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On the approximate point spectra of m-complex symmetric operators, $[m, C]$-symmetric operators and others

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Abstract. In this paper we study properties of approximate point spectra of m-complex symmetric operators, $[m, C]$-symmetric operators, and others on a complex Hilbert space $\mathcal{H}$.

[thm]Lemma

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space, $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$ and $\langle \cdot, \cdot \rangle$ be the inner product of $\mathcal{H}$. Let $\mathbb{N}$ be the set of all natural numbers.

For $T \in B(\mathcal{H})$ and $m \in \mathbb{N}$, the operator $\alpha_m(T)$ is defined by

$$\alpha_m(T) := \sum_{j=0}^{m} (-1)^j \binom{m}{j} T^{*j} \cdot T^{m-j}.$$

An operator $T \in B(\mathcal{H})$ is said to be *m-symmetric* if $\alpha_m(T) = 0$.

Though Helton [6] called such T *coadjoint order* $(m-1)$ and McCullough and Rodman [10] called it *m-selfadjoint*. It holds that 1-symmetric is Hermitian and if T is m-symmetric, then $\sigma(T) \subset \mathbb{R}$. For results of m-symmetric operators, see Gu and Stankus [5], Helton [6] and McCullough and Rodman [10].

Definition 1.1. *An antilinear operator C on $\mathcal{H}$ is said to be a conjugation if C satisfies $C^2 = I$ and $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$.*

Notice that if C is a conjugation on $\mathcal{H}$, then C is an isometry, so C is continuous.

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Definition 1.2. *Let C be a conjugation on $\mathcal{H}$. An operator $T \in B(\mathcal{H})$ is said to be complex symmetric if $CTC = T^*$.*

In [3] and [4], Garcia and Putinar showed nice results of complex symmetric operators and in [7] and [8], Jung, Ko, and Lee studied spectral properties of complex symmetric operators.

Definition 1.3. *For an operator $T \in B(\mathcal{H})$ and a conjugation C, we define the operator $\delta_m(T;C)$ by*

$$\delta_m(T;C) := \sum_{j=0}^{m} (-1)^j \binom{m}{j} T^{*j} \cdot CT^{m-j}C.$$

An operator $T \in B(\mathcal{H})$ is said to be m-complex symmetric if $\delta_m(T;C) = 0$.

It is clear that 1-complex symmetric is complex symmetric and, actually, an $n \times n$ Toeplitz matrix is complex symmetric. See Chō, Ko, and Lee [1] about results of m-complex symmetric operators.

Lemma 1. *Let $T \in B(\mathcal{H})$ and C be a conjugation on $\mathcal{H}$. Then it holds that*

$$\delta_m(T;C) \cdot (CTC) - T^* \cdot \delta_m(T;C) = \delta_{m+1}(T;C).$$

Hence, if T is m-complex symmetric, then T is n-complex symmetric for all $n \geq m$.

Definition 1.4. *For an operator $T \in B(\mathcal{H})$ and a conjugation C, we define the operator $\alpha_m(T;C)$ by*

$$\alpha_m(T;C) := \sum_{j=0}^{m} (-1)^j \binom{m}{j} CT^{m-j}C \cdot T^j.$$

An operator $T \in B(\mathcal{H})$ is said to be $[m, C]$-symmetric if $\alpha_m(T;C) = 0$.

See Chō, Lee, Tanahashi, and Tomiyama [2] about results of $[m, C]$-symmetric operators. In particular, if T is complex symmetric and $[m, C]$-symmetric, then T is m-symmetric.

Lemma 2. *Let $T \in B(\mathcal{H})$ and C be a conjugation on $\mathcal{H}$. Then it holds that*

$$CTC \cdot \alpha_m(T;C) - \alpha_m(T;C) \cdot T = \alpha_{m+1}(T;C).$$

Hence, if T is $[m, C]$-symmetric, then T is $[n, C]$-symmetric for all $n \geq m$.

