \omega_0.$$

Repeating literally the arguments given in the proof of [35, Theorem 3.2.1, p. 95], we obtain that:

$$B_\lambda x = \sum_{k=0}^{\infty}\sum_{j=0}^{k} \binom{k}{j} \sum_{m=0}^{l_{kj}} \lambda^{\beta_m} A_{kjm} \left[\frac{(-1)^j}{\lambda^{(k-j)\sigma}} \big(\lambda^\sigma + A_{n-1}\big)^{-j-1} x\right],\ x \in X,\ \lambda > \omega_0.$$

Having in mind (12) and the equality $\alpha_n - \alpha_{n-1} = \sigma$, it can be easily seen that

$$P_\lambda B_\lambda x = \lambda^{\alpha_{n-1}} x, \quad x \in X,\ \lambda > \omega_0, \tag{14}$$

as well as that (cf. (i)): $A_i(\lambda+A_{n-1})^{-2}=(\lambda+A_{n-1})^{-1}A_i(\lambda+A_{n-1})^{-1}$, $\lambda>\omega$, $1\leq i\leq n-2$ and $A_i(\lambda+A_{n-1})^{-1}A_j(\lambda+A_{n-1})^{-1}=A_j(\lambda+A_{n-1})^{-1}A_i(\lambda+A_{n-1})^{-1}$, $\lambda>\omega$, $1\leq i,\ j\leq n-2$. In combination with (14), this implies that P_λ is injective for $\lambda>\omega_0$ and $B_\lambda x=\lambda^{\alpha_{n-1}}P_\lambda^{-1}x$, $x\in X$, $\lambda>\omega_0$. Then the existence of strong solutions simply follows from Lemma 1.2 (cf. (8)-(9)) if we prove that, for every $k\in\mathbb{N}^0_{m_n-1}$ and $j\in\mathbb{N}_{n-1}$,

$$\lambda^{\alpha_j}A_j\sum_{l\in\mathbb{N}_{n-1}\setminus D_k}\lambda^{\alpha_l-k-1}P_\lambda^{-1}A_lu_k\in LT-X \tag{15}$$

and

$$\lambda^{\alpha_n}\sum_{l\in\mathbb{N}_{n-1}\setminus D_k}\lambda^{\alpha_l-k-1}P_\lambda^{-1}A_lu_k\in LT-X. \tag{16}$$

Clearly, the relation (15) with $j=n-1$ is equivalent to say that

$$\lambda^{\alpha_{n-1}}\bigl(\mu_0-A_{n-1}\bigr)\sum_{l\in\mathbb{N}_{n-1}\setminus D_k}\lambda^{\alpha_l-k-1}P_\lambda^{-1}A_lu_k\in LT-X. \tag{17}$$

Suppose first that (b) holds. We will prove that $\lambda^{\sigma-1}B_\lambda Cx\in LT-X$ for every fixed element $x\in X$. Owing to [11, Theorem 2.7], we have

$$\bigl(z+A_{n-1}\bigr)^{-1}Cx=z^{(1-\sigma)/\sigma}\int_0^\infty e^{-z^{1/\sigma}t}T_\sigma(t)x\,dt,\quad x\in X,\ z>\omega^\sigma.$$

This equality in combination with (6) and

$$(-1)^j\bigl(z+A_{n-1}\bigr)^{-j-1}Cx=j!^{-1}\frac{d^j}{dz^j}\bigl(z+A_{n-1}\bigr)^{-1}Cx,\ z>\omega^\sigma,\ j\in\mathbb{N}_0,\ x\in X,$$

implies

$$\begin{aligned}\lambda^{\sigma-1}\bigl(\lambda^\sigma+A_{n-1}\bigr)^{-j-1}Cx&=\frac{(-1)^j}{j!}\sum_{l=0}^{j}\binom{j}{l}\frac{1-\sigma}{\sigma}\cdots\Bigl(\frac{1-\sigma}{\sigma}-(j-l-1)\Bigr)\\&\times\sum_{l_0=1}^{l}c_{l_0,l,\sigma}\lambda^{l_0-j\sigma}\int_0^\infty e^{-\lambda t}t^{l_0}T_\sigma(t)x\,dt,\quad x\in X,\ \lambda>\omega,\end{aligned} \tag{18}$$

where we have put, by common consent,
$\frac{1-\sigma}{\sigma}\cdots(\frac{1-\sigma}{\sigma}-(j-l-1))\equiv\begin{cases}1, & \text{if }\sigma=1\text{ and }j=l,\\ 0, & \text{if }\sigma=1\text{ and }j>l,\end{cases}$ and
$\sum_{l_0=1}^{l}c_{l_0,l,\sigma}\lambda^{l_0-j\sigma}\int_0^\infty e^{-\lambda t}t^{l_0}T_\sigma(t)x\,dt\equiv\lambda^{-j\sigma}\int_0^\infty e^{-\lambda t}T_\sigma(t)x\,dt$, $t\geq0$, $x\in X$, for $l=0$. If $k\in\mathbb{N}_0$, $0\leq j\leq k$, $t\geq0$ and $x\in X$, then we define

$$H^\sigma_{C,kj0}(t;0,0)x:=j!^{-1}\sum_{l=0}^{j}\sum_{l_0=1}^{l}\binom{j}{l}\frac{1-\sigma}{\sigma}\cdots\Bigl(\frac{1-\sigma}{\sigma}-(j-l-1)\Bigr)c_{l_0,l,\sigma}\Bigl[g_{k\sigma-l_0}*\cdot^{l_0}T_\sigma(\cdot)x\Bigr](t).$$

Using the estimate (7) and an elementary argumentation, it is checked at once that, for every $x\in X$, the series

$$H^\sigma_C(t;0,0)x:=\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\sum_{m=0}^{l_{kj}}A_{kjm}\bigl(g_{-\beta_m}*H^\sigma_{C,kj0}(\cdot;0,0)x\bigr)(t)$$

converges uniformly on compacts of $[0,\infty)$. By definition of $H^{\sigma}_{C,kj0}(\cdot;0,0)$ and (16), it readily follows that:

$$\int_0^{\infty} e^{-\lambda t} H^{\sigma}_{C,kj0}(t;0,0)x\,dt = (-1)^j \lambda^{\sigma-1-(k-j)\sigma}\bigl(\lambda^{\sigma}+A_{n-1}\bigr)^{-j-1}Cx, \tag{19}$$

provided $k\in\mathbb{N}_0$, $0\leq j\leq k$, $x\in X$ and $\lambda>\omega_0$. Furthermore, there exists $c_\sigma>0$ such that

$$\Bigl|\frac{1-\sigma}{\sigma}\cdots\Bigl(\frac{1-\sigma}{\sigma}-(j-l-1)\Bigr)\Bigr|\leq c_\sigma (j-l)!,$$

provided $j\in\mathbb{N}_0$ and $0\leq l\leq j$. Taken together with the inequality (7), the last estimate yields the existence of a number $\eta\geq 1$ such that, for every $p\in\circledast$, there exist $c_p>0$ and $q_p\in\circledast$ such that

$$p\bigl(H^{\sigma}_{C,kj0}(t;0,0)x\bigr)\leq c_p e^{\omega t}\eta^k g_{k\sigma+1}(t)q_p(x),\quad x\in X,\ t\geq 0,\ k\in\mathbb{N}_0,\ 0\leq j\leq k. \tag{20}$$

Since $E_\sigma(a(bt)^\sigma)=O(a^{1/\sigma}e^{bt})$, $t\geq 0$ $(a,\ b>0)$, it is not difficult to show that the series appearing in the definition of $H^{\sigma}_{C}(t;0,0)$ converges uniformly on compacts of $[0,\infty)$ and that there exists $\omega'>\omega$ such that, for every $p\in\circledast$, there exist $c_p>0$ and $q_p\in\circledast$ satisfying $p(H^{\sigma}_{C}(t;0,0)x)\leq c_p e^{\omega' t}q_p(x)$, $x\in X$, $t\geq 0$. Clearly,

$$\int_0^{\infty} e^{-\lambda t}H^{\sigma}_{C}(t;0,0)x\,dt=\lambda^{\sigma-1}B_\lambda Cx,\quad x\in X,\ \lambda>\omega_0.$$

If $k\in\mathbb{N}_0$, $0\leq j\leq k$, $l\in\mathbb{N}_{n-1}\setminus D_k$, $x\in X$ and $t\geq 0$, then we set

$$F^{\sigma,l}_{C,kj0}(t)x:=\begin{cases} -g_{k\sigma+k+1-\alpha_l}(t)Cx, & \text{if } j=0,\\ \bigl(g_{k-\alpha_l}*H^{\sigma}_{C,k(j-1)0}(\cdot;0,0)x\bigr)(t), & \text{if } j>0\end{cases}$$
$$+\Bigl(\Bigl[g_{k-\alpha_l}(\cdot)+\mu_0 g_{k+\sigma-\alpha_l}(\cdot)\Bigr]*H^{\sigma}_{C,kj0}(\cdot;0,0)x\Bigr)(t).$$

Using the resolvent equation and (19), it is checked at once that

$$\mathcal{L}^{-1}\Bigl(\bigl(\mu_0-A_{n-1}\bigr)\lambda^{\alpha_l-k-1}B_\lambda Cx\Bigr)(t)=\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\sum_{m=0}^{l_{kj}}A_{kjm}\Bigl(g_{-\beta_m}*F^{\sigma,l}_{C,kj0}(\cdot;0,0)x\Bigr)(t),$$

provided $k\in\mathbb{N}^0_{m_n-1}$, $l\in\mathbb{N}_{n-1}\setminus D_k$, $x\in X$ and $t\geq 0$. Since $A_j(\mu_0-A_{n-1})^{-1}(\mu_0-A_{n-1})x=A_jx$, $1\leq j\leq n-2$, $x\in D(A_{n-1})$, the above ensures that

$$\lambda^{\alpha_j}A_j\sum_{l\in\mathbb{N}_{n-1}\setminus D_k}\lambda^{\alpha_l-k-1}P_\lambda^{-1}CA_lC^{-1}u_k\in LT-X.$$

