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On Bilateral Fuzzy Contractions

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Abstract. Not long ago, a hybrid type of contraction called bilateral contraction is introduced in the literature. Following this idea, in this paper, we initiate a novel type of contraction, known as bilateral fuzzy contraction, which combines two important results in fixed point theory. These are the Jaggi and Caristi type contractions. Our main motivation herein is to augment the literature by combining the aforementioned approaches, and thereby, extending and modifying these results from their crisp setting into the frame of fuzzy set theory. In support of the obtained results, an example is provided.

1. Introduction

Let (X, d) be a complete metric space. The well-celebrated Banach contraction principle (also called the Banach fixed point theorem) (see[6]) guarantees a unique fixed point if a mapping $T : X \longrightarrow X$ is a contraction, that is, if there exists a real number $\alpha \in (0, 1)$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq \alpha d(x, y).$$

As simple as the Banach fixed point theorem, it is the most applied result in the study of existence and uniqueness of solution of nonlinear problems arising in mathematics and its applications to engineering and life sciences. Shortly, in 1930, Caccioppli [9] published an analogue sort of Banach fixed point theorem. Due to the similarity between the two results, they are sometimes jointly called the Banach-Cacciopoli fixed point theorem. In the case of single-valued mappings, the aforementioned theorems have been generalized by many researchers in various ways (see, for example, [1, 2, 11, 14, 15, 18]) and the references therein. Two obvious intersecting properties of most generalizations of the Banach fixed point theorem is that their proofs are similar and the contractive conditions consist of linear combinations of the distances between two distinct points and their images. The first-two most embraced extensions of Banach-Cacciopoli principle involving rational inequalities were presented by Dass-Gupta [13] and Jaggi [17]. For our purpose in this paper, we recall the main result in [17] as follows.

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Theorem 1.1. *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be a continuous mapping. Further, Let T satisfies the condition:*

$$d(Tx, Ty) \leq \frac{\alpha d(x, Tx)d(y, Ty)}{d(x, y)} + \eta d(x, y),$$

for all $x, y \in X$, $x \neq y$ and for some $\alpha, \eta \in [0, 1)$ with $\alpha + \eta < 1$. Then T has a unique fixed point in X.

On the other hand, in 1976, Caristi [10] presented a fixed point theorem whose statement and proof are significantly different from Banach-Cacciopoli theorem. The main result of [10, Theorem 2.1] is as follows.

Theorem 1.2. *[10] Let (X, d) be complete metric space, A be a closed subset of X. Suppose that $\Lambda : A \longrightarrow A$ is an arbitrary function and $T : A :\longrightarrow X$ is continuous. If there exists a real number $\eta < 0$ such that*

$$d(\Lambda(x), T\Lambda(x)) \leq d(x, Tx) + \eta d(x, \Lambda(x)),$$

for all $x \in A$, then Λ has a fixed point.

Recently, a novel type of contraction called bilateral contraction was introduced by Chi-Ming et al [12]. The presented idea combined two well-known results in fixed point theory due to Caristi [10] and Jaggi [17]. For convenience, we recall the notion of bilateral contraction as follows (for details, see [12]).

Definition 1.3. *[12] Let (X, d) be a metric space. A mapping $T : X \longrightarrow X$ is called a Jaggi type bilateral contraction if there exists a function $\vartheta : X \longrightarrow [0, \infty)$ such that*

$$d(x, Tx) > 0 \text{ implies } d(Tx, Ty) \leq (\vartheta(x) - \vartheta(Tx))R_T(x, y),$$

for all $x, y \in X$, where

$$R_T(x, y) = \max\left\{d(x, y), \frac{d(x, Tx)d(y, Ty)}{d(x, y)}\right\}.$$

Definition 1.4. *[12] Let (X, d) be a metric space. A mapping $T : X \longrightarrow X$ is called a Dass-Gupta type bilateral contraction if there exists a function $\vartheta : X \longrightarrow [0, \infty)$ such that*

$$d(x, Tx) > 0 \text{ implies } d(Tx, Ty) \leq (\vartheta(x) - \vartheta(Tx))Q_T(x, y),$$

for all $x, y \in X$, where

$$Q_T(x, y) = \max\left\{d(x, y), \frac{1 + d(x, Tx)d(y, Ty)}{1 + d(x, y)}\right\}.$$

Away from single-valued mappings, in 1969, Nadler [20] initiated the study of fixed point theorems for multi-valued mappings. Nadler's contraction principle motivated many researchers and hence, the idea has been refined in different directions (see, for instance, [3, 4, 7, 19]).

Along the line, the arena of applied mathematics witnessed tremendous developments as a result of the introduction of fuzzy sets by Zadeh [23]. Classically, a fuzzy set is characterized by a membership function which assigns to each of its elements a grade of membership ranging between zero and one. In this continuation, Weiss [22] and Butnairu [8] pioneered the study of fixed points of fuzzy mappings. Whereas, fixed point theorems for fuzzy set-valued mappings have been investigated by Heilpern [16] who initiated the idea of fuzzy contractions and proved a fixed point theorem parallel to the Banach-Cacciopoli principle in the frame of fuzzy sets.