For a subset $A \subset \mathbb{C}$, let $A^* = \{\bar{z} : z \in A\}$. Let $\sigma(T)$ and $\sigma_p(T)$ be the spectrum and the point spectrum of $T \in B(\mathcal{H})$, respectively. The approximate point spectrum of T is defined by

$$\sigma_a(T) := \{ z \in \mathbb{C} : T - z \text{ is not bounded below} \}.$$

In this paper, we show the following results.

Theorem 1.5. *Let $T \in B(\mathcal{H})$ be an m-complex symmetric operator on $\mathcal{H}$ and let $\{x_n\}$ be a sequence of unit vectors. If, for $\lambda \in \mathbb{C}$, $\|(T-\lambda)x_n\| \to 0$ $(n \to \infty)$, then $\langle (T-\lambda)^m Cx_n, Cx_n \rangle \to 0$ $(n \to \infty)$. Hence, if $(T-\lambda)x = 0$, then $\langle (T-\lambda)^m Cx, Cx \rangle = 0$.*

Theorem 1.6. *Let $T \in B(\mathcal{H})$ be an $[m, C]$-symmetric operator on $\mathcal{H}$ and let $\{x_n\}$ be a sequence of unit vectors. If, for $\lambda \in \mathbb{C}$, $\|(T-\lambda)x_n\| \to 0$ $(n \to \infty)$, then $\|(T-\overline{\lambda})^m Cx_n\| \to 0$ $(n \to \infty)$. Hence, if, for $\lambda \in \mathbb{C}$, $(T-\lambda)x = 0$, then $(T-\overline{\lambda})^m Cx = 0$.*

2. Proof

Proof of Theorem 1.7. Let $\{x_n\}$ be a sequence of unit vectors of $\mathcal{H}$. Since $T \in B(\mathcal{H})$ is an m-complex symmetric operator with a conjugation C, it holds

$$CT^mC = -\sum_{j=1}^{m}(-1)^j\binom{m}{j}T^{*j}\cdot CT^{m-j}C.$$

Hence we have

$$\begin{aligned}
C(T-\lambda)^m\, C &= C\Big(\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{m-j}\lambda^j\Big)C \\
&= \sum_{j=0}^{m}(-1)^j\binom{m}{j}\overline{\lambda}^j\, CT^{m-j}C \\
&= CT^mC + \sum_{j=1}^{m}(-1)^j\binom{m}{j}\overline{\lambda}^j\, CT^{m-j}C \\
&= -\sum_{j=1}^{m}(-1)^j\binom{m}{j}T^{*j}\, CT^{m-j}C + \sum_{j=1}^{m}(-1)^j\binom{m}{j}\overline{\lambda}^j\, CT^{m-j}C \\
&= -\sum_{j=1}^{m}(-1)^j\binom{m}{j}\Big(T^{*j}-\overline{\lambda}^j\Big)CT^{m-j}C. \qquad (1)
\end{aligned}$$

Since $\|(T-\lambda)x_n\| \to 0$, it holds $\|(T^k-\lambda^k)x_n\| \to 0$ for all $k \in \mathbb{N}$ and hence, from (1)

$$\begin{aligned}
\langle (T-\lambda)^m\, Cx_n,\, Cx_n\rangle &= \langle x_n,\, C(T-\lambda)^m\, Cx_n\rangle \\
&= \langle x_n,\, -\sum_{j=1}^{m}(-1)^j\binom{m}{j}\Big(T^{*j}-\overline{\lambda}^j\Big)CT^{m-j}Cx_n\rangle \\
&= -\sum_{j=1}^{m}(-1)^j\binom{m}{j}\langle (T^j-\lambda^j)x_n,\, CT^{m-j}Cx_n\rangle \;\to\; 0 \;\; (n\to\infty).
\end{aligned}$$

The last statement holds by a similar way. □

Corollary 2.1. *Under the assumption of Theorem* 1.5*, the following properties hold.*
(i) $\langle (T^*-\overline{\lambda})^m x_n,\, x_n\rangle \to 0 \; (n\to\infty)$.
(ii) $\langle (T^k-\lambda^k)^m Cx_n,\, Cx_n\rangle \to 0 \; (n\to\infty)$ *for all* $k \in \mathbb{N}$.