Suppose now that (a) holds and fix an element $x\in X$. If $k\in\mathbb{N}_0$, $0\leq j\leq k$ and $t\geq 0$, then we define, as in the proof of theorem in the case that the initial values satisfy the condition (a),

$$H^{\sigma}_{kj0}(t;0,\sigma r+1-\sigma):=j!^{-1}\sum_{l=0}^{j}\sum_{l_0=1}^{l}\binom{j}{l}$$
$$\times\frac{\sigma r+1-\sigma}{\sigma}\cdots\Bigl(\frac{\sigma r+1-\sigma}{\sigma}-(j-l-1)\Bigr)c_{l_0,l,\sigma}\Bigl[g_{k\sigma-l_0}*\cdot^{l_0}S_{\sigma,r}(\cdot)x\Bigr](t),$$

and

$$H^{\sigma}(t;0,\sigma r+1-\sigma):=\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\sum_{m=0}^{l_{kj}}A_{kjm}\bigl(g_{-\beta_m}*H^{\sigma}_{kj0}(\cdot;0,\sigma r+1-\sigma)x\bigr)(t),$$

for any $t \geq 0$. Clearly,

$$\left(z + A_{n-1}\right)^{-1} x = z^{(\sigma r+1-\sigma)/\sigma} \int\limits_0^\infty e^{-z^{1/\sigma} t} S_{\sigma,r}(t)x\, dt, \quad z > \omega^\sigma. \tag{21}$$

By definition of $H^\sigma_{kj0}(\cdot; 0, \sigma r + 1 - \sigma)$ and (21), it readily follows that (cf. also the equation (18)):

$$\int\limits_0^\infty e^{-\lambda t} H^\sigma_{kj0}(t; 0, \sigma r + 1 - \sigma)x\, dt = (-1)^j \lambda^{\sigma-1-(k-j)\sigma} \lambda^{\sigma-\sigma r-1} \left(\lambda^\sigma + A_{n-1}\right)^{-j-1} x,$$

provided $k \in \mathbb{N}_0$, $0 \leq j \leq k$, $\lambda > \omega_0$, and

$$\int\limits_0^\infty e^{-\lambda t} H^\sigma(t; 0, \sigma r + 1 - \sigma)x\, dt = \lambda^{\sigma-\sigma r-1} B_\lambda x, \quad \lambda > \omega_0.$$

Assume $r_0 \in \mathbb{N}_0 \cup \{-1\}$, $r_1 \in \mathbb{R}$ and $r_1 + r_0\sigma \geq \sigma r + 1 - \sigma$; notice that in the previous analysis we have considered the case $r_0 = 0$. If $r_0 = -1$, then it is very simple to construct, with the help of resolvent equation and the arguments given in the case $r_0 = 0$, the continuous function $t \mapsto H^\sigma(t; -1, \sigma r + 1)x$, $t \geq 0$ such that $H^\sigma(t; -1, \sigma r + 1) \in L(X)$ for $t \geq 0$ and

$$B_\lambda x = \lambda^{r\sigma+1} \left(\mu_0 - A_{n-1}\right) \int\limits_0^\infty e^{-\lambda t} H^\sigma\left(t; -1, \sigma r + 1\right) x\, dt, \quad \lambda > \omega_0.$$

Suppose now $r_0 > 0$. Then the identities

$$S_{\sigma,r}(t)y = \sum_{l=0}^{r_0-1} (-1)^l g_{\sigma r+1+l\sigma}(t) A^l_{n-1} y + (-1)^{r_0} \left(g_{\sigma r_0} * S_{\sigma,r}(\cdot) A^{r_0}_{n-1} y\right)(t), \quad t \geq 0,\ y \in D\left(A^{r_0}\right),$$

and

$$\begin{aligned}
&\int\limits_0^\infty e^{-\lambda t} t^{l_0} \left(g_{r_0\sigma} * S_{\sigma,r_0}(\cdot)x\right)(t)\, dt \\
&= \lambda^{\sigma+r_1-\sigma r-1} \int\limits_0^\infty e^{-\lambda t} \Bigg\{ \sum_{l_1=0}^{l_0} \binom{l_0}{l_1} (\sigma r + 1 - \sigma - r_1) \cdots (\sigma r - \sigma - r_1 + l_0 - l_1) \\
&\times \Big[g_{l_1-l_0}(\cdot) * (-1)^{r_0} \cdot^{l_1} \left(g_{r_0\sigma+\sigma+r_1-\sigma r-1} * S_{\sigma,r}(\cdot) A^{r_0}_{n-1} \left(\mu_0 - A_{n-1}\right)^{-r_0} x\right) \Big](t) \Bigg\}\, dt,
\end{aligned}$$

which holds for any $l_0 \in \mathbb{N}$ and $\lambda > \omega$ suff. large, imply that

$$
\begin{aligned}
&\lambda^{(j-k)\sigma - r_1}\bigl(\mu_0 - A_{n-1}\bigr)^{-r_0}\bigl(\lambda^{\sigma} + A_{n-1}\bigr)^{-j-1}x \\
&= (-1)^j \lambda^{(j-k)\sigma - r_1} j!^{-1} \sum_{l=0}^{j}\sum_{l_0=1}^{l}\binom{j}{l}\frac{\sigma r + 1 - \sigma}{\sigma}\cdots\Bigl(\frac{\sigma r + 1 - \sigma}{\sigma} - (j-l-1)\Bigr) \\
&\times c_{l_0,l,\sigma}\lambda^{\sigma r + 1 - \sigma - (j-l)\sigma}\lambda^{l_0 - l\sigma}\int_0^{\infty} e^{-\lambda t} t^{l_0} S_{\sigma,r}(t)\bigl(\mu_0 - A_{n-1}\bigr)^{-r_0}x\,dt \\
&= (-1)^j j!^{-1}\sum_{l=0}^{j}\sum_{l_0=1}^{l}\binom{j}{l}\frac{\sigma r + 1 - \sigma}{\sigma}\cdots\Bigl(\frac{\sigma r + 1 - \sigma}{\sigma} - (j-l-1)\Bigr) l_0! c_{l_0,l,\sigma} \\
&\times \lambda^{-k\sigma - r_1 + \sigma r + 1 - \sigma + l_0}\sum_{m=0}^{r_0-1}(-1)^m g_{\sigma \widetilde{r+1+l_0}+\sigma m}(\lambda) A_{n-1}^{m}\bigl(\mu_0 - A_{n-1}\bigr)^{-r_0}x \\
&+ (-1)^j j!^{-1}\sum_{l=0}^{j}\sum_{l_0=1}^{l}\binom{j}{l}\frac{\sigma r + 1 - \sigma}{\sigma}\cdots\Bigl(\frac{\sigma r + 1 - \sigma}{\sigma} - (j-l-1)\Bigr) c_{l_0,l,\sigma} \\
&\times \lambda^{l_0 - k\sigma}\int_0^{\infty} e^{-\lambda t}\Biggl\{\sum_{l_1=0}^{l_0}\binom{l_0}{l_1}(\sigma r + 1 - \sigma - r_1)\cdots(\sigma r - \sigma - r_1 + l_0 - l_1) \\
&\Bigl[g_{l_1 - l_0}(\cdot) * (-1)^{r_0} \cdot^{l_1} \bigl(g_{r_0\sigma + \sigma + r_1 - \sigma r - 1} * S_{\sigma,r}(\cdot)A_{n-1}^{r_0}\bigl(\mu_0 - A_{n-1}\bigr)^{-r_0}x\bigr)\Bigr](t)\Biggr\}\,dt,
\end{aligned}
$$

so that $\lambda^{(j-k)\sigma - r_1}\bigl(\mu_0 - A_{n-1}\bigr)^{-r_0}\bigl(\lambda^{\sigma} + A_{n-1}\bigr)^{-j-1}x \in LT - X$. Put, for every $t \geq 0$,

$$
H^{\sigma}_{kj0}\bigl(t; r_0, r_1\bigr)x := \mathcal{L}^{-1}\Bigl(\lambda^{(j-k)\sigma - r_1}\bigl(\mu_0 - A_{n-1}\bigr)^{-r_0}\bigl(\lambda^{\sigma} + A_{n-1}\bigr)^{-j-1}x\Bigr)(t)
$$

and

$$
H^{\sigma}\bigl(t; r_0, r_1\bigr)x := \sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\sum_{m=0}^{l_{kj}} A_{kjm}\bigl(g_{-\beta_m} * H^{\sigma}_{kj0}\bigl(\cdot; r_0, r_1\bigr)x\bigr)(t).
$$

Since $r_1 + r_0\sigma \geq \sigma r + 1 - \sigma$, we obtain by the foregoing arguments that the mapping $t \mapsto H^{\sigma}(t; r_0, r_1)x$, $t \geq 0$ is continuous as well as that $H^{\sigma}(t; r_0, r_1) \in L(X)$, $t \geq 0$ and

$$
B_{\lambda}x = \lambda^{r_1}\bigl(\mu_0 - A_{n-1}\bigr)^{r_0}\int_0^{\infty} e^{-\lambda t} H^{\sigma}\bigl(t; r_0, r_1\bigr)x\,dt, \quad \lambda > \omega \quad \text{suff. large.} \tag{22}
$$