In the present paper, our aim is twofold. First, motivated by the results of [12] and the ideas of Caristi [10], Dass-Gupta [13] and Theorem 1.1 due to Jaggi [17], we define the notion of Jaggi-type and Dass-Gupta-type bilateral fuzzy contractions. Thereafter, we show that the existence of fuzzy fixed points for such contractions is guaranteed under certain suitable conditions. Our technique of presentation is adopted from [12, Theorem 1 and 2].

2. Preliminary

In this section, we record some basic concepts/results that are needed in the sequel. For these preliminaries, we follow [5, 16, 20, 23]. Throughout this paper, we denote the set of natural numbers, nonnegative reals and real numbers by $\mathbb{N}$, $\mathbb{R}^+$ and $\mathbb{R}$, respectively. Let (X, d) be a metric space, $\mathbb{K}(X)$ be the class of nonempty compact subsets of X. For $A, B \in \mathbb{K}(X)$, let $H : \mathbb{K}(X) \times \mathbb{K}(X) \longrightarrow \mathbb{R}$ be a mapping defined by

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(A, y)\right\},$$

where $d(x, A) = \inf_{y\in A} d(x, y)$. The mapping H is called the Hausdorff metric on $\mathbb{K}(X)$ induced by d.

Definition 2.1. *Let X be the family of objects denoted generically by x, then a fuzzy set A in X is a set of ordered pairs:*

$$A = \{(x, \mu_A(x)) : x \in X\},$$

where $\mu_A : X \longrightarrow I = [0, 1]$ *is called the membership function and* $\mu_A(x) = A(x)$ *is known as the membership value of x in A. We denote by* I^X*, the family of all fuzzy sets in X. The* α*-level set and the strong* α*-level set of a fuzzy set A in X are represented by* $[A]_\alpha$*, and* $[A]^*_\alpha$*, respectively, and are define by*

$$[A]_\alpha = \{x \in X : A(x) \geq \alpha\}, \text{ if } \alpha \in (0, 1],$$

and

$$[A]^*_\alpha = \{x \in X : A(x) > \alpha\}.$$

Example 2.2. *An organizing committee of an international conference wants to know the facilities needed to play host to the event. An indicated facility by some of the members is accommodation. If*

$$X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

is the set of all available accommodations, then the fuzzy set

$$A = \{\textit{few rooms are available}\}$$

can be seen graphically as in Figure 1. [title=Figure 1: Graphical representation of the fuzzy set in Example 2.2, xlabel=Available rooms, ylabel=Membership values, xmin=1, xmax=10, ymin=0, ymax=1, xtick=1,2,3,4,5,6, 7,8,9,10, ytick=0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1, legend pos=north west, ymajorgrids=true, grid style=dashed,] [color=green, mark=square,] coordinates (1,0.2)(2,0.4)(3,0.3)(4,0.8)(5,0.7)(6,0.1)(7,0.5)(8, 0.6)(9, 0.3) (10,1); Notice that in Figure 1, the α*-level set and strong* α*-level set for* $\alpha = 0.6$*, are respectively given by*

$$[A]_\alpha = \{4, 5, 8, 10\} \text{ and } [A]^*_\alpha = \{4, 5, 10\}.$$

Example 2.3. *The expression "real numbers considerably larger than 30" can be modelled as a fuzzy set A, given by*

$$A = \{(x, \mu_A(x)) : x \in X\},$$

where

$$\mu_A(x) = \begin{cases} 0, & \text{if } x \leq 30 \\ \left(1 + (x - 30)^{-2}\right)^{-1}, & \text{if } x > 30. \end{cases}$$

Definition 2.4. *Let X be an arbitrary set, Y be a metric space. A mapping* $T : X \longrightarrow I^Y$ *is called a fuzzy mapping. If T is a fuzzy mapping from X into* I^Y*, and* $x \in X$, $y \in Y$*, then the function value* $T(x)(y)$ *is the membership value of y in* $T(x)$*. The* α*-level set of T is represented by* $[Tx]_\alpha$.

Definition 2.5. *Let $S, T : X \longrightarrow I^X$ be fuzzy mappings. An element $u \in X$ is called a fuzzy fixed point of S if $u \in [Su]_\alpha$. The point u is called a common fuzzy fixed point of S and T if $u \in [Su]_\alpha \cap [Tu]_\alpha$.*

Example 2.6. *Let $X = [-5, 5]$ and $Y = [-4, 4]$. Define $T : X \longrightarrow I^Y$ by*

$$T(x)(y) = \frac{\sin^2 x + \cos^2 y}{20}.$$

Then T is a fuzzy mapping. Notice that $T(x)(y) \in [0, 1]$, for all $x \in X$ and $y \in Y$. The graphical representation showing the possible membership values of y in $T(x)$ is shown in Figure 2.

Example 2.7. *Let $X = Y = [-10, 10]$. Define $T : X \longrightarrow I^Y$ by*

$$T(x)(y) = \frac{|x|}{|x| + |y| + 5},$$

for all $x \in X$ and $y \in Y$. Then T is a fuzzy mapping. Figure 2 depicts the graphical representation of the membership values of y in $T(x)$.

