Functional Analysis,
Approximation and
Computation
11 (2) (2019), 13–21

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/faac

Class of (A, n)-power-hyponormal operators in semi-hilbertian space

Cherifa Chellali[a], **Abdelkader Benali**[b]

[a] *The Higher School of Economics Oran, Algeria.*
[b] *Faculty of The Exact sciences And computer, Mathematics Department, University of Hassiba Benbouali, Chlef Algeria. B.P. 151 Hay Essalem, chlef 02000, Algeria*

Abstract. In this paper, the concept of n-power- hyponormal operators on a Hilbert space defined by Messaoud Guesba and Mostefa Nadir in [11] is generalized when an additional semi-inner product is considered. This new concept is described by means of oblique projections. For a Hilbert space operator $T \in B(H)$ is (A, n)-power-hyponormal operator for some positive operator A and for some positive integer n if

$$T^{\sharp}T^{n} - T^{n}T^{\sharp} \geq_{A} 0, n = 1, 2, ...$$

1. Introduction

A bounded linear operator T on a complex Hilbert space is n-hyponormal operator if $T^{*}T^{n} - T^{n}T^{*} \geq 0$. The class of p-hyponormal operator was introduced and studied by A. Aluthge [2], from the definition, it is easily seen that this class contrains hyponormal operators, in [8] the authors Messaoud Guesba and Mostefa Nadir introduced the class of n-power-hyponormal operators and study some proprietes of such class for different values of the parameter n, in particular for $n = 2, n = 3$ in Hilbert space.
The propose of this paper is to study the class of (A, n)-power-hyponormal operators in semi-hilbertian spaces.

2. (A, n)-power-hyponormal operators

Definition 2.1. *An operator $T \in \mathcal{B}_{A}(H)$ is said to be (A, n)-power-hyponormal operator for a positive integer n, if*

$$T^{n}T^{\sharp} \leq_{A} T^{\sharp}T^{n}$$

We denote the set of all (A, n)-power-hyponormal operators by $[nH]_{A}$

2010 *Mathematics Subject Classification*. Primary 46C05; Secondary 47A05.
Keywords. Semi-Hilbertian space; $(A, 2)$-power-hyponormal operators; (A, n)-power-hyponormal operators; A-positive operators; idempotent.
Received: 30 September 2018; Accepted: 26 June 2019
Communicated by Dragan S. Djordjević
Email addresses: chchellali@gmail.com (Cherifa Chellali), benali4848@gmail.com (Abdelkader Benali)

Remark 2.2. *Clearly $n = 1$, then $(A, 1)$-power-hyponormal operator is precisely A-hyponormal operator.*

Definition 2.3. *An operator $T \in \mathcal{B}_A(\mathcal{H})$ is said to be (A, n)-power-hyponormal if $T^\sharp T^n - T^n T^\sharp$ is A-positive i.e., $T^\sharp T^n - T^n T^\sharp \geq_A 0$ or equivalently*

$$\langle \left(T^\sharp T^n - T^n T^\sharp\right) u \mid u\rangle_A \geq 0 \ \textit{for all} \ \ u \in \mathcal{H}.$$

Example 2.4. *Let* $T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, S = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \in \mathcal{B}(\mathbb{R}^2)$. *A simple computation shows that*

$$T^\sharp = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, S^\sharp = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}, .$$

Then T is not $(A, 2)$-power-hyponormal operator, because

$$\left\langle \left(T^\sharp T^2 - T^2 T^\sharp\right) \begin{pmatrix} u \\ v \end{pmatrix} \mid \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle_A \quad = \quad -2u^2 - 2v^2 \leq 0.$$

For all $(u, v) \in (\mathbb{R}^2)$
and S is $(A, 1)$-power-hyponormal operator, because

$$\left\langle \left(S^\sharp S - S S^\sharp\right) \begin{pmatrix} u \\ v \end{pmatrix} \mid \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle_A \quad = \quad u^2 + v^2 \geq 0.$$

For all $(u, v) \in (\mathbb{R}^2)$

Proposition 2.5. *If S, $T \in B_A(H)$ are unitarily equivalent and if T is (A, n)-power-hyponormal operators then so is S*