Put $s_{l,k,\sigma} := \max(\lceil \frac{1}{\sigma}(\sigma r + \alpha_l - k)\rceil, 0)$. Using the first part of proof, it is not difficult to see that there exists $\omega' \geq 0$ such that, for every $p \in ⊛$, there exist $c_p > 0$ and $q_p \in ⊛$ such that $p(H^{\sigma}(t; r_0, r_1)x) \leq c_p e^{\omega' t} q_p(x)$, $x \in X$, $t \geq 0$. Fix now an index $k \in \mathbb{N}^0_{m_n - 1}$ and $l \in \mathbb{N}_{n-1} \setminus D_k$. Then (17) follows on account of (22), the inequality $(\sigma + \alpha_l - k - 1) + (\sigma r + 1 - s_{l,k,\sigma}\sigma) \leq 0$ and the following relation:

$$
\begin{aligned}
&\lambda^{\alpha_{n-1} + \alpha_l - k - 1}\bigl(\mu_0 - A_{n-1}\bigr)P_{\lambda}^{-1}A_l u_k = \lambda^{\sigma + \alpha_l - k - 1}\lambda^{\sigma r + 1 - s_{l,k,\sigma}\sigma} \\
&\times \Bigl[\lambda^{-(\sigma r + 1 - s_{l,k,\sigma}\sigma)}\bigl(\mu_0 - A_{n-1}\bigr)^{1 - s_{k,l,\sigma}} B_{\lambda}\bigl(\mu_0 - A_{n-1}\bigr)^{s_{k,l,\sigma}} A_l u_k\Bigr] \in LT - X;
\end{aligned}
$$

one can simply prove (15) by using (17) and decomposition $A_j x = A_j(\mu_0 - A_{n-1})^{-1}(\mu_0 - A_{n-1})x$, $1 \leq j \leq n-2$, $x \in D(A_{n-1})$. Similarly, we have by (22) and the inequality $(\sigma + \alpha_l - k - 1) + (\sigma r + 1 - \sigma - s_{l,k,\sigma}\sigma) \leq 0$

that

$$\lambda^{\alpha_n+\alpha_l-k-1}P_\lambda^{-1}A_lu_k = \lambda^{\sigma+\alpha_l-k-1}\lambda^{\sigma r+1-\sigma-s_{l,k,\sigma}\sigma}$$
$$\times\Big[\lambda^{-(\sigma r+1-\sigma-s_{l,k,\sigma}\sigma)}\big(\mu_0-A_{n-1}\big)^{-s_{k,l,\sigma}}B_\lambda\big(\mu_0-A_{n-1}\big)^{s_{k,l,\sigma}}A_lu_k\Big]\in LT-X.$$

Hence, (16) holds and the proof of theorem is thereby completed. □

Remark 2.2. *(i) For every $i\in\mathbb{N}_{n-2}$, the operator $\tilde{A}_i$ is closed, linear and defined on the whole space X. If we assume that $\alpha_{n-2}-\alpha_{n-1}+\sigma r<0$ as well as that X is a webbed bornological space (this holds provided that X is a Fréchet space) and that there exists $M\geq 1$ such that*

$$p\big(S_{\sigma,r}(t)x\big)\leq Me^{\omega t}p(x),\quad p\in\circledast,\ t\geq 0,\ x\in X, \tag{23}$$

then (11) holds.

(ii) Suppose that (a) holds with some $r>0$. Then [12, Corollary 2.4] implies that the operator $-A_{n-1}$ is the integral generator of an exponentially equicontinuous $(g_\sigma,(\mu_0-A_{n-1})^{-\lceil r\rceil})$-regularized resolvent family. By Theorem 2.1(b), we obtain that there exists a unique strong solution of (1) provided that the initial values satisfy the condition $u_k\in(\mu_0-A_{n-1})^{-\lceil r\rceil}(\bigcap_{l\in\mathbb{N}_{n-1}\setminus D_k}D(A_l))$ for $0\leq k\leq m_n-1$. Since $s_{l,k,\sigma}\leq\lceil r\rceil$ (in many concrete situations, the above inequality is strict), the use of integrated operator solution families produces here better results, so that the choice $C\neq(\mu_0-A_{n-1})^{-\lceil r\rceil}$ is inevitable for obtaining larger initial data sets $\mathfrak{T}_k$ such that the equation (1) has a unique strong solution provided $u_k\in\mathfrak{T}_k$ $(0\leq k\leq m_n-1)$.

(iii) Set, for every $k\in\mathbb{N}^0_{m_n-1}$ and $l\in\mathbb{N}_{n-1}\setminus D_k$, $Q_{k,l}:=\max(\lceil\frac{1}{\sigma}(\sigma r+\alpha_l-k-\alpha_n)\rceil,0)$. Suppose that (23) holds with $(S_{\sigma,r}(t))_{t\geq 0}$, and with $(S_{\sigma,r}(t))_{t\geq 0}$ replaced by $(T_\sigma(t))_{t\geq 0}$ therein. Then it is not difficult to see that the assumptions $0\leq k\leq m_n-1$ and $D_k=\emptyset$ imply

$$x-\sum_{l\in\mathbb{N}_{n-1}\setminus D_k}\lambda^{\alpha_l}P_\lambda^{-1}A_lx=\lambda^{\alpha_n}P_\lambda^{-1}x,\quad x\in X.$$

In this case, the Laplace transform of strong solution $u_k(t)$ of (1) with $u_k^{(j)}(0)=\delta_{jk}u_k$ can be also computed by

$$\int_0^\infty e^{-\lambda t}u_k(t)\,dt=\lambda^{\sigma-k-1}B_\lambda u_k$$
$$=\lambda^{\sigma-k-1}\Bigg[\lambda^{k+1-\sigma}\int_0^\infty e^{-\lambda t}H^\sigma\Big(t;\max\big(\lceil\sigma^{-1}(\sigma r-k)\rceil,0\big),k+1-\sigma\Big)u_k\,dt\Bigg],$$

for $\lambda>\omega$ suff. large; cf. Lemma 1.2, the equation (10). Then the proof of Theorem 2.1, taken together with (10) and the equality

$$g_{r_0}*H^\sigma_{k'j0}\big(\cdot;r_0,\sigma r+1-\sigma-r_0\big)\big(\mu_0-A_{n-1}\big)^{r_0}x$$
$$=g_{r_0'}*H^\sigma_{k'j0}\big(\cdot;r_0',\sigma r+1-\sigma-r_0'\big)\big(\mu_0-A_{n-1}\big)^{r_0'}x,$$

which holds provided $x\in D(A_{n-1}^{\max(r_0,r_0')})$ and $r_0,\ r_0'\in\mathbb{N}_0$, implies that the strong solution $u(t)$ of (1) has the following form:

$$u(t)=\sum_{k=0}^{m_{n-1}-1}\Bigg[g_{k+1}(t)u_k-\sum_{l\in\mathbb{N}_{n-1}\setminus D_k}\Bigg(g_{\alpha_n+k-r+\sigma Q_{k,l}-\alpha_l-\sigma r}*\sum_{k'=0}^{\infty}\sum_{j=0}^{k'}\binom{k'}{j}\sum_{m=0}^{l_{k'j}}A_{k'jm}\Big(g_{-\beta_m}*$$
$$\times H^\sigma_{k'j0}\big(\cdot;Q_{k,l},\sigma r+1-\sigma-Q_{k,l}\big)\big(\mu_0-A_{n-1}\big)^{Q_{k,l}}A_lu_k\Big)\Bigg)(t)\Bigg]+\sum_{k=m_{n-1}}^{m_n-1}u_k(t),\quad t\geq 0, \tag{24}$$

where $H^{\sigma}_{k'j0}(\cdot; Q_{k,l}, r-Q_{k,l})(t)$ can be further expressed in terms of $(S_{\sigma,r}(t))_{t\geq 0}$. Then a straightforward computation shows that there exist $M' \geq 1$ and $\omega' > \omega$ such that, for every $p \in ⊛$ and $t \geq 0$,

$$p(u(t)) \leq M' e^{\omega' t}\Biggl\{ \sum_{k=m_{n-1}}^{m_n-1} \sum_{l=0}^{\max(\lceil \frac{1}{\sigma}(\sigma r-k)\rceil,0)} p\bigl(A^l_{n-1}u_k\bigr) + \sum_{k=0}^{m_{n-1}-1}\Biggl[p\bigl(u_k\bigr) + \sum_{l\in \mathbb{N}_{n-1}\setminus D_k} \sum_{s=0}^{Q_{k,l}} p\bigl(A^s_{n-1}A_l u_k\bigr)\Biggr]\Biggr\}.$$

Similarly, if (b) holds, then $\int_0^\infty e^{-\lambda t}u_k(t)\,dt = \lambda^{-k}\lambda^{\sigma-1}B_\lambda CC^{-1}u_k$ provided that $\lambda > \omega$ is suff. large and $m_{n-1} \leq k \leq m_n - 1$. The strong solution $u(t)$ of (1) has the following form

$$u(t) = \sum_{k=m_{n-1}}^{m_n-1} u_k(t) + \sum_{k=0}^{m_{n-1}-1}\Biggl[g_{k+1}(t)u_k - \sum_{l\in\mathbb{N}_{n-1}\setminus D_k}\Bigl(g_{k-\alpha_l} * H^{\sigma}_C(\cdot;0,0)A_l C^{-1}u_k\Bigr)(t)\Biggr], \qquad (25)$$

for any $t \geq 0$, and the following estimate holds

$$p(u(t)) \leq M' e^{\omega' t}\Biggl\{\sum_{k=m_{n-1}}^{m_n-1} p\bigl(C^{-1}u_k\bigr) + \sum_{k=0}^{m_{n-1}-1} \sum_{l\in\mathbb{N}_{n-1}\setminus D_k}\Bigl[p\bigl(u_k\bigr) + p\bigl(A_l C^{-1}u_k\bigr)\Bigr]\Biggr\},$$

for any $p \in ⊛$ and $t \geq 0$.