Proof. Let $\{x_n\}$ be a sequence of unit vectors of $\mathcal{H}$. Then $\{Cx_n\}$ is a sequence of unit vectors of $\mathcal{H}$.

(i) If $\|(T-\lambda)x_n\| \to 0 \; (n\to\infty)$, then we get $\|(T^*-\overline{\lambda})Cx_n\| \to 0 \; (n\to\infty)$ from Theorem 4.1 (iii) in [1]. Since T^* is also m-complex symmetric, it follows from Theorem 1.5 that $\langle (T^*-\overline{\lambda})^m x_n,\, x_n\rangle \to 0 \; (n\to\infty)$.

(ii) Since T^k is an m-complex symmetric operator for all $k \in \mathbb{N}$ by Theorem 4.5 of [1] and $\|(T^k-\lambda^k)^m x_n\| \to 0 \; (n\to\infty)$, by Theorem 1.5 we have

$$\langle (T^k-\lambda^k)^m Cx_n,\, Cx_n\rangle \to 0 \; (n\to\infty).$$

□

Remark 2.2. Theorem 1.7 does not hold in general. Because let $T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $Cx = \begin{pmatrix} \overline{x_2} \\ \overline{x_1} \end{pmatrix}$ for $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ on $\mathbb{C}^2$. Then for a vector $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, it holds $Tx = 0$. But since $Cx = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we have

$$\langle TCx,\, Cx\rangle = 1 \neq 0.$$

If λ is a real number, then we have the following result.

Theorem 2.3. *Let $T \in B(\mathcal{H})$ be an m-complex symmetric operator on $\mathcal{H}$ and let $\{x_n\}$ be a sequence of unit vectors. If $\lambda \in \mathbb{R}$ and $\|(T-\lambda)x_n\| \to 0$ $(n \to \infty)$, then $\|(T^*-\lambda)^m Cx_n\| \to 0$ $(n \to \infty)$. Hence, if $(T-\lambda)x = 0$, then $(T^*-\lambda)^m Cx = 0$.*

Proof. Let $\{x_n\}$ be a sequence of unit vectors of $\mathcal{H}$. Since T is an m-complex symmetric operator with a conjugation C, it holds

$$(CT^mC)^* = CT^{*m}C = -\sum_{j=1}^{m}(-1)^j\binom{m}{j}CT^{*m-j}C \cdot T^j.$$

Hence, by the same calculation of the proof of Theorem 1.7 and $\lambda \in \mathbb{R}$, we have

$$\begin{aligned} C(T^*-\lambda)^m\, C &= C\Big(\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{*m-j}\lambda^j\Big)C \\ &= CT^{*m}C + \sum_{j=1}^{m}(-1)^j\binom{m}{j}C\Big(T^{*m-j}\lambda^j\Big)C \\ &= -\sum_{j=1}^{m}(-1)^j\binom{m}{j}CT^{*m-j}C\Big(T^j-\lambda^j\Big). \end{aligned} \tag{2}$$

Since $\|(T-\lambda)x_n\| \to 0$, it follows that $\|(T^k-\lambda^k)x_n\| \to 0$ for all $k \in \mathbb{N}$ and hence, from (2) we have

$$\begin{aligned} \|(T^*-\lambda)^m\, Cx_n\| &= \|C(T^*-\lambda)^m\, Cx_n\| \\ &= \|\sum_{j=1}^{m}(-1)^j\binom{m}{j}CT^{*m-j}C\cdot(T^j-\lambda^j)x_n\| \to 0 \end{aligned}$$

as $n \to \infty$. The last statement holds by a similar method. □

Proof of Theorem 1.8. Let $\{x_n\}$ be a sequence of unit vectors of $\mathcal{H}$ such that $\|(T-\lambda)x_n\| \to 0$. Since

$$(CTC-\overline{\lambda})Cx_n = C(T-\lambda)Cx_n \to 0,$$

for all $j \in \mathbb{N}$ it holds

$$(CT^jC-\overline{\lambda}^j)Cx_n \to 0 \ \ (n \to \infty). \tag{3}$$

By Theorem 3.2 of [2], T^* is $[m, C]$-symmetric. Hence

$$\alpha_m(T^*;C) = \sum_{j=0}^{m}(-1)^j\binom{m}{j}CT^{*j}C\cdot T^{*m-j} = 0$$

and

$$\alpha_m(T^*;C)^* = \sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{m-j}\cdot CT^jC = 0.$$