Proof. Let T be an (A, n)-power-hyponormal operator and S be unitary equivalent of T. Then there exists unitary operator U such that $S = UTU^\sharp$ so $S^n = UT^nU^\sharp$
We have

$$\begin{aligned} S^n S^\sharp &= UT^nU^\sharp \left(UTU^\sharp\right)^\sharp \\ &= UT^nU^\sharp U T^\sharp U^\sharp \\ &= UT^n P_{\overline{\mathcal{R}(A)}} T^\sharp U^\sharp \\ &= UT^n T^\sharp U^\sharp \\ &\leq UT^\sharp T^n U^\sharp \\ &= S^\sharp S^n \end{aligned}$$

Hence, $S^n S^\sharp \leq_A S^\sharp S^n$ □

Theorem 2.6. *If S, $T \in B_A(H)$ are commuting (A, n)-power-hyponormal operators and $ST^\sharp = T^\sharp S$ is an (A, n)-power-hyponormal operators*

Proof. Since $ST = TS$, so $S^nT^n = (ST)^n$ and $S^nT^\sharp = T^\sharp S^n$.
Now,
$ST^\sharp = T^\sharp S \Rightarrow TS^\sharp = S^\sharp T$

Then $T^nS^\sharp = S^\sharp T^n$.
We have

$$\begin{aligned}(ST)^n (ST)^\sharp &= S^nT^nT^\sharp S^\sharp \\ &\leq_A S^nT^\sharp T^nS^\sharp \\ &= T^\sharp S^nS^\sharp T^n \\ &\leq_A T^\sharp S^\sharp S^nT^n\end{aligned}$$

Hence
$(ST)^n (ST)^\sharp \leq_A (ST)^\sharp (ST)^n$.
Then ST is an (A, n)-power-hyponormal operator. □

Proposition 2.7. *Let $T \in B_A(H)$ be an (A, n)-power-hyponormal operator. Then $T^\sharp$ is co-(A, n)-power-hyponormal operator*

Proof. Since T is (A, n)-power-hyponormal operator. We have

$$\begin{aligned}T^nT^\sharp \leq_A T^\sharp T^n \Rightarrow \left(T^\sharp T^n\right)^n \leq_A \left(T^nT^\sharp\right)^n &\Rightarrow \left(T^\sharp\right)^\sharp (T^n)^\sharp \leq_A (T^n)^\sharp \left(T^\sharp\right)^\sharp \\ &\Rightarrow T(T^n)^\sharp \leq_A (T^n)^\sharp T \\ &\Rightarrow (T^n)^\sharp T \geq_A T(T^n)^\sharp .\end{aligned}$$

Hence, $T^\sharp$ is co-(A, n)-power-hyponormal operator. □

Theorem 2.8. *If T, $T^\sharp$ are two (A, n)-power-hyponormal operator, then $T^\sharp$ is an (A, n)-power-hyponormal operator.*

Proposition 2.9. *If T is $(A, 3)$-power-hyponormal operator and $T^2 = -\left(T^\sharp\right)^2$. Tthen T is $(A, 3)$-power-normal operator.*

Proof. Since $T^3T^\sharp = TT^2T^\sharp = -T\left(T^\sharp\right)^3$
and
$T^\sharp T^3 = T^\sharp T^2T = -\left(T^\sharp\right)^3 T$
T is $(A, 3)$-power-hyponormal, then

$$\begin{aligned}T^3T^\sharp \leq_A T^\sharp T^3 &\Rightarrow -T\left(T^\sharp\right)^3 \leq_A -\left(T^\sharp\right)^3 T \\ &\Rightarrow T\left(T^\sharp\right)^3 \geq_A \left(T^\sharp\right)^3 T \\ &\Rightarrow \left(T\left(T^\sharp\right)^3\right)^\sharp \geq_A \left(\left(T^\sharp\right)^3 T\right)^\sharp \\ &\Rightarrow T^3T^\sharp \geq_A T^\sharp T^3.\end{aligned}$$

Hence $T^3T^\sharp = T^\sharp T^3$. □

Proposition 2.10. *If T is $(A, 4)$-power-hyponormal and T is skew-normal operator, then T is $(A, 4)$-normal operator.*