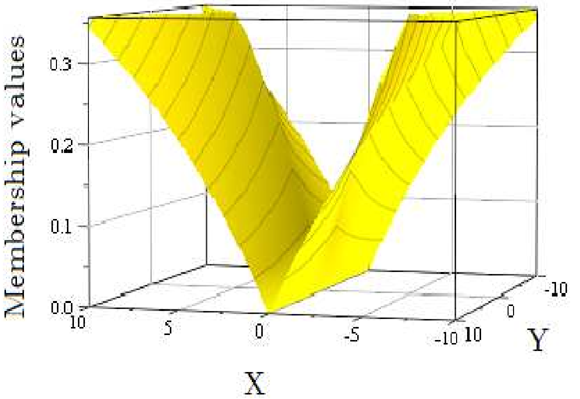

Figure 2: Graphical representation of the fuzzy mapping in Example 2.7

Lemma 2.8. *[20] Let (X, d) be a metric space and $A, B \in \mathbb{K}(X)$. If $a \in A$, then there exists $b \in B$ such that $d(a, b) \leq H(A, B)$.*

3. Main Result

Motivated by the famous result of Jaggi [17, Theorem 1], we start this section by giving the following definition.

Definition 3.1. *Let (X, d) be a metric space and $S, T : X \longrightarrow I^X$ be any two fuzzy mappings. Then, the pair (S, T) is said to form a Jaggi-type bilateral fuzzy contraction, if for every $x \in X$, there exists $\alpha_S(x), \alpha_T(x) \in (0, 1]$ and a continuous function $\Lambda : X \longrightarrow \mathbb{R}^+$ such that*

$$d(x, [Sx]_{\alpha_S(x)}) > 0 \text{ and } d(y, [Ty]_{\alpha_T(y)}) > 0$$

imply

$$H([Sx]_{\alpha_S(x)}, [Ty]_{\alpha_T(y)}) \leq (\Lambda(x) - \Lambda(y))\, M_{(S,T)}(x, y), \tag{1}$$

for all $x, y \in X$, $x \neq y$, where

$$M_{(S,T)}(x, y) = \max\left\{d(x, y), \frac{d(x, [Sx]_{\alpha_S(x)})d(y, [Ty]_{\alpha_T(y)})}{1 + d(x, y)}\right\}.$$

Theorem 3.2. *Let (X, d) be a complete metric space and $S, T : X \longrightarrow I^X$ be any two fuzzy mappings. Assume that for each $x \in X$, there exist $\alpha_S(x), \alpha_T(x) \in (0, 1]$ such that $[Sx]_{\alpha_S(x)}, [Tx]_{\alpha_T(x)} \in \mathbb{K}(X)$. If the pair (S, T) forms a Jaggi-type bilateral fuzzy contraction, then there exists $u \in X$ such that $u \in [Su]_{\alpha_S(u)} \cap [Tu]_{\alpha_T(u)}$.*

Proof. Let $x \in X$ be arbitrary and rename it as $x := x_0$. Then, by hypothesis, there exists $\alpha_S(x_0) \in (0, 1]$ such that $[Sx_0] \in \mathbb{K}(X)$. For simplicity, denote $\alpha_S(x_0)$ by α_1. Then, since $[Sx_0]_{\alpha_1} \in \mathbb{K}(X)$, there exists $x_1 \in [Sx_0]_{\alpha_1}$ such that $d(x_0, x_1) = d(x_0, [Sx_0]_{\alpha_1})$. Similarly, we have $\alpha_T(x_1) \in (0, 1]$ such that $[Tx_1]_{\alpha_T(x_1)} \in \mathbb{K}(X)$. Denote $\alpha_T(x_1)$ by α_2, then, since $[Tx_1]_{\alpha_2} \in \mathbb{K}(X)$, we can find $x_2 \in [Tx_1]_{\alpha_2}$ such that $d(x_1, x_2) = d(x_1, [Tx_1]_{\alpha_2})$. Recursively, we construct a sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of X, with

$$x_{2k+1} \in [Sx_{2k}]_{2k+1} \text{ and } x_{2k+2} \in [Tx_{2k+1}]_{2k+2},\ k \in \mathbb{N}.$$

Notice that if there exists some $k^* \in \mathbb{N}$ such that $x_{k^*+1} = x_{k^*} \in [Sx_{k^*}]_{\alpha_S(x)} \cap [Tx_{k^*}]_{\alpha_T(x)}$, then the proof is finished. So, suppose $x_k \neq x_{k+1}$ for all $k \in \mathbb{N}$. Then