Therefore we have

$$0 = \Big(\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{m-j}\cdot CT^jC\Big)Cx_n$$

$$= T^m Cx_n + \sum_{j=1}^{m} (-1)^j \binom{m}{j} T^{m-j} \cdot CT^j C \cdot Cx_n$$
$$= (T - \overline{\lambda})^m Cx_n + \sum_{j=1}^{m} (-1)^j \binom{m}{j} T^{m-j} \cdot (CT^j C - \overline{\lambda}^j) Cx_n. \tag{4}$$

Therefore, from (3) and (4) we have

$$\lim_{n\to\infty} \|(T - \overline{\lambda})^m \, Cx_n\| = 0$$

Moreover, the last assertion holds by a similar way. □

Corollary 2.4. *Under the assumption of Theorem* 1.8, *it holds* $\|(T^k - \overline{\lambda}^k)^m Cx_n\| \to 0$ $(n \to \infty)$ *for all* $k \in \mathbb{N}$.

Proof. Since T^k is an $[m, C]$-symmetric operator for all $k \in \mathbb{N}$ by Theorem 3.2 of [2] and $\|(T^k - \lambda^k)^m x_n\| \to 0$ $(n \to \infty)$, by Theorem 1.8 we have

$$\|(T^k - \overline{\lambda}^k)^m Cx_n\| \;\to\; 0 \;\; (n \to \infty).$$

□

Remark 2.5. In general, for a $[1, C]$-symmetric operator T, it does not hold that if $(T - \lambda)x = 0$, then $(T - \lambda)Cx = 0$. Indeed, let

$$T = \begin{pmatrix} 2i & 1 \\ 1 & -2i \end{pmatrix} \text{ and } Cx = \begin{pmatrix} \overline{x_2} \\ \overline{x_1} \end{pmatrix} \text{ for } x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ on } \mathbb{C}^2.$$

Then it holds $CTC = T$ and hence T is $[1, C]$-symmetric. For the eigenvalue $\sqrt{3}\, i$ of T and the corresponding eigenvector $x = \begin{pmatrix} 1 \\ (\sqrt{3} - 2)i \end{pmatrix}$, we have

$$(T - \sqrt{3}\, i)Cx = \begin{pmatrix} 4\sqrt{3} - 6 \\ -2\sqrt{3}\, i \end{pmatrix} \neq 0 \text{ and } (T + \sqrt{3}\, i)Cx = 0.$$

Remark 2.6. The operator T in Remark 2.5 is an example of $[m, C]$-symmetric operator which is $\sigma(T) \not\subset \mathbb{R}$. But, $n \times n$ Toeplitz matrix is not [1,C]-symmetric, in general. Indeed, if

$$T = \begin{pmatrix} a_0 & a_{-1} \\ a_1 & a_0 \end{pmatrix} \text{ for } a_{-1}, a_0, a_1 \in \mathbb{C} \text{ and } Cx = \begin{pmatrix} \overline{x_2} \\ \overline{x_1} \end{pmatrix} \text{ for } x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ on } \mathbb{C}^2,$$

then $CTC = \begin{pmatrix} \overline{a_0} & \overline{a_1} \\ \overline{a_{-1}} & \overline{a_0} \end{pmatrix}$. Therefore, $CTC \neq T$ if $a_0 \neq \overline{a_0}$ or $a_1 \neq \overline{a_{-1}}$. On the other hand, since an $n \times n$ Toeplitz matrix is 1-complex symmetric, there are many examples of m-complex symmetric operator T which is $\sigma(T) \not\subset \mathbb{R}$.

3. Skew m-complex and skew $[m, C]$-symmetric

For an operator $T \in B(\mathcal{H})$ and a conjugation C, T is said to be *skew complex symmetric* if $CTC = -T^*$. Please see Li and Zhu [9] about skew complex symmetric operators.