Proof. T is skew-normal operator, then $\left(T^\sharp\right)^2 = T^2$. Since
$T^4T^\sharp = T^2T^2T^\sharp = \left(T^\sharp\right)^5$
and
$T^\sharp T^4 = T^\sharp T^2T^2 = \left(T^\sharp\right)^5$.
Hence $T^4T^\sharp = T^\sharp T^4$. □

Example 2.11. *Let* $T = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \in \mathcal{B}(\mathbb{C}^2)$. *A simple computation shows that*

$$T^{\sharp} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix},.$$

Then T *is* $(A,2)$*-power-hyponormal, but is not* $(A,3)$*-power-hyponormal*

Example 2.12. *Let* $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *and* $T = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^3)$. *It easy to check that*

$$A \geq 0, \mathcal{R}(T^*A) \subset \mathcal{R}(A) and \;\; T^{\sharp} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix}, T^{\sharp}T \neq TT^{\sharp} and \;\; \|Tu\|_A \ngeq \|T^{\sharp}u\|_A.$$

Proposition 2.13. *Let* $T, S \in \mathcal{B}_A(\mathcal{H})$ *are* $(A,1)$*-hyponormal operator operators Then If.*
$S(T^{\sharp}T) = (T^{\sharp}T)S$ *and* $T(S^{\sharp}S) = (S^{\sharp}S)T$ *then* ST *and* TS *are* $(A,1)$*-power-hyponormal operators.*

Proof. We have

$$\begin{aligned} (ST)\,(ST)^{\sharp} &= STT^{\sharp}S^{\sharp} \\ &= TT^{\sharp}SS^{\sharp} \\ &\leq_A T^{\sharp}TSS^{\sharp} \\ &\leq_A T^{\sharp}SS^{\sharp}T \\ &\leq_A T^{\sharp}S^{\sharp}ST \\ &= (ST)^{\sharp}\,ST \end{aligned}$$

Then ST is $(A,1)$-power-hyponormal operator.
Now,

$$\begin{aligned} TS\,(TS)^{\sharp} &= TSS^{\sharp}T^{\sharp} \\ &= TT^{\sharp}SS^{\sharp} \\ &\leq_A T^{\sharp}TSS^{\sharp} \\ &\leq_A T^{\sharp}SS^{\sharp}T \\ &\leq_A T^{\sharp}S^{\sharp}ST \\ &= (ST)^{\sharp}\,ST \end{aligned}$$

Then ST is $(A,1)$-power-hyponormal operator.
□

this proposition remains true for any natural numbers n

Proposition 2.14. *Let* $T, S \in \mathcal{B}_A(\mathcal{H})$ *are* (A,n)*-power-hyponormal operator operators Then If.*
$S^n(T^nT^{\sharp}) = (T^nT^{\sharp})S^n$ *and* $T^n(S^{\sharp}S^n) = (S^{\sharp}S^n)T^n$ *then* $(ST)^n$ *and is* $(TS)^n$ *are* (A,n)*-power-hyponormal operators.*

Proof. We have

$$\begin{aligned} (ST)^n\,(ST)^{\sharp} &= S^nT^nT^{\sharp}S^{\sharp} \\ &= T^nT^{\sharp}S^nS^{\sharp} \\ &\leq_A T^{\sharp}T^nS^{\sharp}S^n \\ &\leq_A T^{\sharp}S^{\sharp}S^nT^n \\ &\leq_A T^{\sharp}S^{\sharp}\,(ST)^n \\ &= (ST)^{\sharp}\,(ST)^n \end{aligned}$$

Then $(ST)^n$ is (A,n)-power-hyponormal operator.
Now,

$$\begin{array}{rcl} (TS)^n\,(TS)^\sharp & = & T^nS^nS^\sharp T^\sharp \\ & = & T^nT^\sharp S^nS^\sharp \\ & \leq_A & T^\sharp T^nS^\sharp S^n \\ & \leq_A & T^\sharp SS^\sharp T \\ & \leq_A & T^\sharp S^\sharp ST \\ & = & (ST)^\sharp\, ST \end{array}$$