(iv) *Suppose that (a) holds with $(S_{\sigma,r}(t))_{t\geq 0}$ being an exponentially equicontinuous analytic $(g_\sigma, g_{\sigma r+1})$-regularized resolvent family of angle $\theta \in (0, \pi/2]$. Then the formula appearing in the brackets of the second addend on the right hand side of (24) represents the solution $u_k(t)$ for each $k \in \mathbb{N}^0_{m_n-1}$. Using this fact and [11, Lemma 3.3, Theorem 3.4(i)], it is not difficult to prove that the mapping $t \mapsto u_k(t)$, $t > 0$ can be analytically extended to the sector Σ_θ. Similarly, if (b) holds with $(T_\sigma(t))_{t\geq 0}$ being an exponentially equicontinuous analytic (g_σ, C)-regularized resolvent family of angle θ, then the solution $u_k(t)$ of (1) can be analytically extended to the sector Σ_θ.*

(v) *It is worth noting that we do not assume in the formulation of Theorem 2.1(a) that $r \in \mathbb{N}_0$. In the case of abstract Cauchy problem (ACP_n), we cannot use this fact for obtaining some better results on the wellposedness of (1); the situation is quite different in the case of a general multi-term fractional differential equation (1), and we shall illustrate this by the following example. Consider the equation*

$$\begin{aligned} &u'''(t) + A_3u''(t) + A_2\mathbf{D}^{1/2}_t u(t) + A_1u(t) = 0, \quad t > 0,\\ &u(0) = 0,\ u'(0) = u_1,\ u''(0) = 0. \end{aligned} \qquad (26)$$

Assuming that the operator $-A_3$ generates an exponentially equicontinuous r-times integrated semigroup $(S_{1,r}(t))_{t\geq 0}$ for some $r \in (0, 1/2]$, the abstract Cauchy problem (26) has a unique solution for any $u_1 \in D(A_1) \cap D(A_2)$. If $r = 1$, then we obtain a weaker result on the wellposedness of (26) since we must impose the condition that $u_1 \in D(A_1) \cap D(A_3A_2)$.

Example 2.3. *(i) The conditions of [5, Theorem 3.3] (cf. also [36, Theorem (∗)]) are not fulfilled in the situation of [35, Example 6.2.5, Example 6.2.6], Theorem 2.1 produces here much better results compared with [35, Theorem 6.3.1]. In order to illustrate this, we shall first consider the equation*

$$\begin{aligned} &\frac{\partial^2 u(t,x)}{\partial t^2} + \left(\rho_1\frac{\partial^3}{\partial x^3} - \rho_2\frac{\partial^2}{\partial x^2}\right)\frac{\partial u(t,x)}{\partial t} + c\frac{\partial^2 u(t,x)}{\partial x^2} = 0, \quad t \geq 0,\ x \in \mathbb{R},\\ &u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x), \quad x \in \mathbb{R}, \end{aligned} \qquad (27)$$

where $\rho_1 \in \mathbb{R}$, $\rho_2 > 0$ *and* $c \in \mathbb{C}$. *Let* $X = L^p(\mathbb{R})$ *for some* $p \in (1, \infty)$, *and let the fractional Sobolev space* $\mathrm{S}^{\alpha,p}(\mathbb{R}^n)$ *be defined in the sense of [25, Definition 12.3.1, p. 297]* ($n \in \mathbb{N}$, $\alpha \in \mathbb{C}_+$). *By [35, Theorem 1.5.10], the operator* $-(\rho_1 \frac{\partial^3}{\partial x^3} - \rho_2 \frac{\partial^2}{\partial x^2})$, *considered with its maximal distributional domain, generates an exponentially bounded* $(I - \Delta)^{-(3/2)|1/p-1/2|}$*-regularized semigroup* $(T_1(t))_{t\geq 0}$ *on* X. *Applying Theorem 2.1, we obtain that there exists a unique solution of problem (27) provided that* $\varphi \in \mathrm{S}^{2+3|1/p-1/2|,p}(\mathbb{R})$ *and* $\psi \in \mathrm{S}^{3+3|1/p-1/2|,p}(\mathbb{R})$; *observe, however, that the existence and uniqueness of solutions of (27) have been proved in [35, Example 6.2.5] under the assumptions* $\varphi \in \mathrm{S}^{5,p}(\mathbb{R})$, $\psi \in \mathrm{S}^{6,p}(\mathbb{R})$. *Furthermore, [16, Theorem 2.18] and the analysis given in the example preceding [16, Remark 3.9] imply that the mapping* $t \mapsto T_1(t) \in L(X)$, $t > 0$ *is infinitely differentiable and that, for every compact set* $K \subseteq (0, \infty)$, *there exists* $h_K > 0$ *such that* $\sup_{p'\in\mathbb{N}_0, t\in K}(h_K^{p'}||\frac{d^{p'}}{dt^{p'}}T_1(t)||/p'!^{3/2}) < \infty$, *i.e.,* $(T_1(t))_{t\geq 0}$ *is* $\frac{3}{2}$*-hypoanalytic in the sense of [16, Definition 2.14]. Now we will prove that, for every* $\varphi \in \mathrm{S}^{2+3|1/p-1/2|,p}(\mathbb{R})$ *and* $\psi \in \mathrm{S}^{3+3|1/p-1/2|,p}(\mathbb{R})$, *the corresponding solutions* $u_0(t)$ *and* $u_1(t)$ *of problem (1) are also* $\frac{3}{2}$*-hypoanalytic (with the clear meaning). Let* $K \subseteq (0, \infty)$ *be a compact set. By the proofs of [16, Lemma 2.15, Theorem 2.10] and the representation formula (25), it suffices to prove that, for every* $x \in X$, *the mapping* $t \mapsto H_C^1(t; 0, 0)x$, $t > 0$ *is* $\frac{3}{2}$*-hypoanalytic. With the notation used so far, we have that the mapping* $t \mapsto H_C^1(t; 0, 0)x$, $t > 0$ *is infinitely differentiable with*

$$\frac{d^{p'}}{dt^{p'}} H_C^1(\cdot; 0, 0)x = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \sum_{s=0}^{j} \frac{(-1)^j}{j!} \binom{k}{j} \binom{j}{s} \mu_0^s \tilde{A}_1{}^k \frac{d^{p'-(s+k-j)}}{dt^{p'-(s+k-j)}} \Big[\cdot^j T_1(\cdot)x \Big], \tag{28}$$

for any $p' \in \mathbb{N}_0$, *where we have put* $\frac{d^v}{dt^v}[\cdot^j T_1(\cdot)x] \equiv g_{-v} * [\cdot^j T_1(\cdot)x]$ *if* $-v \in \mathbb{N}$. *The* $\frac{3}{2}$*-hypoanalyticity of above mapping now follows from the equality (28), the estimate*

$$\begin{aligned}
&\sup_{t\in K} \left\| \frac{d^{p'-(s+k-j)}}{dt^{p'-(s+k-j)}} \Big[\cdot^j T_1(\cdot)x \Big](t) \right\| \\
&\quad \leq \big(1 + c_K\big)^{k+p'} \Big[p'!^{3/2} (s + k - j)!^{(-3/2)} + (k - j + s - p' + 1)!^{-1} \Big],
\end{aligned}$$

which holds for any $p' \in \mathbb{N}_0$ *and an appropriately chosen constant* $c_K > 0$, *and a simple computation involving the* $\frac{3}{2}$*-hypoanalyticity of* $(T_1(t))_{t\geq 0}$. *We want also to note, without carrying out a deeper and detailed analysis, that our results can be applied in the analysis of equation*

$$\begin{aligned}
&\frac{\partial^2 u(t, x)}{\partial t^2} + \left(\rho_1 \frac{\partial^3}{\partial x^3} - \rho_2 \frac{\partial^2}{\partial x^2} \right) \frac{\partial u(t, x)}{\partial t} + \left(c \frac{\partial^2}{\partial x^2} + a(x) \right) u(t, x) = 0, \ t \geq 0, \\
&u(0, x) = \varphi(x), \ u_t(0, x) = \psi(x), \quad x \in \mathbb{R},
\end{aligned} \tag{29}$$

where $a \in L^\infty(\mathbb{R})$; *cf. [37, Example 4.2] and [14, Example 5.3] for more details. Speaking-matter-of-factly, the estimates obtained in the proof of Theorem 2.1(b), in combination with [37, Theorem 2.7(a)] (cf. also [14, Theorem 3.5(b)]), indicate that there exists an exponentially bounded* $(I - \Delta)^{-(3/2)|1/p-1/2|}$*-existence family* $(E(t))_{t\geq 0}$ *for (27), in the sense of [37, Definition 2.1], and that there exist* $M \geq 1$ *and* $\omega \geq 0$ *such that* $||E(t)|| + ||E'(t)|| \leq Me^{\omega t}$, $t \geq 0$. *Designate* $\mathrm{S}^{0,2}(\mathbb{R}) := L^\infty(\mathbb{R})$. *Then the perturbation result [17, Theorem 2.3(i)] implies that there exists an exponentially bounded* $(I - \Delta)^{-(3/2)|1/p-1/2|}$*-existence family* $(E_0(t))_{t\geq 0}$ *for (29), provided that* $a \in L^\infty(\mathbb{R}) \cap \mathrm{S}^{3|1/p-1/2|,p}(\mathbb{R})$. *If the function* $a(x)$ *satisfies the above condition, then there exists a unique solution of (29) provided* $\varphi \in \mathrm{S}^{2+3|1/p-1/2|,p}(\mathbb{R})$, $\psi \in \mathrm{S}^{3+3|1/p-1/2|,p}(\mathbb{R})$, $a\varphi \in \mathrm{S}^{3|1/p-1/2|,p}(\mathbb{R})$ *and* $a\psi \in \mathrm{S}^{3|1/p-1/2|,p}(\mathbb{R})$. *Notice that T.-J. Xiao and J. Liang have imposed in [37, Example 4.2] much stronger conditions* $a \in W^{3,\infty}(\mathbb{R})$ *and* $\varphi \in \mathrm{S}^{5,p}(\mathbb{R})$, $\psi \in \mathrm{S}^{6,p}(\mathbb{R})$. *Consider now the problem*

