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A note on topological direct sum of subspaces

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Abstract. Some properties of topological direct sum of subspaces of a normed space X are discussed. Using the connection between this sum and the decomposition of the identity operator, we consider the appropriate matrix form of a bounded linear operator.

1. Introduction and preliminaries

The aim of this note is to present some (mainly known, but somehow less used) facts about the notion of a topological direct sum of linear subspaces. The case of Banach spaces, as the most important, is specially discussed. Also, for a given bounded linear operator, we consider its matrix form.

If X and Y are normed spaces then $\mathcal{B}(X,Y)$ denotes the set of all bounded linear operators from X to Y. We write $\mathcal{B}(X)$ instead of $\mathcal{B}(X,X)$. For $A \in \mathcal{B}(X,Y)$, the image space and the null space of A are denoted by $\operatorname{Im} A$ and $\operatorname{Ker} A$, respectively. Suppose that X and Y are Banach spaces such that

$$X = X_1 \oplus X_2 \oplus X_3 \text{ and } Y = Y_1 \oplus Y_2 \oplus Y_3, \tag{1}$$

where X_i are closed subspaces of X, and Y_i are closed subspaces of Y, $i = 1,2,3$. Let $A \in \mathcal{B}(X,Y)$ and let

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} : \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \to \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}, \tag{2}$$

be the matrix form of A with respect to direct sums (1). Namely, for $x = x_1 + x_2 + x_3 \in X$, $x_j \in X_j$, let $Ax_j = y_{1j} + y_{2j} + y_{3j}$, where $y_{ij} \in Y_i$. The operator $A_{ij} : X_j \to Y_i$ is defined by $A_{ij}x_j = y_{ij}$.

It is widely used the fact that $A \in \mathcal{B}(X,Y)$ if and only if $A_{ij} \in \mathcal{B}(X_j, Y_i)$, for all $i, j = 1,2,3$.

We want to point to the following questions:

1. What actually means the direct sum $X = X_1 \oplus X_2 \oplus X_3$, when X is Banach space? Is it sufficient to require the unique representation of $x \in X$ in the form $x = x_1 + x_2 + x_3$ where $x_i \in X_i$? Do we have to suppose that X_i are closed subspaces of X? Perhaps we need that sums $X_1 \oplus X_2$, $X_1 \oplus X_3$, $X_2 \oplus X_3$ (or only one of them) must be closed?

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2. If $A \in \mathcal{B}(X, Y)$ why is then $A_{ij} \in \mathcal{B}(X_j, Y_i)$? What about the operator

$$B = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} : \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \to \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix},$$

which is obtained from representation (2)? Is it bounded? But before that, is the subspace $X_1 \oplus X_2$ closed, i.e. is the operator B defined on a Banach space?

3. Does the condition $A_{ij} \in \mathcal{B}(X_j, Y_i)$, for all $i, j = 1, 2, 3$ imply $A \in \mathcal{B}(X, Y)$?

4. Which conditions we have to require when X and Y are not Banach spaces? When we deal with Hilbert spaces and orthogonal sums, do we have to suppose that X_i and Y_i are closed?

5. Which conditions we have to require when X is the sum of n subspaces, $X = X_1 \oplus + \cdots + \oplus X_n$? Do we have to suppose that the sum of any k, $1 \leq k \leq n$, of these n subspaces is closed? Perhaps from the closeness of X_i, the closeness of the sum of any k subspaces follows.

6. How to define the direct sum $X_1 \oplus \cdots \oplus X_n$ in the case when $X_1 + \cdots + X_n \neq X$?

In what follows, we will consider these questions. We will explain a close connection between the direct sum of subspaces and two-sided Peirce decomposition of the identity of the ring. Thus we will establish a connection between analytic-topologic and algebraic notions.

2. Direct sum of linear subspaces

The notion of a direct product of finite number of linear spaces (sometimes called exterior direct sum) and the notion of a direct sum of finite number of subspaces are closely related, but still different. In the most general case, the direct product of "structures" (groups, rings, vector spaces etc.) $S_1, \ldots, S_n$ is the Cartesian product

$$\prod = S_1 \times \cdots \times S_n$$

together with appropriate operations which are defined coordinately. The direct product $\prod = S_1 \times \cdots \times S_n$ of normed spaces is defined as a direct product of linear spaces. The norm on $\prod$ can be defined in many different ways. Recall the following well-known theorem.