Then $(ST)^n$ is (A,n)-hyponormal operator.
□

Lemma 2.15. *Let $T_k, S_k \in \mathcal{B}(\mathcal{H}),\ k = 1,2$ and Let $A, B \in \mathcal{B}(\mathcal{H})^+$, such that $T_1 \geq_A T_2 \geq_A 0$ and $S_1 \geq_B S_2 \geq_B 0$, then*

$$\left(T_1 \otimes S_1\right) \geq_{A\otimes B} \left(T_2 \otimes S_2\right) \geq_{A\otimes B} 0.$$

Theorem 2.16. *Let $A, B \in \mathcal{B}(\mathcal{H})^+$. If $T \in \mathcal{B}_A(\mathcal{H})$ and $S \in \mathcal{B}_B(\mathcal{H})$ are nonzero operators, then .*

$T \otimes S$ is $(A\otimes B, n)$-power-hyponormal if and only if T is (A,n)-power-hyponormal and S is (B,n)-power-hyponormal.

.

Proof. Assume that T is (A,n)-power-hyponormal and S is (B,n)-power-hyponormal operators. Then

$$\begin{array}{rcl} \left(T\otimes S\right)^\sharp\left(T\otimes S\right)^n & = & \left(T^\sharp\otimes S^\sharp\right)\left(T^n\otimes S^n\right) \\ & = & T^\sharp T^n \otimes S^\sharp S^n \\ & \geq_{A\otimes B} & T^nT^\sharp \otimes S^nS^\sharp = \left(T\otimes S\right)^n\left(T\otimes S\right)^\sharp. \end{array}$$

Which implies that $T \otimes S$ is $(A \otimes B, n)$-power-hyponormal operator.

Conversely, assume that $T \otimes S$ is $(A \otimes B, n)$-power-hyponormal operator.We aim to show that T is (A,n)-power-hyponormal and S is (B,n)-power-hyponormal. Since $T \otimes S$ is a $(A \otimes A, n)$-power-hyponormal operator, we obtain

$$\begin{array}{rcl} (T\otimes S) \text{ is } (A\otimes B, n)\text{-power-hyponormal} & \Longleftrightarrow & (T\otimes S)^\sharp (T\otimes S)^n \geq_{(A\otimes B,n)} (T\otimes S)^n(T\otimes S)^\sharp \\ & \Longleftrightarrow & T^\sharp T^n \otimes S^nS^\sharp \geq_{A\otimes B} T^nT^\sharp \otimes S^nS^\sharp. \end{array}$$

Then, there exists $d > 0$ such that

$$\begin{cases} d\,T^\sharp T^n \geq_A T^nT^\sharp \\ \text{and} \\ d^{-1}S^\sharp S^n \geq_B S^nS^\sharp \end{cases}$$

a simple computation shows that $d = 1$ and hence

$$T^\sharp T^n \geq_A T^nT^\sharp \quad \text{and} \quad S^\sharp S^n \geq_B S^nS^\sharp.$$

Therefore, T is (A,n)-power-hyponormal and S is (B,n)-power-hyponormal.
□

Proposition 2.17. *If $T, S \in \mathcal{B}_A(\mathcal{H})$ are (A,n)-power-hyponormal, then $TS \otimes T, TS \otimes S, ST \otimes T$ and $ST \otimes S \in \mathcal{B}_{A\otimes A}(\mathcal{H}\overline{\otimes}\mathcal{H})$ are $(A \otimes A, n)$-power-hyponormal if the following assertions hold:*

(1) $S^nT^nT^\sharp = T^nT^\sharp S^n$.

(2) $T^nS^\sharp S^n = S^\sharp S^nT^n$.