$$\begin{aligned}
&u_{ttt}(t, x) + i\rho\Delta u_{tt}(t, x) + \sum_{|\alpha|\leq 2} a_\alpha D^\alpha u_t(t, x) + \sum_{|\beta|\leq 2} b_\beta D^\beta u(t, x) = 0, \ t \geq 0, \\
&u(0, x) = \varphi(x), \ u_t(0, x) = \psi(x), \ u_{tt}(0, x) = \phi(x), \quad x \in \mathbb{R}^n,
\end{aligned} \tag{30}$$

where $\rho \in \mathbb{R} \setminus \{0\}$ and $a_\alpha,\ b_\beta \in \mathbb{C}$ ($|\alpha|,\ |\beta| \leq 2$). Let $X = L^p(\mathbb{R}^n)$ for some $p \in (1,\infty)$. Then the operator $-i\rho\Delta$ generates an exponentially bounded $(I-\Delta)^{-n|1/p-1/2|}$-regularized semigroup on X, and

$$\lim_{\lambda\to+\infty} \lambda^{1-\kappa}\Big\|\big(\lambda+i\rho\Delta\big)^{-1}\Big\| = 0, \quad \kappa > 0,$$

because the operator Δ generates a bounded analytic semigroup of angle $\pi/2$ on X. By Theorem 2.1, we know that there exists a unique solution of (30) provided $\varphi,\ \psi,\ \phi \in \mathrm{S}^{2+2n|1/p-1/2|,p}(\mathbb{R}^n)$. In [35, Example 6.2.6], the authors have considered the case $n = 3$ and $p \in (6/5, 6)$, where the assumptions $\varphi,\ \psi,\ \phi \in \mathrm{S}^{4,p}(\mathbb{R}^3)$ have been required for the existence and uniqueness of solutions of (30); notice that our result produces better results here since $2 + 6|1/p - 1/2| < 4$ for any $p \in (6/5, 6)$.

(ii) *Let X be one of the spaces $L^p(\mathbb{R}^n)$ ($1 \leq p \leq \infty$), $C_0(\mathbb{R}^n)$, $C_b(\mathbb{R}^n)$, $BUC(\mathbb{R}^n)$, and let $0 \leq l \leq n$. Put $\mathbb{N}_0^l := \{\eta \in \mathbb{N}_0^n : \eta_{l+1} = \cdots = \eta_n = 0\}$. Then the space X_l is defined by $X_l := \{f \in X; f^{(\eta)} \in X$ for all $\eta \in \mathbb{N}_0^l\}$, and totality of seminorms $(q_\eta(f) := ||f^{(\eta)}||_X,\ f \in X_l;\ \eta \in \mathbb{N}_0^l)$ induces a Fréchet topology on X_l. Let the symbol $\mathbf{T_1}\langle\cdot\rangle$ possess the same meaning as in [18, Remark 2.2]; cf. also [35, Chapter 1] for more details. Suppose $1 \leq \sigma < 2$, $n = 3$, $A_2 := -e^{i(2-\sigma)\frac{\pi}{2}}\Delta$, $A_1 := \sum_{|\beta|\leq 1} a_\beta D^\beta$ ($a_\beta \in \mathbb{C}$, $|\beta| \leq 1$), $\gamma > n/2$, resp. $\gamma = n|1/p-1/2|$ if $1 < p < \infty$ and $X = L^p(\mathbb{R}^n)$. Set $C := \mathbf{T_0}\langle(1+|x|^2)^{-\gamma}\rangle$ and consider the equation (1) with $\alpha_3 = \alpha_2 + \sigma$ and $\alpha_2 \in [\sigma, 2)$. By [18, Theorem 2.1, Remark 2.2], we know that the operator $-A_2$ is the integral generator of a global (g_σ, C)-regularized resolvent family $(R_\sigma(t))_{t\geq 0}$ satisfying that there exists $M \geq 1$ such that*

$$\begin{aligned} &q_\eta\big(R_\sigma(t)f\big) \leq M\big(1+t^{n/2}\big)q_\eta(f),\ t \geq 0,\ f \in X_l,\ \eta \in \mathbb{N}_0^l,\ \textit{resp.,} \\ &q_\eta\big(R_\sigma(t)f\big) \leq M\big(1+t^{n|\frac{1}{p}-\frac{1}{2}|}\big)q_\eta(f),\ t \geq 0,\ f \in X_l,\ \eta \in \mathbb{N}_0^l. \end{aligned} \tag{31}$$

The estimate (11) is also valid since, for every $\zeta > 0$, the operator Δ generates an exponentially bounded analytic ζ-times integrated semigroup of angle $\pi/2$ on X, satisfying additionally an estimate like (31). If $1 < \sigma < 2$, resp. $\sigma = 1$, then Theorem 2.1(b) shows that the equation (1) has a unique strong solution provided that $u_0,\ u_1 \in C(D(A_1))$ and $u_2 \in C(D(A_2))$, resp. $u_0 \in C(D(A_1))$ and $u_1 \in C(D(A_2))$; if $X = L^p(\mathbb{R}^n)$ for some $p \in (1,\infty)$, and $l = 0$, this simply means that $u_0,\ u_1 \in \mathrm{S}^{2n|1/p-1/2|+1,p}(\mathbb{R}^n)$ and $u_2 \in \mathrm{S}^{2n|1/p-1/2|+2,p}(\mathbb{R}^n)$, resp., $u_0 \in \mathrm{S}^{2n|1/p-1/2|+1,p}(\mathbb{R}^n)$ and $u_1 \in \mathrm{S}^{2n|1/p-1/2|+2,p}(\mathbb{R}^n)$. It can be easily seen that the use of integrated operator solution families produces here weaker results; however, it should be noted that the non-existence of an appropriate reference which systematically treats the generation of $(g_\sigma, g_{\sigma r+1})$-regularized resolvent families by coercive differential operators (cf. [18] for the notion) additionally hinders possibility of proper applications of Theorem 2.1(a). As an illustrative example, we would like to quote the operator $e^{i(2-\sigma)\frac{\pi}{2}}\Delta$ acting on $L^1(\mathbb{R})$ with its maximal distributional domain ($1 < \sigma < 2$); then it is not clear whether there exists a number $\zeta \in (0,1)$ such that $e^{i(2-\sigma)\frac{\pi}{2}}\Delta$ generates an exponentially bounded $(g_\sigma, g_{1+\zeta})$-regularized resolvent family, see e.g. [3, Example 3.7] and [13, Example 23]. Notice, finally, that Theorem 2.1(b) can be applied with $C = I$ if $X = L^2(\mathbb{R}^n)$ and the operator $-A_2$ satisfies the conditions clarified in the formulation of [18, Theorem 2.2].

Before stating the following theorem, we would like to recall that the number $s_{l,k,\sigma} = \max(\lceil\frac{1}{\sigma}(\alpha_l - k + \sigma r)\rceil, 0)$ has been already defined in the proof of Theorem 2.1, for any $k \in \mathbb{N}^0_{m_n-1}$ and $l \in \mathbb{N}_{n-1} \setminus D_k$.

Theorem 2.4. *Suppose $n \in \mathbb{N} \setminus \{1,2\}$, $\sigma \in (0,2]$, $r \geq 0$, $\alpha_n - \alpha_{n-1} = \sigma$, $M \geq 1$, $\omega \geq 0$, $D(A_{n-1}) \subseteq \bigcap_{i=0}^{n-2} D(A_i)$ and $(\omega^\sigma, \infty) \subseteq \rho(-A_{n-1})$. Put $\grave{A}_i(\lambda)x := \lambda^{\alpha_i-\alpha_{n-1}}(\lambda^\sigma + A_{n-1})^{-1}A_ix$, $b_i := \max(\lceil\sigma^{-1}(\alpha_i - \alpha_{n-1} + \sigma r + 1)\rceil, 0)$ and $v_i := \max(\lceil\sigma^{-1}(\alpha_i - \alpha_{n-1} + 1)\rceil, 0)$ for $x \in D(A_{n-1})$, $\lambda > \omega$ and $i \in \mathbb{N}_{n-2}$. Let $\mu_0 < -\omega^\sigma$. If*

(a) *The operator $-A_{n-1}$ is the integral generator of a $(g_\sigma, g_{\sigma r+1})$-regularized resolvent family $(S_{\sigma,r}(t))_{t\geq 0}$ satisfying (23) as well as*

$$p\Big(\big(\mu_0 - A_{n-1}\big)^{b_i}A_ix\Big) \leq M\Big[p(x) + p\big(A_{n-1}x\big)\Big], \tag{32}$$

for any $x \in D\big(A_{n-1}\big)$, $p \in \circledast$, $i \in \mathbb{N}_{n-2}$, and $A_lu_k \in D(A_{n-1}^{s_{l,k,\sigma}})$, provided $0 \leq k \leq m_n - 1$ and $l \in \mathbb{N}_{n-1} \setminus D_k$, *or*

(b) *The operator* $-A_{n-1}$ *is the integral generator of a* (g_σ, C)*-regularized resolvent family* $(T_\sigma(t))_{t\geq 0}$ *satisfying (23) with* $(S_{\sigma,r}(t))_{t\geq 0}$ *replaced by* $(T_\sigma(t))_{t\geq 0}$ *therein, as well as*

$$p\Big(\big(\mu_0 - A_{n-1}\big)^{v_i} C^{-1} A_i x\Big) \leq M\Big[p(x) + p\big(A_{n-1}x\big)\Big], \tag{33}$$

for any $x \in D\big(A_{n-1}\big)$, $p \in \circledast$, $i \in \mathbb{N}_{n-2}$, *and* $A_l u_k \in R(C)$, *provided* $0 \leq k \leq m_n - 1$ *and* $l \in \mathbb{N}_{n-1} \setminus D_k$,

or

(c) *The operator* $-A_{n-1}$ *is the integral generator of a* (g_σ, C)*-regularized resolvent family* $(T_\sigma(t))_{t\geq 0}$ *satisfying (23) with* $(S_{\sigma,r}(t))_{t\geq 0}$ *replaced by* $(T_\sigma(t))_{t\geq 0}$ *therein, as well as (a) holds and* $A_l u_k \in R(C)$, *provided* $0 \leq k \leq m_n - 1$ *and* $l \in \mathbb{N}_{n-1} \setminus D_k$,

then the abstract Cauchy problem (1) has a unique strong solution.