$$d(x_{2k}, x_{2k+1}) = d(x_{2k}, [Sx_{2k}]_{\alpha_{2k+1}}) > 0$$

and

$$d(x_{2k+1}, x_{2k+2}) = d(x_{2k+1}, [Tx_{2k+1}]_{\alpha_{2k+2}}) > 0.$$

Hence, by Lemma 2.8, and the contractive condition (1), we have

$$\begin{aligned} d(x_{2k}, x_{2k+1}) \leq & H([Sx_{2k-1}]_{\alpha_{2k}}, [Tx_{2k}]_{\alpha_{2k+1}}) \\ & \leq (\Lambda(x_{2k-1}) - \Lambda(x_{2k}))\, M_{(S,T)}(x_{2k-1}, x_{2k}) \\ & \leq (\Lambda(x_{2k-1}) - \Lambda(x_{2k})) \max\Bigg\{ d(x_{2k-1}, x_{2k}), \\ & \quad \frac{d(x_{2k-1}, [Sx_{2k-1}]_{\alpha_{2k}}) d(x_{2k}, [Tx_{2k}]_{\alpha_{2k+1}})}{1 + d(x_{2k-1}, x_{2k})} \Bigg\} \\ & \leq (\Lambda(x_{2k-1}) - \Lambda(x_{2k})) \max\left\{ d(x_{2k-1}, x_{2k}), \frac{d(x_{2k-1}, x_{2k}) d(x_{2k}, x_{2k+1})}{1 + d(x_{2k-1}, x_{2k})} \right\} \\ & \leq (\Lambda(x_{2k-1}) - \Lambda(x_{2k})) \max\left\{ d(x_{2k-1}, x_{2k}), d(x_{2k}, x_{2k+1}) \right\}. \end{aligned} \tag{2}$$

Now, we consider the following cases:

Case 1: Assume that $\max\{d(x_{2k-1}, x_{2k}), d(x_{2k}, x_{2k+2})\} = d(x_{2k-1}, x_{2k})$. Then, on account of (2), we have

$$d(x_{2k}, x_{2k+1}) \leq (\Lambda(x_{2k-1}) - \Lambda(x_{2k}))\, d(x_{2k-1}, x_{2k}).$$

In other words,

$$0 < \frac{d(x_{2k}, x_{2k+1})}{d(x_{2k-1}, x_{2k})} \leq (\Lambda(x_{2k-1}) - \Lambda(x_{2k})). \tag{3}$$

In this case, by repeating the above steps, for all $n \in \mathbb{N}$, we have

$$0 < \frac{d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} \leq \Lambda(x_{n-1}) - \Lambda(x_n). \tag{4}$$

(4) implies that $\Lambda(x_n) \leq \Lambda(x_{n-1})$, for all $n \in \mathbb{N}$. Hence, the sequence $\{\Lambda(x_n)\}_{n\in\mathbb{N}}$ is decreasing and positive, and thus converges to some $p \geq 0$. Further, notice that

$$\begin{aligned} 0 < \sum_{i=1}^{n} \frac{d(x_i, x_{i+1})}{d(x_{i-1}, x_i)} \leq & \sum_{i=1}^{n} (\Lambda(x_{i-1}) - \Lambda(x_i)) \\ & = (\Lambda(x_0) - \Lambda(x_1)) + (\Lambda(x_1) - \Lambda(x_2)) + \cdots + (\Lambda(x_{n-1}) - \Lambda(x_n)) \\ & = \Lambda(x_0) - \Lambda(x_n) \longrightarrow \Lambda(x_0) - p < \infty \text{ as } n \longrightarrow \infty. \end{aligned} \tag{5}$$

It follows that

$$0 < \frac{d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} < \infty, \text{ for all } n \in \mathbb{N}.$$

Consequently,

$$\lim_{n\longrightarrow\infty} \frac{d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} = 0. \tag{6}$$

By (6), for $\eta \in (0,1)$, there exists $m \in \mathbb{N}$ such that for all $n \geq m$,

$$\frac{d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} \leq \eta.$$

Equivalently,

$$d(x_n, x_{n+1}) \leq \eta d(x_{n-1}, x_n). \tag{7}$$

Case 2: Assume that $\max\{d(x_{2k-1}, x_{2k}), d(x_{2k}, x_{2k+1})\} = d(x_{2k}, x_{2k+1})$. Then, using (2), we get

$$d(x_{2k}, x_{2k+1}) \leq (\Lambda(x_{2k-1}) - \Lambda(x_{2k}))\, d(x_{2k}, x_{2k+1}). \tag{8}$$

Taking $2k = n \in \mathbb{N}$ in (8), yields

$$d(x_n, x_{n+1}) \leq (\Lambda(x_{n-1} - \Lambda(x_n)))\, d(x_n, x_{n+1}). \tag{9}$$

As $n \longrightarrow \infty$ in (9),

$$1 \leq (\Lambda(x_{n-1}) - \Lambda(x_n)) \longrightarrow 0,$$

gives a contradiction. Combining the results from Case 1 and Case 2, and using (7), we conclude that the sequence $\{d(x_n, x_{n+1})\}_{n\in\mathbb{N}}$ is decreasing and bounded below, and hence converges to its infimum, say $\rho \geq 0$. Since $\eta < 1$, then clearly, $\eta = 0$. Now, from (7), for each $i, j \in \mathbb{N}$ with $i > j$, we get

$$d(x_i, x_j) \leq \sum_{t=i}^{j-1} d(x_t, x_{t+1}) \leq \frac{\eta^i}{1-\eta} d(x_0, x_1).$$

