Skew exactness and range-kernel orthogonality III

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Abstract. We record some “range-kernel orthogonality” for the elementary operators defining Hankel and Toeplitz operators.

Introduction

This note is a sort of follow-up visit to our earlier paper [H3], in which we feel we may have left gaps in a couple of the arguments, specifically Theorem 5 and Theorem 6; we have also belatedly noticed the connection between the “derivations” involved and the Brown-Halmos definitions [BH] of Hankel and Toeplitz operators.

1. Hermitian elements of a Banach algebra

Recall [BD] that a Banach algebra element \(a \in A\) is said to be hermitian, written \(a \in \text{Re}(A)\), provided its numerical range is real:

\[ V_A(a) \equiv \{ \varphi(a) : \varphi \in \text{State}(A) \} \subseteq \mathbb{R} , \]

where the “states” of \(A\)

\[ \text{State}(A) = \{ \varphi \in A^\dagger : \|\varphi\| = 1 = \varphi(1) \} \]

are the norm one linear functionals which achieve their norm at the identity \(1 \in A\). The real-linear subspace \(\text{Re}(A) = H \subseteq A\) is an example of a “hermitian subspace” [H4] of \(A\): we have

\[ 1 \in H ; H \cap iH = O \equiv \{0\} . \]

We introduce the complex-linear Palmer subspace

\[ \text{Reim}(A) = \text{Re}(A) + i\text{Re}(A) \]

and observe that the “real and imaginary parts” of \(a = h + ik \in \text{Reim}(A)\) are well-defined, with

\[ x : \text{Reim}(A) \rightarrow \text{Re}(A) , \ y : \text{Reim}(A) \rightarrow \text{Re}(A) \]
given by setting
\[ a = x(a) + iy(a) \ (a \in \text{Reim}(A)), \]
and then there is an involution \( * : \text{Reim}(A) \to \text{Reim}(A) \) given by
\[ (h + ik)^* = h - ik \ (\{h, k\} \subseteq \text{Re}(A)). \]
If we further observe \([BD],[Sp]\) that \( H = \text{Re}(A) \) satisfies
\[ H = \text{cl}(H) \perp iH, \]
the subspaces of “real” and “imaginary” elements are closed and mutually “orthogonal”, then it follows that \( \text{Reim}(A) \) is also a closed subspace of \( A \), on which the involution \( a \to a^* \) is a bounded (real-)linear operator. Here, for \( E + F \subseteq X \), “orthogonality” \( E \perp F \) is according to James and Birkhoff:
\[ E \perp F \iff \text{AND} x \in E (\|x\| = \text{dist}(x,F)). \]
The Palmer subspace is not in general closed under multiplication; it is however \([BD],[Sp]\) closed under the Lie bracket: if \( \{a, b\} \subseteq \text{Reim}(A) \) then
\[ [a, b] \equiv ab - ba \in \text{Reim}(A), \]
with
\[ [a, b]^* = [b^*, a^*]. \]

2. Self-commutant approximants

Specializing to the algebra \( A = B(X) \) of bounded operators on a Banach space \( X \), we claim that

Lemma 1 If \( X \) is a Banach space then
\[ T \in \text{Reim} B(X) \implies T^{-1}(0) \cap T^{-1}(0) \subseteq \text{Re}(T)^{-1}(0) \perp \text{Re}(T)(X). \]

Proof. If \( \{H, K\} \subseteq \text{Re} B(X) \) then
\[ T = H + iK \implies T^{-1}(0) \cap T^{-1}(0) = H^{-1}(0) \cap K^{-1}(0), \]
and now by Sinclair’s theorem ([Si] Proposition 1; [F] Corollary 7)
\[ H^{-1}(0) \perp H(X). \]

If in particular \( T \in B(X) \) satisfies the Fuglede condition,
\[ T^{-1}(0) \subseteq T^{-1}(0), \]
then Lemma 1 says that
\[ T^{-1}(0) \perp \text{Re}(T)(X). \]

If in particular \( A \) and \( B \) are Banach algebras and \( M \) a Banach \((A,B)\)-bimodule then \([H4]\) tuples \( a \in A^n \) and \( b \in B^n \) combine to give “elementary” operators
\[ L_a \circ R_b : m \mapsto \sum_j a_j m b_j \ (M \to M); \]
we focus particularly on the “generalized inner derivation”

\[ L_a - R_b : m \mapsto am - mb \]

and its multiplicative analogue

\[ L_a R_b - I : m \mapsto amb - m \]

associated with \((a, b) \in A \times B\). For example if \((a, b) = (v, u)\) are the backward and forward shifts in \(A = B(X)\) with \(X = \ell_2\) then [BH] \((L_a R_b - I)^{-1}(0)\) and \((L_a - R_b)^{-1}(0)\) are respectively the Toeplitz and the Hankel operators.

Generally when \(A = M = B\),

**Theorem 2** If \(A\) is a Banach algebra and \([a, b] \subseteq \text{Reim}(A)\) then

\[ b \in (L_a - R_a)^{-1}(0) \cap (L_a' - R_a')^{-1}(0) \implies ||a|| \leq ||a - [b, b']|| \text{ and } ||b|| \leq ||b - [a, a']||. \]

**Proof.** This is Lemma 1 with \(X = A\) and \(T = L_a - R_b\), together with the observation that

\[ (L_a - R_a)(b) = 0 \iff (L_b - R_b)(a) = 0, \]

and also (1.10)

\[ (L_a' - R_a')(b) = (L_a - R_b)(b^*) = 0. \]

Indeed if \((L_b - R_b)(a) = 0 = (L_b - R_b)'(a)\) then, with \(b = h + ik\) and arbitrary \(c \in A\),

\[ ||a|| \leq ||a - \text{Re}(T)(c)|| \equiv ||a - (L_b - R_b)(c)|| : \]

but now

\[ c = 2ik \implies (L_b - R_b)(c) = (L_b - R_b)(b^*) = [b, b^*]; \]

for the second part interchange \(a\) and \(b\) •

Theorem 2 is a cosmetic improvement of Theorem 5 of [H3], in that the Fuglede condition (2.4) is withheld from \(T = L_a - R_a\), and also plugs a small gap (2.7) in the argument there.