Definition 3. For an operator $T \in B(\mathcal{H})$ and a conjugation C, we define the operator $\gamma_m(T; C)$ by

$$\gamma_m(T; C) := \sum_{j=0}^{m} \binom{m}{j} T^{*j} \cdot CT^{m-j} C.$$

An operator $T \in B(\mathcal{H})$ is said to be *skew m-complex symmetric* if $\gamma_m(T;C) = 0$. It is clear that skew 1-complex symmetric is skew complex symmetric. It holds that $T^* \gamma_m(T;C) + \gamma_m(T;C) \cdot CTC = \gamma_{m+1}(T;C)$. Therefore if T is skew m-complex symmetric, then T is skew n-complex symmetric for all $n \geq m$.

Theorem 3.1. *Let $T \in B(\mathcal{H})$ be a skew m-complex symmetric operator on $\mathcal{H}$ and $\{x_n\}$ be a sequence of unit vectors. If, for $\lambda \in \mathbb{C}$, $\|(T-\lambda)x_n\| \to 0$ $(n \to \infty)$, then $\langle (T+\lambda)^m Cx_n, Cx_n\rangle \to 0$ $(n \to \infty)$. Hence, if $(T-\lambda)x = 0$, then $\langle (T+\lambda)^m Cx, Cx\rangle = 0$.*

Proof. Since T is skew m-complex symmetric, we have

$$CT^mC = -\sum_{j=1}^{m} \binom{m}{j} T^{*j} \cdot CT^{m-j}C.$$

Since

$$C(T+\lambda)^m C = \sum_{j=0}^{m} \binom{m}{j} \overline{\lambda}^j \cdot CT^{m-j}C,$$

it holds

$$\begin{aligned} C(T+\lambda)^m C &= CT^mC + \sum_{j=1}^{m} \binom{m}{j} \overline{\lambda}^j\, CT^{m-j}C \\ &= -\sum_{j=1}^{m} \binom{m}{j} T^{*j}\, CT^{m-j}C + \sum_{j=1}^{m} \binom{m}{j} \overline{\lambda}^j\, CT^{m-j}C \\ &= -\sum_{j=1}^{m} \binom{m}{j} \left(T^{*j} - \overline{\lambda}^j\right) CT^{m-j}C. \end{aligned}$$

Since $\|(T-\lambda)x_n\| \to 0$, it holds $\|(T^k - \lambda^k)x_n\| \to 0$ for all $k \in \mathbb{N}$ and hence

$$\begin{aligned} &\langle (T+\lambda)^m\, Cx_n, Cx_n\rangle \\ &= \langle x_n, C(T+\lambda)^m\, Cx_n\rangle = \langle x_n, -\sum_{j=1}^{m} \binom{m}{j} \left(T^{*j} - \overline{\lambda}^j\right) CT^{m-j}Cx_n\rangle \\ &= -\sum_{j=1}^{m} \binom{m}{j} \langle (T^j - \lambda^j)x_n, CT^{m-j}Cx_n\rangle \to 0 \ \ (n \to \infty). \end{aligned}$$

□

Remark 3.2. By Theorem 4.5 of [1], if T is m-complex symmetric, then so is T^n for every $n \in \mathbb{N}$. But there exists a skew 1-complex symmetric operator T such that T^2 is not skew 1-complex symmetric. For example, let

$$T = \begin{pmatrix} 1+i & 0 \\ 0 & -1-i \end{pmatrix} \text{ and } Cx = \begin{pmatrix} \overline{x_2} \\ \overline{x_1} \end{pmatrix} \text{ for } x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ on } \mathbb{C}^2.$$

Then it is easy to see $CTC = \begin{pmatrix} -1+i & 0 \\ 0 & 1-i \end{pmatrix} = -T^*$ and hence T is skew 1-complex symmetric. But since $T^2 = \begin{pmatrix} 2i & 0 \\ 0 & 2i \end{pmatrix}$, we have $CT^2C = T^{2*}$ and hence T^2 is complex symmetric and not skew 1-complex symmetric. □