Hence, $\lim_{i\longrightarrow\infty} \sup\left\{d(x_i, x_j) : i > j\right\} = 0$. This proves that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy. By completeness of X, there exists $u \in X$ such that $x_n \longrightarrow u$ as $n \longrightarrow \infty$. To show that $u \in [Su]_{\alpha_S(u)}$, we apply Lemma 2.8 as follows:

$$\begin{aligned}
d(u, [Su]_{\alpha_S(u)}) &\leq d(u, x_{2n}) + d(x_{2n}, [Su]_{\alpha_S(u)}) \\
&\leq d(u, x_{2n}) + H([Tx_{2n-1}]_{\alpha_{2n}}, [Su]_{\alpha_S(u)}) \\
&\leq d(u, x_{2n}) + (\Lambda(u) - \Lambda(x_{2n-1})) \max\left\{d(u, x_{2n-1}), \frac{d(u, [Su]_{\alpha_S(u)}) d(x_{2n-1}, [Tx_{2n-1}]_{\alpha_{2n}})}{1 + d(u, x_{2n-1})}\right\} \\
&\leq d(u, x_{2n}) + (\Lambda(u) - \Lambda(x_{2n-1})) \max\left\{d(u, x_{2n}), \frac{d(u, [Su]_{\alpha_S(u)}) d(x_{2n-1}, x_{2n})}{1 + d(u, x_{2n-1})}\right\}.
\end{aligned} \tag{10}$$

Letting $n \longrightarrow \infty$ in (10), and using the continuity of Λ, we have

$$\begin{aligned}
d(u, [Su]_{\alpha_S(u)}) &\leq d(u, u) + (\Lambda(u) - \Lambda(u)) \max\left\{d(u, u), \frac{d(u, [Su]_{\alpha_S(u)}) d(u, u)}{1 + d(u, u)}\right\} \\
&\leq 0.
\end{aligned}$$

This implies that $u \in [Su]_{\alpha_S(u)}$. On similar steps, by using

$$d(u, [Tu]_{\alpha_T(u)}) \leq d(u, x_{2n}) + d(x_{2n}, [Tu]_{\alpha_T(u)}),$$

one can show that $u \in [Tu]_{\alpha_T(u)}$. Consequently, $u \in [Su]_{\alpha_S(u)} \cap [Tu]_{\alpha_T(u)}$. □

Corollary 3.3. *Let (X,d) be a complete metric space and $S : X \longrightarrow I^X$ be a fuzzy mapping. Assume that for each $x \in X$, there exists $\alpha_S(x) \in (0,1]$ such that $[Sx]_{\alpha_S(x)} \in \mathbb{K}(X)$. If there exists a continuous function $\Lambda : X \longrightarrow \mathbb{R}^+$ such that*

$$d(x,[Sx]_{\alpha_S(x)}) > 0 \text{ and } d(y,[Sy]_{\alpha_S(y)}) > 0$$

imply

$$H([Sx]_{\alpha_S(x)},[Sy]_{\alpha_S(y)}) \leq (\Lambda(x) - \Lambda(y))M_{(S,S)},$$

for all $x,y \in X$, $x \neq y$, where

$$M_{(S,S)} = \max\left\{d(x,y), \frac{d(x,[Sx]_{\alpha_S(x)})d(y,[Sy]_{\alpha_S(y)})}{1+d(x,y)}\right\},$$

then, there exists $u \in X$ such that $u \in [Su]_{\alpha_S(u)}$.

Proof. Putting $S = T$ in Theorem 3.2 completes the proof. □

Corollary 3.4. *Let (X,d) be a complete metric space and $S,T : X \longrightarrow I^X$ be any two fuzzy mappings. Assume that for each $x \in X$, there exists $\alpha_S(x), \alpha_T(x) \in (0,1]$ such that $[Sx]_{\alpha_S(x)}, [Tx]_{\alpha_T(x)} \in \mathbb{K}(X)$. If there exists a continuous function $\Lambda : X \longrightarrow \mathbb{R}^+$ such that*

$$d(x,[Sx]_{\alpha_S(x)}) > 0 \text{ and } d(y,[Ty]_{\alpha_T(y)}) > 0$$

imply

$$H([Sx]_{\alpha_S(x)},[Ty]_{\alpha_T(y)}) \leq (\Lambda(x) - \Lambda(y))\left[r_1 d(x,y) + r_2\frac{d(x,[Sx]_{\alpha_S(x)})d(y,[Ty]_{\alpha_T(y)})}{1+d(x,y)}\right],$$

for all $x,y \in X$, $x \neq y$, where $\sum_{i=1}^{2} r_i = 1$, then, there exists $u \in X$ such that $u \in [Su]_{\alpha_S(u)} \cap [Tu]_{\alpha_T(u)}$.

Corollary 3.5. *Let (X,d) be a complete metric space and $S,T : X \longrightarrow I^X$ be any two fuzzy mappings. Assume that for each $x \in X$, there exists $\alpha_S(x), \alpha_T(x) \in (0,1]$ such that $[Sx]_{\alpha_S(x)}, [Tx]_{\alpha_T(x)} \in \mathbb{K}(X)$. If there exists a continuous function $\Lambda : X \longrightarrow \mathbb{R}^+$ such that*

$$d(x,[Sx]_{\alpha_S(x)}) > 0 \text{ and } d(y,[Ty]_{\alpha_T(y)}) > 0$$

imply

$$H([Sx]_{\alpha_S(x)},[Ty]_{\alpha_T(y)}) \leq (\Lambda(x) - \Lambda(y))\left(\frac{d(x,[Sx]_{\alpha_S(x)})d(y,[Ty]_{\alpha_T(y)})}{1+d(x,y)}\right),$$

for all $x,y \in X$, $x \neq y$, then, there exists $u \in X$ such that $u \in [Su]_{\alpha_S(u)} \cap [Tu]_{\alpha_T(u)}$.