Proof. We shall only consider the case in which X is a Banach space; although technically complicated, the proof of theorem in general case is quite similar and follows from the proofs of [35, Theorem 1.1.11] and Theorem 2.1, along with the dominated convergence theorem and the sequential completeness of X. In any of the cases (a), (b) or (c) set out above, the uniqueness of strong solutions is a simple consequence of Lemma 1.2(ii); because of that, we shall only prove the existence of such solutions. Suppose first that (a) holds. Using the generalized resolvent equation

$$\big(z - A_{n-1}\big)^{-1}\big(\lambda - A_{n-1}\big)^{-k} x = \frac{(-1)^k}{(z-\lambda)^k}\big(z - A_{n-1}\big)^{-1} x + \sum_{i=1}^{k} \frac{(-1)^{k-i}\big(\lambda - A_{n-1}\big)^{-i} x}{\big(z-\lambda\big)^{k+1-i}}, \tag{34}$$

for any $x \in X$, $k \in \mathbb{N}_0$ and $\lambda,\ z \in \rho(A)$ with $z \neq \lambda$, we easily infer that, for every $m \in \{0,1\}$, $i \in \mathbb{N}_{n-2}$ and $x \in X$, we have $\lambda^{\alpha_i - \alpha_{n-1}} A_{n-1}^m (\lambda^\sigma + A_{n-1})^{-1}(\mu_0 - A_{n-1})^{-b_i} x \in LT - X$. Keeping in mind (32), it readily follows that there exist $M' \geq M$ and $\omega' \geq \omega$ (universal constants in the remaining part of proof, possibly different from line to line) such that, for every $i \in \mathbb{N}_{n-2}$, $m \in \{0,1\}$ and $x \in D(A_{n-1})$, there exists a continuous function $t \mapsto F_{m,i}(t;x)$, $t \geq 0$ so that $F_{1,i}(t;x) = A_{n-1}F_{0,i}(t;x)$, $t \geq 0$, $x \in D(A_{n-1})$,

$$\big\|F_{m,i}(t;x)\big\| \leq M' e^{\omega' t}\Big\|\big(\mu_0 - A_{n-1}\big)^{b_i} A_i x\Big\| \leq M' e^{\omega' t}\Big[\|x\| + \big\|A_{n-1}x\big\|\Big],$$

provided $t \geq 0$, $x \in D(A_{n-1})$, and

$$\lambda^{\alpha_i - \alpha_{n-1}} A_{n-1}^m \big(\lambda^\sigma + A_{n-1}\big)^{-1} A_i x = \int\limits_0^\infty e^{-\lambda t} F_{m,i}(t;x)\,dt,\ x \in D\big(A_{n-1}\big),\ \lambda > \omega'.$$

Setting $F_{0,i}(t)x := F_{0,i}(t;x)$, $t \geq 0$, $x \in D(A_{n-1})$, it is not difficult to prove that $(F_{0,i}(t))_{t\geq 0} \subseteq L([D(A_{n-1})])$ is exponentially bounded, strongly continuous and that

$$\sum_{i=1}^{n-2} \grave{A}_i(\lambda)x = \int\limits_0^\infty e^{-\lambda t} \sum_{i=1}^{n-2} F_{0,i}(t)x\,dt, \quad x \in D\big(A_{n-1}\big),\ \lambda > \omega'.$$

In particular, there exists $c \in (0, 1/(n-2))$ such that, for every $x \in D\big(A_{n-1}\big)$, $\lambda > \omega'$ and $i \in \mathbb{N}_{n-2}$,

$$\big\|\grave{A}_i(\lambda)x\big\| + \big\|A_{n-1}\grave{A}_i(\lambda)x\big\| \leq c\Big[\|x\| + \big\|A_{n-1}x\big\|\Big]. \tag{35}$$

Then the proof of Theorem 2.1 in combination with (35) shows that, for every $x \in D(A_{n-1})$ and $\lambda > \omega'$, the series

$$\mathcal{B}_\lambda x := \sum_{k=0}^{\infty}\left[-\sum_{i=1}^{n-2} \grave{A}_i(\lambda)\right]^k x$$

is convergent in the topology of $[D(A_{n-1})]$. Taking into account the equality $\alpha_n - \alpha_{n-1} = \sigma$, it can be easily seen that the operator P_λ is injective for $\lambda > \omega'$ as well as that

$$\mathcal{B}_\lambda \big(\lambda^\sigma + A_{n-1}\big)^{-1} x = \lambda^{\alpha_{n-1}} P_\lambda^{-1} x, \quad x \in X, \ \lambda > \omega'. \tag{36}$$

Define now $F_0(t) := -\sum_{i=1}^{n-2} F_{0,i}(t)$, $t \geq 0$. The foregoing arguments in combination with the proof of [35, Theorem 1.1.11] imply that

$$\mathcal{B}_\lambda x - x = \int\limits_0^\infty e^{-\lambda t} \sum_{k=1}^\infty F_0^{*,k}(t) x \, dt, \quad x \in D\big(A_{n-1}\big), \ \lambda > \omega'. \tag{37}$$

Since $A_l u_k \in D(A_{n-1}^{s_{l,k,\sigma}})$ for $0 \leq k \leq m_n - 1$ and $l \in \mathbb{N}_{n-1} \setminus D_k$, it is very simple to prove with the help of (34) that there exists a continuous function $t \mapsto G(t) \in [D(A_{n-1})]$, $t \geq 0$ such that $\|G(t)\| + \|A_{n-1}G(t)\| \leq M' e^{\omega' t}$, $t \geq 0$ and

$$\sum_{k=0}^{m_n-1} \sum_{l \in \mathbb{N}_{n-1} \setminus D_k} \lambda^{\alpha_l - k - 1} \big(\lambda^\sigma + A_{n-1}\big)^{-1} A_l u_k = \int\limits_0^\infty e^{-\lambda t} G(t) \, dt, \quad \lambda > \omega'. \tag{38}$$

Define $v(t) := G(t) + (\sum_{k=1}^\infty F_0^{*,k} * G)(t)$, $t \geq 0$. Using (36)-(38), we get that the mapping $t \mapsto v(t) \in [D(A_{n-1})]$, $t \geq 0$ is continuous, exponentially bounded and that

$$\tilde{v}(\lambda) = -\lambda^{\alpha_{n-1}} P_\lambda^{-1} \sum_{k=0}^{m_n-1} \sum_{l \in \mathbb{N}_{n-1} \setminus D_k} \lambda^{\alpha_l - k - 1} A_l u_k, \quad \lambda > \omega'. \tag{39}$$

Taken together with [35, Theorem 1.1.10], the equalities

$$\begin{aligned} A_i \int_0^\infty & e^{-\lambda t} \big(g_{\alpha_{n-1} - \alpha_i} * v\big)(t) \, dt \\ &= A_i \big(\mu_0 - A_{n-1}\big)^{-1} \lambda^{\alpha_i - \alpha_{n-1}} \big(\mu_0 - A_{n-1}\big) \tilde{v}(\lambda) \\ &= A_i \big(\mu_0 - A_{n-1}\big)^{-1} \mathcal{L}\big(g_{\alpha_{n-1} - \alpha_i} * \big[\mu_0 v(\cdot) - A_{n-1} v(\cdot)\big]\big)(\lambda), \quad \lambda > \omega', \end{aligned}$$

show that the mapping $t \mapsto A_i(g_{\alpha_{n-1}-\alpha_i} * v)(t)$, $t \geq 0$ is well defined and continuous ($i \in \mathbb{N}_{n-1}$). Keeping in mind that $\mathbf{D}_t^{\alpha_j} g_{k+1}(t)$ identically equals 0, if $m_j - 1 \geq k$ and $t \geq 0$, resp. $g_{k+1-\alpha_j}(t)$ if $m_j - 1 < k$ and $t \geq 0$ ($j \in \mathbb{N}_n$, $k \in \mathbb{N}_{m_n-1}^0$), a straightforward computation involving (39) yields that

$$\lambda^\sigma \tilde{v}(\lambda) + \mathcal{L}\left(\sum_{i=1}^{n-1} A_i \big(g_{\alpha_{n-1}-\alpha_i} * v\big)(t)\right)(\lambda) + \widetilde{I_0}(\lambda) = 0,$$

where $I_0(t) := \sum_{l=1}^{n-1} \sum_{k=m_l}^{m_n-1} A_l g_{k+1-\alpha_j}(t) u_k$, $t \geq 0$. The above simply implies that there exists a continuous, exponentially bounded function $t \mapsto V(t)$, $t \geq 0$ such that $v \in C^{(\lceil \sigma \rceil - 1)}([0,\infty) : X)$, $v^{(k)}(0) = 0$ for $0 \leq k \leq \lceil \sigma \rceil - 1$ and $\mathbf{D}_t^\sigma v(t) = V(t)$, $t \geq 0$. Then the uniqueness theorem for Laplace transform, along with the equality [3, (1.26)], shows that $v^{(k)}(0) = 0$, $k = 0, \cdots, \lceil \sigma \rceil - 1$ and

$$\mathbf{D}_t^\sigma v(t) + \sum_{i=1}^{n-1} A_i \big(g_{\alpha_{n-1}-\alpha_i} * v\big)(t) + I_0(t) = 0, \quad t \geq 0.$$