Theorem 3.3. *Let T be a skew m-complex symmetric operator on $\mathcal{H}$ and $\{x_n\}$ be a sequence of unit vectors. If $\lambda \in \mathbb{C}$ and $\|(T-\lambda)x_n\| \to 0$ $(n \to \infty)$, then $\|(T^* + \overline{\lambda})^m Cx_n\| \to 0$ $(n \to \infty)$. Hence, if $(T-\lambda)x = 0$, then $(T^* + \overline{\lambda})^m Cx = 0$.*

Proof. By the assumption, we have, for any $j \in \mathbb{N}$,

$$(CT^jC - \overline{\lambda}^j)Cx_n \to 0 \ \ (n \to \infty). \tag{5}$$

Since $\gamma_m(T;C) = \sum_{j=0}^{m} \binom{m}{j} T^{*j} \cdot CT^{m-j}C = 0$, it holds

$$C\,(\gamma_m(T;C))^* C = \sum_{j=0}^{m} \binom{m}{j} T^{*m-j} \cdot CT^jC = 0.$$

Therefore we have

$$\begin{aligned} 0 &= \Big(\sum_{j=0}^{m} (-1)^j \binom{m}{j} T^{*m-j} \cdot CT^jC\Big) Cx_n \\ &= T^{*m}Cx_n + \sum_{j=1}^{m} (-1)^j \binom{m}{j} T^{*m-j} \cdot CT^jC \cdot Cx_n \\ &= (T^* + \overline{\lambda})^m Cx_n + \sum_{j=1}^{m} (-1)^j \binom{m}{j} T^{*m-j} \cdot (CT^jC - \overline{\lambda}^j)Cx_n. \end{aligned} \tag{6}$$

From (5) and (6), we have

$$\|(T^* + \overline{\lambda})^m\, Cx_n\| \to 0 \ \ (n \to \infty).$$

□

Corollary 3.4. *Let $T \in B(\mathcal{H})$ be a skew m-complex symmetric operator on $\mathcal{H}$. Then the following properties hold.*
(i) *If $\lambda \in \sigma_a(T)$ for $\lambda \in \mathbb{C}$, then $-\overline{\lambda} \in \sigma_a(T^*)$.*
(ii) *If $\lambda \in \sigma_p(T)$ for $\lambda \in \mathbb{C}$, then $-\overline{\lambda} \in \sigma_p(T^*)$.*

Proof. (i) If $\lambda \in \sigma_a(T)$ for $\lambda \in \mathbb{C}$, then there exists a sequence $\{x_n\}$ of unit vectors of $\mathcal{H}$ such that $\|(T-\lambda)x_n\| \to 0$. Therefore, we obtain $\|(T^* + \overline{\lambda})^m Cx_n\| \to 0$ $(n \to \infty)$ from Theorem 3.3. If $\frac{(T^*+\overline{\lambda})^{m-1}Cx_n}{\|(T^*+\overline{\lambda})^{m-1}Cx_n\|} \not\to 0$, then $-\overline{\lambda} \in \sigma_a(T^*)$. Otherwise, $\|(T^* + \overline{\lambda})^{m-1}Cx_n\| \to 0$. If $\frac{(T^*+\overline{\lambda})^{m-2}Cx_n}{\|(T^*+\overline{\lambda})^{m-2}Cx_n\|} \not\to 0$, then $-\overline{\lambda} \in \sigma_a(T^*)$. Otherwise, $\|(T^* + \overline{\lambda})^{m-2}Cx_n\| \to 0$. By similar process, we get that $\|(T^* + \overline{\lambda})Cx_n\| \to 0$. Hence $-\overline{\lambda} \in \sigma_a(T^*)$.

The assertion (ii) holds by a similar method. □

Definition 4. For an operator $T \in B(\mathcal{H})$ and a conjugation C, we define the operator $\zeta_m(T;C)$ by

$$\zeta_m(T;C) := \sum_{j=0}^{m} \binom{m}{j} CT^{m-j}C \cdot T^j.$$

An operator $T \in B(\mathcal{H})$ is said to be *skew* $[m,C]$*-symmetric* if $\zeta_m(T;C) = 0$.