Remark 3.6. *By setting $S = T$ in corollaries 3.4 and 3.5, we can derive another corollaries analogous to Corollary 3.3.*

Encouraged by the results of Dass-Gupta [13], we introduce the concept of Dass-Gupta type bilateral fuzzy contraction as follows:

Definition 3.7. *Let (X,d) be a metric space and $S,T : X \longrightarrow I^X$ be any two fuzzy mappings. Then, the pair (S,T) is said to form a Dass-Gupta- type bilateral fuzzy contraction, if for every $x \in X$, there exist $\alpha_S(x), \alpha_T(x) \in (0,1]$ and a continuous function $\Lambda : X \longrightarrow \mathbb{R}^+$ such that*

$$d(x,[Sx]_{\alpha_S(x)}) > 0 \text{ and } d(y,[Ty]_{\alpha_T(y)}) > 0$$

imply

$$H([Sx]_{\alpha_S(x)}, [Ty]_{\alpha_T(y)}) \leq (\Lambda(x) - \Lambda(y))\, W_{(S,T)}, \tag{11}$$

for all $x, y \in X$, $x \neq y$, *where*

$$W_{(S,T)} = \max\left\{d(x, y), \frac{(1 + d(x, [Sx]_{\alpha_S(x)}))d(y, [Ty]_{\alpha_T(y)})}{1 + d(x, y)}\right\}.$$

Theorem 3.8. *Let* (X, d) *be a complete metric space and* $S, T : X \longrightarrow I^X$ *be any two fuzzy mappings. Assume that for each* $x \in X$*, there exist* $\alpha_S(x), \alpha_T(x) \in (0, 1]$ *such that* $[Sx]_{\alpha_S(x)}, [Tx]_{\alpha_T(x)} \in \mathbb{K}(X)$*. If the pair* (S, T) *forms a Dass-Gupta -type bilateral fuzzy contraction, then there exists* $u \in X$ *such that* $u \in [Su]_{\alpha_S(u)} \cap [Tu]_{\alpha_T(u)}$*.*

Proof. Following the proof of Theorem 3.2, we generate a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points of X such that

$$x_{2k+1} \in [Sx_{2k}]_{\alpha_{2k+1}} \text{ and } x_{2k+2} \in [Tx_{2k+1}]_{\alpha_{2k+2}},\ k \in \mathbb{N}.$$

Taking

$$d(x_{2k}, x_{2k+1}) = d(x_{2k}, [Sx_{2k}]_{\alpha_{2k+1}}) > 0$$

and

$$d(x_{2k+1}, x_{2k+2}) = d(x_{2k+1}, [Tx_{2k+1}]_{\alpha_{2k+1}}) > 0,$$

Lemma 2.8 and the contractive condition 11 guarantee that

$$\begin{aligned} d(x_{2k}, x_{2k+1}) \leq& H([Sx_{2k-1}]_{\alpha_{2k}}, [Tx_{2k}]_{\alpha_{2k+1}}) \\ \leq& (\Lambda(x_{2k-1}) - \Lambda(x_{2k}))\, W_{(S,T)}(x_{2k-1}, x_{2k}) \\ \leq& (\Lambda(x_{2k-1}) - \Lambda(x_{2k})) \max\left\{d(x_{2k-1}, x_{2k}), \frac{(1 + d(x_{2k-1}, [Sx_{2k-1}]))d(x_{2k}, [Tx_{2k}]_{\alpha_{2k+1}})}{1 + d(x_{2k-1}, x_{2k})}\right\} \\ \leq& (\Lambda(x_{2k-1}) - \Lambda(x_{2k})) \max\left\{d(x_{2k-1}, x_{2k}), \frac{(1 + d(x_{2k-1}, x_{2k}))d(x_{2k}, x_{2k+1})}{(1 + d(x_{2k-1}, x_{2k}))}\right\} \\ \leq& (\Lambda(x_{2k-1}) - \Lambda(x_{2k})) \max\{d(x_{2k-1}, x_{2k}), d(x_{2k}, x_{2k+1})\}. \end{aligned} \tag{12}$$

From (12), we can follow all the steps in Case 1 and Case 2 of Theorem 3.2. After then, we conclude that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Using the completeness of X, there exists $u \in X$ such that $x_n \longrightarrow u$ as $n \longrightarrow \infty$. To prove that $u \in [Tu]_{\alpha_T(u)}$, we appeal to Lemma 2.8 as follows:

$$\begin{aligned} d(u, [Tu]_{\alpha_T(u)}) \leq& d(u, x_{2n}) + d(x_{2n}, [Tu]_{\alpha_T(u)}) \\ \leq& d(u, x_{2n}) + H([Sx_{2n-1}]_{\alpha_{2n}}, [Tu]_{\alpha_T(u)}) \\ \leq& d(u, x_{2n}) + (\Lambda(x_{2n-1}) - \Lambda(u)) \\ &\times \max\left\{d(u, x_{2n-1}), \frac{(1 + d(x_{2n-1}, [Sx_{2n-1}]_{\alpha_{2n}}))d(u, [Tu]_{\alpha_T(u)})}{(1 + d(x_{2n-1}, u))}\right\} \\ \leq& d(u, x_{2n}) + (\Lambda(x_{2n-1}) - \Lambda(u)) \\ &\times \max\left\{d(u, x_{2n-1}), \frac{(1 + d(x_{2n-1}, x_{2n}))d(u, [Tu]_{\alpha_T(u)})}{1 + d(x_{2n-1}, u)}\right\}. \end{aligned} \tag{13}$$