Now one can prove, without any substantial difficulties, that the function

$$u(t) := \sum_{k=0}^{m_n-1} g_{k+1}(t) u_k + \big(g_{\alpha_{n-1}} * v\big)(t), \quad t \geq 0,$$

is a strong solution of (1). Observing that for each $x \in D(A_{n-1})$ there exists $y \in X$ such that (cf. (33)), for every $\lambda > \omega'$ and $m \in \{0,1\}$,

$$\lambda^{\alpha_i-\alpha_{n-1}} A_{n-1}^m \big(\lambda^\sigma + A_{n-1}\big)^{-1} CC^{-1} A_i x \\ = \lambda^{\alpha_i-\alpha_{n-1}} A_{n-1}^m \big(\lambda^\sigma + A_{n-1}\big)^{-1} C\big(\mu_0 - A_{n-1}\big)^{-v_i} y,$$

and

$$\lambda^{\alpha_l-k-1} A_{n-1}^m \big(\lambda^\sigma + A_{n-1}\big)^{-1} CC^{-1} A_l u_k \in LT - X, \quad m \in \{0,1\},$$

provided $0 \leq k \leq m_n - 1$ and $l \in \mathbb{N}_{n-1} \setminus D_k$, the proof of theorem in the case that (b) holds can be deduced similarly. The proof of (c) becomes standard and therefore omitted. □

Remark 2.5. *(i) A careful examination of the proof of Theorem 2.4 shows the following. In the case that (a) holds, we have the following estimate on the growth rate of constructed solution $u(t)$:*

$$p\left(u(t) - \sum_{k=0}^{m_n-1} g_{k+1}(t) u_k\right) + p\left(A_{n-1}\left[u(t) - \sum_{k=0}^{m_n-1} g_{k+1}(t) u_k\right]\right) \\ \leq M' e^{\omega' t} \sum_{k=0}^{m_n-1} \sum_{l \in \mathbb{N}_{n-1} \setminus D_k} \sum_{q=0}^{s_{l,k,\sigma}} p\big(A_{n-1}^q A_l u_k\big), \quad t \geq 0, \ p \in \circledast.$$

Similarly, in the case that (b) or (c) holds, we have that

$$p\left(u(t) - \sum_{k=0}^{m_n-1} g_{k+1}(t) u_k\right) + p\left(A_{n-1}\left[u(t) - \sum_{k=0}^{m_n-1} g_{k+1}(t) u_k\right]\right) \\ \leq M' e^{\omega' t} \sum_{k=0}^{m_n-1} \sum_{l \in \mathbb{N}_{n-1} \setminus D_k} p\big(C^{-1} A_l u_k\big), \quad t \geq 0, \ p \in \circledast.$$

(ii) Keeping in mind the first part of this remark as well as the estimate (32), it can be easily seen that Theorem 2.4(a) provides a generalization of [36, Theorem (∗)] and [35, Theorem 3.4.2], where the cases $\sigma = 1$ and $\sigma = 2$ have been considered. Although formulated with an arbitrary number $r \geq 0$, the choice $\sigma r \notin \mathbb{N}$ does not produce here any refinement of already known results on the wellposedness of abstract Cauchy problems [36, (1.1)] and [35, (4.1), p. 111] (cf. also Remark 2.2(v)). It is also worth noting that [36, Theorem (∗)] has been generalized in [36, Proposition 3.4, Theorem 3.5]; the proofs of these results rely upon a similar analysis on the Banach space $(D(A^p), \|\cdot\|_p)$, where $p \geq 2$ and $\|x\|_p \equiv \|x\| + \cdots + \|A^p x\|$, $x \in D(A^p)$. Witout giving full details, we wish to observe only that Theorem 2.4(b), compared with [36, Theorem 3.5], can produce a larger set of initial date for which a strong solution of problem [36, (1.1)] exists.

(iii) There exists a large number of concrete examples in which the condition (i) stated in the formulation of Theorem 2.1 is not fulfilled, in many of them Theorem 2.4(c) is applicable and produces better results than Theorem 2.4(a). Notice also that Theorem 2.1 can be applied only in the case that $\sigma \in [1,2]$ and $\alpha_{n-1} - \alpha_{n-2} \geq \sigma$. Using the recent results from [18], we can provide several applications of Theorem 2.4 with $\sigma \in (0,1)$.

3. Inhomogeneous abstract multi-term Cauchy problems

In this section, we shall consider the well-posedness results for the inhomogeneous Cauchy problem:

$$\begin{aligned} &\mathbf{D}_t^{\alpha_n} u(t) + \sum_{i=1}^{n-1} A_i \mathbf{D}_t^{\alpha_i} u(t) = f(t), \quad t > 0, \\ &u^{(k)}(0) = u_k, \quad k = 0, \cdots, m_n - 1, \end{aligned} \tag{40}$$

where $f \in C([0,\infty) : X)$. Let the estimate (23) hold with $(S_{\sigma,r}(t))_{t\geq 0}$, and with $(S_{\sigma,r}(t))_{t\geq 0}$ replaced by $(T_\sigma(t))_{t\geq 0}$ therein. Suppose first that the assumptions of Theorem 2.1(a) hold as well as that the mapping $t \mapsto (\mu_0 - A_{n-1})^{\lceil \sigma^{-1}(\sigma r+1)\rceil} f(t)$, $t \geq 0$ is continuous and satisfies that, for every $p \in ⊛$, there exists $c_p > 0$ such that

$$p\Bigl(\bigl(\mu_0 - A_{n-1}\bigr)^{\lceil \sigma^{-1}(\sigma r+1)\rceil} f(t)\Bigr) \leq c_p e^{\omega t}, \quad t \geq 0. \tag{41}$$

Then (22) implies that the function

$$B_\lambda \tilde{f}(\lambda) = \lambda^\sigma \bigl(\mu_0 - A_{n-1}\bigr)^{\lceil \sigma^{-1}(\sigma r+1)\rceil} \times \int_0^\infty e^{-\lambda t} H^\sigma\bigl(t; \lceil \sigma^{-1}(\sigma r+1)\rceil, -\sigma\bigr) \tilde{f}(\lambda)\, dt$$

belongs to the class $LT - X$. Therefore, there exists a function $v_f \in C([0,\infty) : X)$ such that $\lambda^{-\sigma} B_\lambda \tilde{f}(\lambda) = \int_0^\infty e^{-\lambda t} v_f(t)\, dt$, $\lambda > \omega$ suff. large. Set $u_f(t) := (g_{\alpha_{n-1}-\sigma} * v_f)(t)$, $t \geq 0$. Then $\widetilde{u_f}(\lambda) = P_\lambda^{-1} \tilde{f}(\lambda)$ for $\lambda > \omega$ suff. large, and it is not difficult to prove with the help of (22) that, for every $j \in \mathbb{N}_{n-1}$,

$$\begin{aligned}
&\lambda^{\alpha_j}\bigl(\mu_0 - A_{n-1}\bigr) P_\lambda^{-1} \tilde{f}(\lambda) \\
&= \lambda^{\alpha_j+\sigma-\alpha_n}\bigl(\mu_0 - A_{n-1}\bigr)^{1-\lceil \sigma^{-1}(\sigma r+1)\rceil} B_\lambda \bigl(\mu_0 - A_{n-1}\bigr)^{\lceil \sigma^{-1}(\sigma r+1)\rceil} \tilde{f}(\lambda) \\
&= \lambda^{\alpha_j+\sigma-\alpha_n} \lambda^{\sigma r+1-\sigma-\bigl(\lceil \sigma^{-1}(\sigma r+1)\rceil - 1\bigr)\sigma} \\
&\times \int_0^\infty e^{-\lambda t} H^\sigma \Bigl(t; \lceil \sigma^{-1}(\sigma r+1)\rceil - 1, \sigma r + 1 - \sigma - \bigl(\lceil \sigma^{-1}(\sigma r+1)\rceil - 1\bigr)\sigma\Bigr) \\
&\times \bigl(\mu_0 - A_{n-1}\bigr)^{\lceil \sigma^{-1}(\sigma r+1)\rceil} \tilde{f}(\lambda)\, dt \in LT - X,
\end{aligned}$$

because $\sigma r + 1 - \sigma - \lceil \sigma^{-1}(\sigma r+1)\rceil\sigma + 2\sigma + \alpha_j - \alpha_n \leq 0$. The above implies that, for every $j \in \mathbb{N}_{n-1}$, $\lambda^{\alpha_j} A_j P_\lambda^{-1} \tilde{f}(\lambda) \in LT - X$. By [35, Theorem 1.1.10], we may conclude that, for every $j \in \mathbb{N}_{n-1}$, the mapping $t \mapsto A_j \mathbf{D}_t^{\alpha_j} u_f(t)$, $t \geq 0$ is well defined, continuous and that

$$\int_0^\infty e^{-\lambda t} A_j \mathbf{D}_t^{\alpha_j} u_f(t)\, dt = \lambda^{\alpha_j} A_j P_\lambda^{-1} \tilde{f}(\lambda), \tag{42}$$

for $\lambda > \omega$ suff. large. Now a trivial computation involving the uniqueness theorem for the Laplace transform shows that (40) holds with $u_k = 0$ for $0 \leq k \leq m_n - 1$. Hence, the function $u(t) := u_f(t) + \sum_{k=0}^{m_n-1} u_k(t)$, $t \geq 0$, is a strong solution of (40), with the clear meaning. Furthermore,

$$u_f(\cdot) = H^\sigma\Bigl(\cdot; \max\bigl(\lceil \sigma^{-1}(\sigma r + 1 - \alpha_n)\rceil, -1\bigr), \alpha_{n-1}\Bigr) * \bigl(\mu_0 - A_{n-1}\bigr)^{\max\bigl(\lceil \sigma^{-1}(\sigma r+1-\alpha_n)\rceil, -1\bigr)} f(\cdot),$$

which implies that in the estimate of growth rate of $p(u(t))$, given after the equation (24), we need to add the additional term

$$M e^{\omega' t} \sup_{0\leq s\leq t} p\Biggl(\bigl(\mu_0 - A_{n-1}\bigr)^{\max\bigl(\lceil \sigma^{-1}(\sigma r+1-\alpha_n)\rceil, -1\bigr)} f(s)\Biggr), \quad t \geq 0. \tag{43}$$

In such a way, we have proved a proper extension of [39, Theorem 2.2]. Let us mention that the proof given above is different from that appearing in [39] and, in our opinion, much simpler even for the equations with integer order derivatives.