It holds that $CTC \cdot \zeta_m(T;C) + \zeta_m(T;C) \cdot T = \zeta_{m+1}(T;C)$. Therefore if T is skew $[m,C]$-symmetric, then T is skew $[n,C]$-symmetric for all $n \geq m$. If T is skew $[m,C]$-symmetric, then it holds

$$0 = C(\zeta_m(T;C))^*C = \sum_{j=0}^{m}\binom{m}{j}T^{*j}\cdot CT^{*m-j}C = \zeta_m(T^*;C)$$

and hence so is T^*.

Remark 3.5. By Theorem 3.2 of [2], if T is $[m,C]$-symmetric, then so is T^n for every $n\in\mathbb{N}$. But there exists a skew $[1,C]$-symmetric operator T such that T^2 is not skew $[1,C]$-symmetric. For example, let

$$T = \begin{pmatrix} -1 & -2i \\ -2i & 1 \end{pmatrix} \text{ and } Cx = \begin{pmatrix} \overline{x_2} \\ \overline{x_1} \end{pmatrix} \text{ for } x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ on } \mathbb{C}^2.$$

Then it is easy to see $CTC = \begin{pmatrix} 1 & 2i \\ 2i & -1 \end{pmatrix} = -T$ and hence T is skew $[1,C]$-symmetric. But since $T^2 = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$, we have $CT^2C = T^2$. Hence T^2 is $[1,C]$-symmetric and not skew $[1,C]$-symmetric. □

Theorem 3.6. *Let $T\in B(\mathcal{H})$ be a skew $[m,C]$-symmetric operator on $\mathcal{H}$ and $\{x_n\}$ be a sequence of unit vectors. If, for $\lambda\in\mathbb{C}$, $\|(T-\lambda)x_n\| \to 0\ (n\to\infty)$, then $\|(T+\overline{\lambda})^m Cx_n\| \to 0\ (n\to\infty)$. Hence, if $(T-\lambda)x=0$, then $(T+\overline{\lambda})^m Cx = 0$.*

Proof. By the assumption, we have, for any $j\in\mathbb{N}$,

$$(CT^jC - \overline{\lambda}^j)Cx_n \to 0 \ \ (n \to \infty). \tag{7}$$

Since T^* is skew $[m,C]$-symmetric, it holds $\gamma_m(T^*;C) = \sum_{j=0}^{m}\binom{m}{j}T^{*j}\cdot CT^{*m-j}C = 0$. Therefore it holds

$$C\,(\gamma_m(T^*;C))^*C = \sum_{j=0}^{m}\binom{m}{j}T^{m-j}\cdot CT^jC = 0.$$

Hence it holds

$$\begin{aligned} 0 &= \Big(\sum_{j=0}^{m}\binom{m}{j}T^{m-j}\cdot CT^jC\Big)Cx_n \\ &= T^mCx_n + \sum_{j=1}^{m}\binom{m}{j}T^{m-j}\cdot CT^jC\cdot Cx_n \\ &= (T+\overline{\lambda})^mCx_n + \sum_{j=1}^{m}\binom{m}{j}T^{m-j}\cdot (CT^jC-\overline{\lambda}^j)Cx_n. \end{aligned} \tag{8}$$

From (7) and (8), we have

$$\|(T+\overline{\lambda})^m\,Cx_n\| \to 0 \ \ (n\to\infty).$$

□

Corollary 3.7. *Let $T\in B(\mathcal{H})$ be a skew $[m,C]$-symmetric operator on $\mathcal{H}$. Then the following properties hold.*
(i) *If $\lambda\in\sigma_a(T)$ for $\lambda\in\mathbb{C}$, then $-\overline{\lambda}\in\sigma_a(T)$.*
(ii) *If $\lambda\in\sigma_p(T)$ for $\lambda\in\mathbb{C}$, then $-\overline{\lambda}\in\sigma_p(T)$.*