Taking limit in (13) as $n \longrightarrow \infty$, and using the continuity of Λ, we have

$$d(u, [Tu]_{\alpha_T(u)}) \leq 0\left(d(u, [Tu]_{\alpha_T(u)})\right) = 0.$$

Hence, $u \in [Tu]_{\alpha_T(u)}$. On similar steps, one can show that $u \in [Su]_{\alpha_S(u)}$. Therefore, $u \in [Su]_{\alpha_S(u)} \cap [Tu]_{\alpha_T(u)}$. □

Corollary 3.9. *Let (X, d) be a complete metric space and $S : X \longrightarrow I^X$ be a fuzzy mapping. Assume that for each $x \in X$, there exists $\alpha_S(x) \in (0, 1]$ such that $[Sx]_{\alpha_S(x)} \in \mathbb{K}(X)$. If there exists a continuous function $\Lambda : X \longrightarrow \mathbb{R}^+$ such that*

$$d(x, [Sx]_{\alpha_S(x)}) > 0 \text{ and } d(y, [Sy]_{\alpha_S(y)}) > 0$$

imply

$$H([Sx]_{\alpha_S(x)}, [Sy]_{\alpha_S(y)}) \leq (\Lambda(x) - \Lambda(y))W_{(S,S)},$$

for all $x, y \in X$, $x \neq y$, where

$$M_{(S,S)} = \max\left\{d(x, y), \frac{(1 + d(x, [Sx]_{\alpha_S(x)}))d(y, [Sy]_{\alpha_S(y)})}{1 + d(x, y)}\right\},$$

then there exists $u \in X$ such that $u \in [Su]_{\alpha_S(u)}$.

Proof. Putting $S = T$ in Theorem 3.8 completes the proof. □

Corollary 3.10. *Let (X, d) be a complete metric space and $S, T : X \longrightarrow I^X$ be any two fuzzy mappings. Assume that for each $x \in X$, there exists $\alpha_S(x), \alpha_T(x) \in (0, 1]$ such that $[Sx]_{\alpha_S(x)}, [Tx]_{\alpha_T(x)} \in \mathbb{K}(X)$. If there exists a continuous function $\Lambda : X \longrightarrow \mathbb{R}^+$ such that*

$$d(x, [Sx]_{\alpha_S(x)}) > 0 \text{ and } d(y, [Ty]_{\alpha_T(y)}) > 0$$

imply

$$H([Sx]_{\alpha_S(x)}, [Ty]_{\alpha_T(y)}) \leq (\Lambda(x) - \Lambda(y))\left[r_1 d(x, y) + r_2 \frac{(1 + d(x, [Sx]_{\alpha_S(x)}))d(y, [Ty]_{\alpha_T(y)})}{1 + d(x, y)}\right],$$

for all $x, y \in X$, $x \neq y$, where $\sum_{i=1}^{2} r_i = 1$, then there exists $u \in X$ such that $u \in [Su]_{\alpha_S(u)} \cap [Tu]_{\alpha_T(u)}$.

Corollary 3.11. *Let (X, d) be a complete metric space and $S, T : X \longrightarrow I^X$ be any two fuzzy mappings. Assume that for each $x \in X$, there exists $\alpha_S(x), \alpha_T(x) \in (0, 1]$ such that $[Sx]_{\alpha_S(x)}, [Tx]_{\alpha_T(x)} \in \mathbb{K}(X)$. If there exists a continuous function $\Lambda : X \longrightarrow \mathbb{R}^+$ such that*

$$d(x, [Sx]_{\alpha_S(x)}) > 0 \text{ and } d(y, [Ty]_{\alpha_T(y)}) > 0$$

imply

$$H([Sx]_{\alpha_S(x)}, [Ty]_{\alpha_T(y)}) \leq (\Lambda(x) - \Lambda(y))\left(\frac{(1 + d(x, [Sx]_{\alpha_S(x)}))d(y, [Ty]_{\alpha_T(y)})}{1 + d(x, y)}\right),$$

for all $x, y \in X$, $x \neq y$, then, there exists $u \in X$ such that $u \in [Su]_{\alpha_S(u)} \cap [Tu]_{\alpha_T(u)}$.

Remark 3.12. *By setting $S = T$ in corollaries 3.10 and 3.11, we can derive another corollaries parallel to Corollary 3.9.*

In what follows, we provide an example to support the hypotheses of theorems 3.2.