Suppose now that (b) holds as well as that the mapping $t \mapsto (\mu_0 - A_{n-1})^{\lceil \sigma^{-1}\rceil} C^{-1} f(t)$, $t \geq 0$ is well defined, continuous and satisfies that, for every $p \in ⊛$, there exists $c_p > 0$ such that

$$p\Bigl(\bigl(\mu_0 - A_{n-1}\bigr)^{\lceil \sigma^{-1}\rceil} C^{-1} f(t)\Bigr) \leq c_p e^{\omega t}, \quad t \geq 0.$$

Arguing in a similar fashion, we obtain that there exists a unique strong solution of (40) and that in the estimate of growth rate of $p(u(t))$ we need to add the additional term

$$Me^{\omega' t} \sup_{0\leq s\leq t} p\Big(\big(\mu_0 - A_{n-1}\big)^{-1} C^{-1} f(s)\Big), \quad t \geq 0.$$

Concerning inhomogeneous abstract multi-term Cauchy problems, Theorem 2.1 produces similar results as Theorem 2.4 and we shall explain this fact only in the case that $\sigma \in (0, 2)$ and the assumptions of Theorem 2.4(a) hold. Suppose that $u(t)$ is the solution of homogeneous counterpart of (40) with the initial values u_k $(0 \leq k \leq m_n - 1)$. Let the mapping $t \mapsto (\mu_0 - A_{n-1})^{\lceil \sigma^{-1}(\sigma r+1)\rceil} f(t)$, $t \geq 0$ be continuous, and let the estimate (41) hold, for any $p \in ⊛$ and a corresponding $c_p > 0$. Then the generalized resolvent equation (34) implies, along with the formulae [3, (1.26)-(1.27)] and (41), that

$$\lambda^{\sigma}\big(\lambda^{\sigma} + A_{n-1}\big)^{-1} \tilde{f}(\lambda) \in LT - X. \tag{44}$$

Designate $x_f(t) := \mathcal{L}^{-1}(\lambda^{\sigma}(\lambda^{\sigma} + A_{n-1})^{-1}\tilde{f}(\lambda))(t)$, $t \geq 0$ and $y_f(t) := \mathcal{L}^{-1}((\lambda^{\sigma} + A_{n-1})^{-1}\tilde{f}(\lambda))(t)$, $t \geq 0$. Taking into account [35, Theorem 1.1.10] and (44), it is very simple to prove that $(\mu_0 - A_{n-1})y_f(t) = \mu_0 y_f(t) - f(t) + x_f(t)$, $t \geq 0$. In the sequel, we shall employ the same notation as in the proof of Theorem 2.4; recall that the operator family $(Q(t) \equiv \sum_{k=0}^{\infty} F_0^{*,k}(t))_{t\geq 0} \subseteq L([D(A_{n-1})])$ is exponentially bounded. Then the mappings $t \mapsto \int_0^t Q(t-s)y_f(s)\,ds$, $t \geq 0$ and $t \mapsto \int_0^t (\mu_0 - A_{n-1})Q(t-s)y_f(s)\,ds$, $t \geq 0$ are well defined and exponentially bounded. Applying [35, Theorem 1.1.10] again, we obtain that

$$\big(\mu_0 - A_{n-1}\big) \int_0^{\infty} e^{-\lambda t} Q(t)\big(\lambda^{\sigma} + A_{n-1}\big)^{-1} \tilde{f}(\lambda)\,dt = \int_0^{\infty} e^{-\lambda t}\Big(\big(\mu_0 - A_{n-1}\big)Q * y_f\Big)(t)\,dt,$$

for $\lambda > \omega$ suff. large. For $j \in \mathbb{N}_{n-1}$ fixed, we obtain similarly that

$$\begin{aligned}\lambda^{\alpha_j}\big(\mu_0 - A_{n-1}\big)P_{\lambda}^{-1}\tilde{f}(\lambda) = \lambda^{\alpha_j - \alpha_{n-1}} \Bigg[&\big(\lambda^{\sigma} + A_{n-1}\big)^{-1}\big(\mu_0 - A_{n-1}\big)\tilde{f}(\lambda) \\ &+ \int_0^{\infty} e^{-\lambda t}\big(\mu_0 - A_{n-1}\big)Q(t)\big(\lambda^{\sigma} + A_{n-1}\big)^{-1}\tilde{f}(\lambda)\,dt \Bigg] \in LT - X.\end{aligned}$$

Using the resolvent equation, (44) and the foregoing arguments, we get that $\lambda^{\alpha_j} A_j P_{\lambda}^{-1}\tilde{f}(\lambda) \in LT - X$, $j \in \mathbb{N}_{n-1}$ and that (42) holds. Since

$$\lambda^{\alpha_{n-1}} A_{n-1} P_{\lambda}^{-1}\tilde{f}(\lambda) = \tilde{f}(\lambda) - \lambda^{\alpha_n} P_{\lambda}^{-1}\tilde{f}(\lambda) - \sum_{j=1}^{n-2} \lambda^{\alpha_j} A_j P_{\lambda}^{-1} P_{\lambda}^{-1}\tilde{f}(\lambda), \tag{45}$$

the above yields that $\lambda^{\alpha_n} P_{\lambda}^{-1}\tilde{f}(\lambda) \in LT - X$. Hence, there exists a unique continuous, exponentially bounded function $t \mapsto w_f(t)$, $t \geq 0$ such that $\mathcal{L}(w_f(t))(\lambda) = \lambda^{\alpha_n} P_{\lambda}^{-1}\tilde{f}(\lambda)$ for $\lambda > \omega$ suff. large. Set $U_f(t) := (g_{\alpha_n} * w_f)(t)$, $t \geq 0$. Then $U_f \in C^{m_n - 1}([0, \infty) : X)$, $(U_f)^{(k)}(0) = 0$ for $0 \leq k \leq m_n - 1$ and the Caputo derivative $\mathbf{D}_t^{\zeta} U_f(t)$ is defined for any $\zeta \in [0, \alpha_n]$. Furthermore, a simple computation involving the Laplace transform shows that the function $t \mapsto u(t) + U_f(t)$, $t \geq 0$ is a unique solution of the problem (40). By (45), we have that

$$\begin{aligned}\widetilde{U_f}(\lambda) = P_{\lambda}^{-1}\tilde{f}(\lambda) = \lambda^{-\alpha_n} \Bigg[&\tilde{f}(\lambda) - \sum_{j=1}^{n-2} \lambda^{\alpha_j} A_j \big(\mu_0 - A_{n-1}\big)^{-1}\big(\mu_0 - A_{n-1}\big)P_{\lambda}^{-1}\tilde{f}(\lambda) \\ &+ \lambda_{\alpha_{n-1}}\big(\mu_0 - A_{n-1}\big)P_{\lambda}^{-1}\tilde{f}(\lambda) - \mu_0 \lambda^{\alpha_{n-1}} P_{\lambda}^{-1}\tilde{f}(\lambda) \Bigg],\end{aligned}$$

for $\lambda > \omega$ suff. large. It can be simply checked with the help of (21) and the generalized resolvent equation (34) that

$$\lambda^{-\alpha_{n-1}}\big(\lambda^{\sigma}+A_{n-1}\big)^{-1}\big(\mu_0-A_{n-1}\big)^{-\max\left(\lceil\sigma^{-1}(\sigma r+1-\alpha_n)\rceil,-1\right)} \times \mathcal{L}\left(\big(\mu_0-A_{n-1}\big)^{\max\left(\lceil\sigma^{-1}(\sigma r+1-\alpha_n)\rceil,-1\right)}f\cdot\right)(\lambda)\in LT-X$$

and that the inverse Laplace transform of this function, denoted by $z(\cdot)$, satisfies that, for every $t \geq 0$,

$$\|z(t)\|\leq Me^{\omega' t}\sup_{0\leq s\leq t}\left\|\big(\mu_0-A_{n-1}\big)^{\max\left(\lceil\sigma^{-1}(\sigma r+1-\alpha_n)\rceil,-1\right)}f(s)\right\|,\quad t\geq 0. \tag{46}$$

If $\lceil\sigma^{-1}(\sigma r+1-\alpha_n)\rceil \geq 0$, then we can use (46), (32) and the equality

$$\begin{aligned}\lambda^{-\sigma}\big(\mu_0-A_{n-1}\big)P_{\lambda}^{-1}\tilde{f}(\lambda) &= \mu_0\lambda^{-\alpha_n}\big(\lambda^{\sigma}+A_{n-1}\big)^{-1}\tilde{f}(\lambda)+\lambda^{-\alpha_{n-1}}\big(\lambda^{\sigma}+A_{n-1}\big)^{-1}\tilde{f}(\lambda)-\lambda^{-\alpha_n}\tilde{f}(\lambda)\\ &+\lambda^{-\alpha_n}\int_0^{\infty}e^{-\lambda t}\big(\mu_0-A_{n-1}\big)Q(t)\big(\lambda^{\sigma}+A_{n-1}\big)^{-1}\tilde{f}(\lambda)\,dt,\end{aligned}$$

so as to conclude that, in the final estimate of growth rate of $p(u(t))$, we need to add the term appearing in (43). If $\lceil\sigma^{-1}(\sigma r+1-\alpha_n)\rceil \leq -1$, then the best we can do is show (a slightly weaker estimate than (43)) that, in the final estimate of growth rate of $p(u(t))$, one can add the term

$$Me^{\omega' t}\sup_{0\leq s\leq t}p(f(s)),\quad t\geq 0.$$

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