Proof. (i) If $\lambda \in \sigma_a(T)$ is real, then there exists a sequence $\{x_n\}$ of unit vectors of $\mathcal{H}$ such that $\|(T-\lambda)x_n\| \to 0$. Therefore, we have $\|(T+\overline{\lambda})^m C x_n\| \to 0$ $(n \to \infty)$ from Theorem 3.6. If $\frac{(T+\overline{\lambda})^{m-1}Cx_n}{\|(T+\overline{\lambda})^{m-1}Cx_n\|} \not\to 0$, then $-\overline{\lambda} \in \sigma_a(T)$. Otherwise, $\|(T+\overline{\lambda})^{m-1}Cx_n\| \to 0$. If $\frac{(T+\overline{\lambda})^{m-2}Cx_n}{\|(T+\overline{\lambda})^{m-2}Cx_n\|} \not\to 0$, then $-\overline{\lambda} \in \sigma_a(T)$. Otherwise, $\|(T+\overline{\lambda})^{m-2}Cx_n\| \to 0$. By similar process, we get that $\|(T+\overline{\lambda})Cx_n\| \to 0$. Hence $-\overline{\lambda} \in \sigma_a(T)$.

The assertion (ii) holds by a similar way. □

Remark 3.8. Theorem 3.6 does not hold in general. Because let

$$T = \begin{pmatrix} 1 & 2i \\ 2i & -1 \end{pmatrix} \text{ and } Cx = \begin{pmatrix} \overline{x_2} \\ \overline{x_1} \end{pmatrix} \text{ for } x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ on } \mathbb{C}^2.$$

Then it holds $CTC = -T$ and hence T is skew $[1, C]$-symmetric. For the eigenvalue $\sqrt{3}\, i$ of T and the corresponding eigenvector $x = \begin{pmatrix} 1 \\ \frac{\sqrt{3}+i}{2} \end{pmatrix}$, we have

$$(T + \sqrt{3}\, i)Cx = \begin{pmatrix} 2\sqrt{3}\, i \\ -\sqrt{3} + 3i \end{pmatrix} \neq 0 \textit{ and } (T - \sqrt{3}i)Cx = 0.$$

□

Theorem 3.9. *Let $T \in B(\mathcal{H})$ be a skew $[m, C]$-symmetric operator on $\mathcal{H}$ and $\{x_n\}$ be a sequence of unit vectors. If, for $\lambda \in \mathbb{C}$, $\|(T-\lambda)x_n\| \to 0$ $(n \to \infty)$, then $\langle (T^* + \lambda)^m Cx_n, Cx_n \rangle \to 0$ $(n \to \infty)$. Hence, if $(T-\lambda)x = 0$, then $\langle (T^*+\lambda)^m Cx, Cx \rangle = 0$.*

Proof. Since T is skew $[m, C]$-symmetric, we have

$$CT^{*m}C = -\sum_{j=1}^{m} \binom{m}{j} T^{*j} \cdot CT^{*m-j}C$$

and

$$C(T^* + \lambda)^m C = \sum_{j=0}^{m} \binom{m}{j} \overline{\lambda}^j \cdot CT^{*m-j}C.$$

Hence it holds

$$\begin{aligned} C(T^* + \lambda)^m C &= CT^{*m}C + \sum_{j=1}^{m} \binom{m}{j} \overline{\lambda}^j\, CT^{*m-j}C \\ &= -\sum_{j=1}^{m} \binom{m}{j} T^{*j} \cdot CT^{*m-j}C + \sum_{j=1}^{m} \binom{m}{j} \overline{\lambda}^j\, CT^{*m-j}C \\ &= -\sum_{j=1}^{m} \binom{m}{j} \left(T^{*j} - \overline{\lambda}^j\right) \cdot CT^{*m-j}C. \end{aligned}$$

Therefore we have

$$\begin{aligned} &\langle (T^* + \lambda)^m\, Cx_n,\, Cx_n \rangle \\ &= \langle x_n,\, C(T^* + \lambda)^m\, Cx_n \rangle = \langle x_n,\, -\sum_{j=1}^{m} \binom{m}{j} \left(T^{*j} - \overline{\lambda}^j\right) CT^{*m-j}Cx_n \rangle \\ &= -\sum_{j=1}^{m} \binom{m}{j} \langle (T^j - \lambda^j)x_n,\, CT^{*m-j}Cx_n \rangle \to 0 \ \ (n \to \infty). \end{aligned}$$

□

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