Proof. Assume that the conditions (1) and (2) are hold. Since T and S are (A,n)-power-hyponormal, we have

$$\left(TS \otimes T\right)^\sharp\left(TS \otimes T\right)^n = \left(S^\sharp T^\sharp \otimes T^\sharp\right)\left(T^nS^n \otimes T^n\right) = \left(S^\sharp T^\sharp T^nS^n\right) \otimes \left(T^\sharp T^n\right).$$

Since $T^\sharp T^n \geq_A T^nT^\sharp$ it follows from Lemma 2.1 see [11].
That

$$S^\sharp T^\sharp T^nS^n \geq_A S^\sharp T^nT^\sharp S^n = S^\sharp S^nT^nT^\sharp = T^nS^\sharp S^nT^\sharp \geq_A T^nS^nS^\sharp T^\sharp = (TS)^n\,(TS)^\sharp$$

Thus,

$$\begin{cases} S^\sharp T^\sharp T^nS^n \geq_A T^nS^n(TS)^\sharp \geq_A 0 \\ \text{and} \\ T^\sharp T^n \geq_A T^nT^\sharp \geq_A 0 \end{cases}$$

Lemma 2.1 implies that

$$\left(TS \otimes T\right)^\sharp\left(TS \otimes T\right)^n \geq_{A\otimes A} (TS)^n\,(TS)^\sharp \otimes T^nT^\sharp = \left(TS \otimes T\right)^n\left(TS \otimes T\right)^\sharp.$$

In the same way, we may deduce the $(A \otimes A, n)$-power-hyponormality of $TS \otimes S, ST \otimes T$ and $ST \otimes S$. □

Theorem 2.18. *If $T \in \mathcal{B}_A(\mathcal{H})$ and $S \in \mathcal{B}_A(\mathcal{H})$ such that $\mathcal{N}(A)$ is invariant for T and S. Then :*

S is $(A,1)$-power-hyponormal then $T \boxplus S$ is $(A \otimes A, 1)$-power-hyponormal.

Proof. Firstly, observe that if $T^\sharp T \geq_A TT^\sharp$ and $S^\sharp S \geq_A SS^\sharp$ then we have following inequalities

$$\left(T \otimes I\right)^\sharp\left(T \otimes I\right) \geq_{A\otimes A} \left(T \otimes I\right)\left(T \otimes I\right)^\sharp$$

and

$$\left(S \otimes I\right)^\sharp\left(S \otimes I\right) \geq_{A\otimes A} \left(S \otimes I\right)\left(S \otimes I\right).$$

Taking into account that $TP_{\overline{\mathcal{R}(A)}} = P_{\overline{\mathcal{R}(A)}}T$ and $SP_{\overline{\mathcal{R}(A)}} = P_{\overline{\mathcal{R}(A)}}S$ we infer

$$\begin{aligned} (T \boxplus S)^\sharp(T \boxplus S) &= \left(T \otimes I + I \otimes S\right)^\sharp\left(T \otimes I + I \otimes S\right) \\ &= \left(T \otimes I\right)^\sharp\left(T \otimes I\right) + \left(T \otimes I\right)^\sharp\left(I \otimes S\right) + \left(I \otimes S\right)^\sharp\left(T \otimes I\right) + \left(I \otimes S\right)^\sharp\left(I \otimes S\right) \\ &\geq_{A\otimes A} \left(T \otimes I\right)\left(T \otimes I\right)^\sharp + \left(I \otimes S\right)\left(T \otimes I\right)^\sharp + \left(T \otimes\ \right)\left(I \otimes S\right)^\sharp + \left(I \otimes S\right)\left(I \otimes S\right)^\sharp \\ &\geq_{A\otimes A} \left(T \otimes I + I \otimes S\right)\left(T \otimes I + I \otimes S\right)^\sharp \\ &\geq_{A\otimes A} \left(T \boxplus S\right)\left(T \boxplus S\right)^\sharp, \end{aligned}$$

then $T \boxplus S$ is $(A \otimes A, 1)$-power-hyponormal.