3. **Multiplicative commutants**

Theorem 6 of [H3] needs more than cosmetic adjustment. If we write, for arbitrary \(c \in A\),

\[ D_c = L_c - R_c; \quad \Delta_c = L_cR_c - I, \]

then we recall [H1] that if \([a, b] \subseteq A\) then

\[ \begin{pmatrix} D_{a+b} \\ D_{ab} \end{pmatrix} = \begin{pmatrix} I & I \\ R_b & L_a \end{pmatrix} \begin{pmatrix} D_a \\ D_b \end{pmatrix} \]

and hence always

\[ D_a^{-1}(0) \cap D_b^{-1}(0) \subseteq D_{a+b}^{-1}(0) \cap D_{ab}^{-1}(0). \]

Conversely (taking [H2],[HH] the “adjugate” !)

\[ \begin{pmatrix} L_a & -I \\ -R_b & I \end{pmatrix} \begin{pmatrix} D_{a+b} \\ D_{ab} \end{pmatrix} = \begin{pmatrix} L_a - R_b & 0 \\ 0 & L_a - R_b \end{pmatrix} \begin{pmatrix} D_a \\ D_b \end{pmatrix}; \]

and hence if \(L_a - R_b\) is one one there is equality in (3.3). We also have \(ST - I = S(T - I) + (S - I)\), so that

\[ (S - I)^{-1}(0) \cap (T - I)^{-1}(0) \subseteq (ST - I)^{-1}(0); \]
3.6 \[ \Delta_a \Delta_b = L_b R_b \Delta_a + \Delta_b = L_{ab} R_{ba} - I (\neq \Delta_{ab}), \]

and hence

3.7 \[ \Delta_a^{-1}(0) \cap \Delta_b^{-1}(0) \subseteq (L_{ab} R_{ba} - I)^{-1}(0). \]

As a supplement to the first row of (3.2) we have also

3.8 \[ D_{a+b} = (L_a - R_b) + (L_b - R_a), \]

so that

3.9 \[ (L_a - R_b)^{-1}(0) \cap (L_b - R_a)^{-1}(0) \subseteq D_{a+b,1}^{-1}(0), \]

giving cosmetic improvement of the sort of two-variable extension of Theorem 2 noticed by Mansour and Bouzenada ([MB] Theorem 3.1).

If we make the assumption

3.10 \[ \text{Reim}(A)^2 \subseteq \text{Reim}(A), \]

that the Palmer subspace is a subalgebra (so that [BD],[Sp] it is in fact a C*-algebra) then, in place of Theorem 6 of [H3],

**Theorem 3** If \([a, b] \subseteq \text{Reim}(A)\) then

3.11 \[ b \in (L_a R_a - I)^{-1}(0) \cap (L_{a'} R_{a'} - I)^{-1}(0) \implies \text{AND}_{c_{cA}}(\|b\| \leq \|b + c - a'acaa'\|). \]

**Proof.** This is (3.7) with \(b = a'\):

3.12 \[ (L_a R_a - I)^{-1}(0) \cap (L_{a'} R_{a'} - I)^{-1}(0) \subseteq (L_{a''} R_{a''} - I)^{-1}(0) \perp (L_{a''} R_{a''} - I)(A). \]

The multiplicative assumption of course guarantees that the product of commuting hermitian operators in the middle of (3.12) is again hermitian.

We at the same time have cosmetic improvement of Duggal’s ([Du] Theorem 2.6) version:

3.13 \[ b \in (L_a R_a - I)^{-1}(0) \cap (L_{a'} R_{a'} - I)^{-1}(0) \implies b'c \in (L_a - R_a)^{-1}(0). \]

An old example of Anderson and Foias says that the multiplicative assumption (3.10) cannot be omitted from Theorem 3; if

3.14 \[ 0 \neq a' = a = a^2 \neq 1 \in A = \mathcal{B}(X), \]

then ([AF] Example 5.8) \(L_a R_a \in \mathcal{B}(A)\) is not hermitian, and the orthogonality at the end of (3.12) is liable to fail.

If \(a = u \in A = \mathcal{B}(X)\) with \(X = \ell_2\) is the forward shift then \((L_a - R_a)^{-1}(0)\) consists of the analytic Toeplitz operators while \(a' = v\) is the backward shift, and \((L_{a'} - R_{a'})^{-1}(0)\) the co-analytic Toeplitz operators; in this case the intersection on the left hand side of (2.5) reduces to the scalar multiples of the identity. When \(a = u\) then \((L_a R_a - I)^{-1}(0)\) consists of the Toeplitz operators; \((L_a R_a - I)^{-1}(0)\) however, and therefore the intersection, reduces to the co-analytic Toeplitz operators.

Putting \(a = 1\) in (2.5) shows ([MB] Corollary 3.2) that

3.12 \[ b \in \text{Reim}(A) \implies \|1 - [b, b']\| \geq 1. \]
References