Theorem 2.1. *Let $(X_1, \|\cdot\|_1), \ldots, (X_n, \|\cdot\|_n)$ be normed spaces, $\prod = X_1 \times \cdots \times X_n$ and let*

$$\|(x_1, \ldots, x_n)\|_p = (\|x_1\|_1^p + \cdots + \|x_n\|_n^p)^{\frac{1}{p}}, \quad \text{where } p \geq 1 \text{ is a real number},$$
$$\|(x_1, \ldots, x_n)\|_\infty = \max\{\|x_1\|_1, \ldots, \|x_n\|_n\}.$$

Then each of the functions $\|\cdot\|_p$, $1 \leq p \leq \infty$, defines a norm on $\prod$ and all of them are mutually equivalent, i.e. they define the same topology on $\prod$. The linear space $\prod$ together with one of the above defined norms, is a Banach space if and only if X_i is a Banach space for all $i = 1, 2, \ldots, n$.

Proof. See, for example, Problem 3.9 (pg. 168.), Problem 3.33 (pg. 178.) and Example 4. E (pg. 208.) in [5]. □

Definition 2.2. *Let $X_1, \ldots, X_n$ be linear subspaces of a linear space X. The sum $S = X_1 + \cdots + X_n$ is an algebraic (inner) direct sum (ADS for short) if the map*

$$\varphi : \prod \to S, \quad \varphi(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n \tag{3}$$

is the isomorphism of linear spaces.

Recall that the sum $S = X_1 + \cdots + X_n$ is the ADS if and only if the condition $x_1 + \cdots + x_n = 0$, $x_i \in X_i$, $i = 1, \ldots, n$ implies $x_i = 0$, $i = 1, \ldots, n$. Another well-known characterizations of ADS can be found in [4].

If we have the norm on a linear space X, then it defines a topology on X.

Definition 2.3. *Let $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be normed spaces. The map $\varphi : X_1 \to X_2$ is a topological isomorphism (or homeomorphism) if φ is the isomorphism of linear spaces and both φ and φ^{-1} are continuous.*

It follows that a topological isomorphism φ maps open sets to open sets and closed sets to closed sets. Note that in general φ does not preserve the norm in the sense that

$$\|\varphi(x)\|_2 = \|x\|_1, \ \forall x \in S_1. \tag{4}$$

A linear map that satisfies the condition (4), i.e. the map that "preserves" distances, is called an isometry. An isomorphism which is an isometry is called isometric isomorphism. Of course, if φ is an isometric isomorphism then $\|\varphi\| = 1$ and $\|\varphi^{-1}\| = 1$, so φ is a topological isomorphism. In general case the converse is not true.

Definition 2.4. *Let $X_1, \ldots, X_n$ be linear subspaces of a normed space $(X, \|\cdot\|)$. Let the product $\prod = X_1 \times \cdots \times X_n$ be equipped with one of the norms $\|\cdot\|_p$, $1 \le p \le \infty$.*

The sum $S = X_1 + \cdots + X_n$ is a topological direct sum (TDS for short) of subspaces if the map $\varphi : \prod \to S$ defined by (3) is a topological isomorphism of normed spaces.

From Theorem 2.1 it follows that it does not matter which one of norms $\|\cdot\|_p$, $1 \le p \le \infty$ we choose, and it justifies the present formulation of this definition.

We can use idempotent operators to characterize TDS.

Theorem 2.5. *Let $X_1, \ldots, X_n$ be linear subspaces of a normed space X, such that the sum $S = X_1 + X_2 + \cdots + X_n$ is ADS. The following statements are equivalent:*

(i) *S is TDS.*

(ii) *The map $E_i : S \to S$ defined by*

$$E_i(x_1 + x_2 + \cdots + x_n) = x_i, \qquad x_j \in X_j, \ j = 1, 2, \ldots, n$$

is continuous for every $i = 1, 2, \ldots, n$.

Proof. First, note that E_i is a linear idempotent with $\operatorname{Im} E_i = X_i$, $i = 1, \ldots, n$. Let $\prod = X_1 \times \cdots \times X_n$. By Theorem 2.1 we know that $(\prod, \|\cdot\|_1)$ is a normed space. Define

$$\varphi : \prod \to S, \quad \varphi(x_1, \ldots, x_n) = x_1 + \cdots + x_n.$$

Thus, $\varphi(x_1, \ldots, x_n) = E_1 x + \cdots + E_n x$, $x = x_1 + \cdots + x_n$. Since S is ADS it follows that φ is linear and bijective. Moreover,

$$\varphi^{-1}(x_1 + \cdots + x_n) = (E_1(x_1 + \cdots + x_n), \ldots, E_n(x_1 + \cdots + x_n)).$$

Also,

$$\|\varphi(x_1, \ldots, x_n)\| = \|x_1 + \cdots + x_n\| \le \|x_1\| + \cdots + \|x_n\| = \|(x_1, \ldots, x_n)\|_1.$$

It follows that φ is bounded, i.e. continuous. From this fact, by Definition 2.4 we conclude that S is TDS if and only if φ^{-1} is continuous.