□

Theorem 2.19. *Let $T_1, T_2, ..., T_m$ be (A, n)-power-hyponormal operator in $\mathcal{B}_A(\mathcal{H})$. Then $(T_1 \oplus T_2 \oplus .... \oplus T_m)$ is $(A \oplus A \oplus ... \oplus A, n)$ -power-hyponormal operators and $(T_1 \otimes T_2 \otimes .... \otimes T_m)$ is $(A \otimes A \otimes ... \otimes A, n)$-power-hyponormal operators.*

Proof. Since

$$\begin{aligned}
(T_1 \oplus T_2 \oplus .... \oplus T_m)^n (T_1 \oplus T_2 \oplus .... \oplus T_m)^\sharp &= \left(T_1^n \oplus T_2^n \oplus .... \oplus T_m^n\right)\left(T_1^\sharp \oplus T_2^\sharp \oplus .... \oplus T_m^\sharp\right) \\
&= T_1^n T_1^\sharp \oplus T_2^n T_2^\sharp \oplus .... \oplus T_m^n T_m^\sharp \\
&\leq_{A \oplus A ... \oplus A} T_1^\sharp T_1^n \oplus T_2^\sharp T_2^n \oplus .... \oplus T_m^\sharp T_m^n \\
&= \left(T_1^\sharp \oplus T_2^\sharp \oplus .... \oplus T_m^\sharp\right)\left(T_1^n \oplus T_2^n \oplus .... \oplus T_m^n\right) \\
&= (T_1 \oplus T_2 \oplus .... \oplus T_m)^\sharp (T_1 \oplus T_2 \oplus .... \oplus T_m)^n .
\end{aligned}$$

Then $(T_1 \oplus T_2 \oplus .... \oplus T_m)$ is $(A \oplus A \oplus ... \oplus A, n)$-power-hyponormal operators.
Now,

$$\begin{aligned}
(T_1 \otimes T_2 \otimes .... \otimes T_m)^n (T_1 \otimes T_2 \otimes .... \otimes T_m)^\sharp &= \left(T_1^n \otimes T_2^n \otimes .... \otimes T_m^n\right)\left(T_1^\sharp \otimes T_2^\sharp \otimes .... \otimes T_m^\sharp\right) \\
&= T_1^n T_1^\sharp \otimes T_2^n T_2^\sharp \otimes .... \otimes T_m^n T_m^\sharp \\
&\leq_{A \otimes A ... \otimes A} T_1^\sharp T_1^n \oplus T_2^\sharp T_2^n \otimes .... \otimes T_m^\sharp T_m^n \\
&= \left(T_1^\sharp \otimes T_2^\sharp \otimes .... \otimes T_m^\sharp\right)\left(T_1^n \otimes T_2^n \otimes .... \oplus T_m^n\right) \\
&= (T_1 \otimes T_2 \otimes .... \otimes T_m)^\sharp (T_1 \otimes T_2 \otimes .... \otimes T_m)^n .
\end{aligned}$$

Then $(T_1 \otimes T_2 \otimes .... \otimes T_m)$ is $(A \otimes A \otimes ... \otimes A, n)$-power-hyponormal operators. □

Proposition 2.20. *If T is $(A, 2)$-power-hyponormal and T is idempotent. Then T is $(A, 1)$-power-hyponormal operator*

Proof. Since T is $(A, 2)$-power-hyponormal operator, then
$T^2 T^\sharp \leq_A T^\sharp T^2$
since T is idempotent $T^2 = T$, wich implies
$TT^\sharp \leq_A T^\sharp T$
Thus T is is $(A, 1)$-power-hyponormal operator □

Proposition 2.21. *If T is $(A, 3)$-power-hyponormal and T is idempotent. Then T is $(A, 2)$-power-hyponormal operator*

Proof. Since T is $(A, 2)$-power-hyponormal operator, then
$T^3 T^\sharp \leq_A T^\sharp T^3$
since T is idempotent $T^2 = T$, wich implies
$TT^\sharp \leq_A T^\sharp T$
Thus T is is $(A, 1)$-power-hyponormal operator □

Proposition 2.22. *If T, S are $(A, 2)$-power-hyponormal operators, such that $TS^\sharp = S^\sharp T$ and $TS + ST = 0$, then ST and $S + T$ are $(A, 2)$-power-normal operator.*

Proof. Since $ST + TS = 0$, hence $S^2T^2 = T^2S^2$, so $(S + T)^2 = S^2 + T^2$.