Example 3.13. *Let $X = \left\{(2, 2), (2, 3), (4, 4)\right\}$ be equipped with the taxicab metric $d : X \times X \longrightarrow \mathbb{R}$, given by*

$$d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|.$$

Clearly, (X, d) is a complete metric space. Suppose that $S, T : X \longrightarrow I^X$ are fuzzy mappings defined as follows:

$$S(2,2)(t) = S(2,3)(t) = S(4,4)(t) = \begin{cases} \frac{2}{13}, & \text{if } t = (2,2) \\ \frac{4}{5}, & \text{if } t = (2,3) \\ 0, & \text{if } t = (4,4), \end{cases}$$

$$T(2,2)(t) = T(4,4)(t) = \begin{cases} 0, & \text{if } t = (2,2) \\ \frac{3}{7}, & \text{if } t = (2,3) \\ \frac{4}{5}, & \text{if } t = (4,4), \end{cases}$$

and

$$T(2,3)(t) = \begin{cases} \frac{2}{5}, & \text{if } t = (2,2) \\ \frac{3}{5}, & \text{if } t = (2,3) \\ 0, & \text{if } t = (4,4). \end{cases}$$

Also, define $\Lambda : X \longrightarrow \mathbb{R}^+$ as follows:

$$\Lambda(2,2) = 15, \ \Lambda(2,3) = 7, \ \Lambda(4,4) = 20.$$

Take $\alpha_S(x) = 0.2$ and $\alpha_T(x) = 0.5$, for all $x \in X$. Then,

$$\begin{aligned} [S(2,2)]_{\alpha_S(x)} &= \{t \in X : S(2,2)(t) \geq \alpha_S(x)\} \\ &= [S(2,3)]_{\alpha_S(x)} = [S(4,4)]_{\alpha_S(x)} \\ &= \{(2,3)\} \in \mathbb{K}(X), \end{aligned}$$

and

$$\begin{aligned} [T(2,2)]_{\alpha_T(x)} &= \{t \in X : T(2,2)(t) \geq \alpha_T(x)\} \\ &= [T(4,4)]_{\alpha_T(x)} = \{(4,4)\} \in \mathbb{K}(X). \end{aligned}$$

Similarly, $[T(2,3)]_{\alpha_T(x)} = \{(2,3)\} \in \mathbb{K}(X)$.
Now,

$$\begin{aligned} d((2,2), [S(2,2)]_{\alpha_S(x)}) &= \inf\{d((2,2), y) : y \in [S(2,2)]_{\alpha_S(x)}\} \\ &= 1. \end{aligned}$$

In like manner,

$$d((2,3), [S(2,3)]_{\alpha_S(x)}) = 0, \ d((4,4), [S(4,4)]_{\alpha_S(x)}) = 3.$$

Also,

$$\begin{aligned} d((2,2), [T(2,2)]_{\alpha_T(x)}) &= \inf\{d((2,2), y) : y \in [T(2,2)]_{\alpha_T(x)}\} \\ &= 4. \end{aligned}$$

On identical steps,

$$d((2,3), [T(2,3)]_{\alpha_T(x)}) = 3 \text{ and } d((4,4), [T(4,4)]_{\alpha_T(x)}) = 0.$$

Now, considering

$$d((2,2), [S(2,2)]_{\alpha_S(x)}) > 0 \text{ and } d((2,3), [T(2,3)]_{\alpha_T(x)}) > 0,$$

we have

$$H([S(2,2)]_{\alpha_S(x)}, [T(2,3)]_{\alpha_T(x)}) = 3,$$

$$\Lambda((2,2)) - \Lambda((2,3)) = 8, \text{and } M_{(S,T)}((2,2),(2,3)) = \frac{3}{2}.$$

Therefore,

$$H([S(2,2)]_{\alpha_S(x)}, [T(2,3)]_{\alpha_T(x)}) \leq (\Lambda((2,2)) - \Lambda((2,3)))\, M_{(S,T)}((2,2),(2,3)).$$

Again, using

$$d((4,4), [S(4,4)]_{\alpha_S(x)}) > 0 \text{ and } d((2,2), [T(2,2)]_{\alpha_T(x)}) > 0,$$

we get

$$H([S(4,4)]_{\alpha_S(x)}, [T(2,3)]_{\alpha_T(x)}) = 4,$$

$$\Lambda((4,4)) - \Lambda((2,2)) = 5, \text{and } M_{(S,T)}((4,4),(2,2)) = 4.$$

Hence,

$$H([S(4,4)]_{\alpha_S(x)}, [T(2,2)]_{\alpha_T(x)}) \leq (\Lambda((4,4)) - \Lambda((2,2)))\, M_{(S,T)}((4,4),(2,2)).$$

Thus, for all $x, y \in X$ with $x \neq y$,

$$d(x, [Sx]_{\alpha_S(x)}) > 0 \text{ and } d(y, [Ty]_{\alpha_T(y)}) > 0$$

imply

$$H([Sx]_{\alpha_S(x)}, [Ty]_{\alpha_T(y)}) \leq (\Lambda(x) - \Lambda(y))\, M_{(S,T)}(x,y),$$

where

$$M_{(S,T)}(x,y) = \max\left\{d(x,y), \frac{d(x,[Sx]_{\alpha_S(x)})d(y,[Ty]_{\alpha_T(y)})}{1+d(x,y)}\right\}.$$

It follows that all the conditions of Theorem 3.2 are satisfied. In this case, we see that there exists $(2,3) \in X$ such that $(2,3) \in [S(2,3)]_{\alpha_S} \cap [T(2,3)]_{\alpha_T}$.

Competing Interests

The authors declare that they have no competing interests.

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