$$\begin{aligned}
(S + T)^2 (S + T)^\sharp &= \left(S^2 + T^2\right)\left(S^\sharp + T^\sharp\right) \\
&= S^2S^\sharp + S^2T^\sharp + T^2S^\sharp + T^2T^\sharp \\
&= S^2S^\sharp + T^\sharp S^2 + S^\sharp T^2 + T^2T^\sharp \\
&\leq_A S^\sharp S^2 + T^\sharp S^2 + S^\sharp T^2 + T^\sharp T^2 \\
&= (S + T)^\sharp (S + T)^2
\end{aligned}$$

Now,

$$\begin{array}{rcl} (ST)^2\,(ST)^\sharp & = & S^2T^2T^\sharp S^\sharp \\ & \leq_A & S^2T^\sharp T^2 S^\sharp \\ & = & T^\sharp S^2 S^\sharp T^2 \\ & \leq_A & T^\sharp S^\sharp S^2 T^2 \\ & = & (ST)^\sharp\,(ST)^2 \end{array}$$

Hence
$(ST)^2\,(ST)^\sharp \leq_A (ST)^\sharp\,(ST)^2$.
Then ST is an $(A,2)$-power-hyponormal operator. □

Example 2.23. *Let* $T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, S = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^2)$. *A simple computation shows that*

$$T^\sharp = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, S^\sharp = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix},.$$

Then T *is* $(A,2)$*-power-hyponormal operator, but*

$$\left\langle \left(T^\sharp T^2 - T^2 T^\sharp\right)\begin{pmatrix} u \\ v \end{pmatrix} \mid \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle_A \quad = \quad 0.$$

For all $(u,v) \in (\mathbb{C}^2)$
and S *is* $(A,2)$*-power-hyponormal operator, but*

$$\left\langle \left(S^\sharp S^2 - S^2 S^\sharp\right)\begin{pmatrix} u \\ v \end{pmatrix} \mid \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle_A \quad = \quad 0.$$

For all $(u,v) \in (\mathbb{C}^2)$
Such that $TS + ST = 0$ *and* $TS^\sharp \neq S^\sharp T$
but $S + T$ *and* ST *are* $(A,2)$*-power-hyponormal operator*
the following example shows that proposition 8.14 *is not necessarily true if* $TS^\sharp \neq S^\sharp T$

Proposition 2.24. *Let* $T \in \mathcal{B}_A(\mathcal{H})$. *If* T *is* (A,n)*-power-hyponormal, then* T *is* $(A,2)$*-power-hyponormal operator*

Proposition 2.25. *Let* $T, S \in \mathcal{B}_A(\mathcal{H})$ *are commuting* (A,n)*-power-hyponormal operators, such that* $TS^\sharp = S^\sharp T$ *and* $(T+S)^\sharp$ *is commutes with*

$$\sum_{1\leq p\leq n-1} \binom{n}{p}\left(T^p S^{n-p}\right).$$

Then $(T+S)$ *is an* (A,n)*-power-hyponormal*

Proof. Since

$$\begin{aligned}
\left(T+S\right)^n\left(T+S\right)^\sharp &= \left[\sum_{0\le p\le n}\binom{n}{p}\left(T^pS^{n-p}\right)\right]\left(T+S\right)^\sharp \\
&= S^nS^\sharp + \sum_{1\le p\le n-1}\binom{n}{p}\left(T^pS^{n-p}\right)\left(T+S\right)^\sharp + T^nS^\sharp + S^nT^\sharp + T^nT^\sharp \\
&= S^nS^\sharp + \sum_{1\le p\le n-1}\binom{n}{p}\left(T^pS^{n-p}\right)\left(T+S\right)^\sharp + S^\sharp T^n + T^\sharp S^n + T^nT^\sharp \\
&\le_A S^\sharp S^n + \left(T+S\right)^\sharp \sum_{1\le p\le n-1}\binom{n}{p}\left(T^pS^{n-p}\right) + S^\sharp T^n + T^\sharp S^n + T^nT^\sharp \\
&\le_A \left(T+S\right)^\sharp\left[\sum_{0\le p\le n}\binom{n}{p}\left(T^pS^{n-p}\right)\right] \\
&= \left(T+S\right)^\sharp\left(T+S\right)^n.
\end{aligned}$$

Then $(T+S)$ is an (A,n)-power-hyponormal. □

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