(i) $\Longrightarrow$ (ii): Suppose that S is TDS, that is, suppose that φ^{-1} is bounded. We have

$$\begin{aligned}
\|E_i(x_1 + \cdots + x_n)\| &\le \|E_1(x_1 + \cdots + x_n)\| + \cdots + \|E_n(x_1 + \cdots + x_n)\| \\
&= \|(E_1(x_1 + \cdots + x_n), \ldots, E_n(x_1 + \cdots + x_n))\|_1 \\
&= \|\varphi^{-1}(x_1 + \cdots + x_n)\|_1 \le \|\varphi^{-1}\| \cdot \|x_1 + \cdots + x_n\|.
\end{aligned}$$

We conclude that E_i is continuous for every $i = 1, 2, \dots, n$.

(ii) $\Longrightarrow$ (i): Suppose now that E_i is continuous for every $i = 1, 2, \dots, n$. We have

$$\begin{aligned} \|\varphi^{-1}(x_1 + \cdots + x_n)\|_1 &= \|(E_1(x_1 + \cdots + x_n), \dots, E_n(x_1 + \cdots + x_n))\|_1 \\ &= \|E_1(x_1 + \cdots + x_n)\| + \cdots + \|E_n(x_1 + \cdots + x_n)\| \\ &\leq \|E_1\| \cdot \|x_1 + \cdots + x_n\| + \cdots + \|E_n\| \cdot \|x_1 + \cdots + x_n\| \\ &= (\|E_1\| + \cdots + \|E_n\|)\|x_1 + \cdots + x_n\|, \end{aligned}$$

so we conclude that φ^{-1} is bounded. □

Corollary 2.6. *Let $X_1, \dots, X_n$ be subspaces of a normed space X. If $S = X_1 + \cdots + X_n$ is TDS and S is closed in X, then every X_i is closed in X.*

Proof. Using notations from previous theorem, we know that $X_i = \operatorname{Im}(E_i) = \operatorname{Ker}(I - E_i)$ is closed in S, since E_i is a continuous projection from S onto X_i. By the assumption of this theorem, S is closed in X, we get that X_i is closed in X. □

If we have n unitary spaces $(H_1, \langle\cdot,\cdot\rangle_1)$, …, $(H_n, \langle\cdot,\cdot\rangle_n)$ then the scalar product on $\prod = H_1 \times \cdots \times H_n$ is defined by

$$\langle (x_1, \dots, x_n), (y_1, \dots, y_n) \rangle = \langle x_1, y_1 \rangle_1 + \cdots + \langle x_n, y_n \rangle_n.$$

Note that in the case of unitary spaces, the norm on $\prod$ is actually $\|\cdot\|_2$ norm. The product $\prod$ is a Hilbert space if and only if H_i is a Hilbert space for all $i = 1, 2, \dots, n$ (Theorem 2.1). For the isomorphism of unitary spaces we may require that it preserves the scalar product. It turns out that it is sufficient that the isomorphism preserves the norm induced by scalar product.

Lemma 2.7. *Let $(H_1, \langle\cdot,\cdot\rangle_1)$ and $(H_2, \langle\cdot,\cdot\rangle_2)$ be unitary spaces and let $\varphi : H_1 \to H_2$ be an isomorphism between linear spaces H_1 and H_2. The following statements are equivalent:*

(i) $\|\varphi(x)\|_2 = \|x\|_1$, *for every $x \in H_1$;*

(ii) $\langle \varphi(x), \varphi(y) \rangle_2 = \langle x, y \rangle_1$, *for every $x, y \in H_1$.*

Proof. It is clear that (ii) implies (i). Using the well-known polarization identity

$$\langle x, y \rangle = \frac{1}{4}\left(\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2\right)$$

we conclude that (i) implies (ii). □

Definition 2.8. *Let $H_1, \dots, H_n$ be subspaces of a unitary space $(H, \langle\cdot,\cdot\rangle)$. Suppose that on the linear space $\prod = H_1 \times \cdots \times H_n$ the scalar product is defined by*

$$\langle (x_1, \dots, x_n), (y_1, \dots, y_n) \rangle = \langle x_1, y_1 \rangle + \cdots + \langle x_n, y_n \rangle.$$

The sum $S = H_1 + \cdots + H_n$ is the orthogonal direct sum (ODS) if the map $\varphi : \prod \to S$ defined by (3) is an isometric isomorphism. ODS is denoted by

$$H_1 \overset{\perp}{\oplus} \cdots \overset{\perp}{\oplus} H_n, \quad \text{i.e.} \quad \overset{\perp}{\bigoplus}{}_{i=1}^{n} H_i.$$

It is clear that every ODS is TDS.

Theorem 2.9. *Let $H_1, \dots, H_n$ be subspaces of a unitary space H and let*

$$S = H_1 + \cdots + H_n.$$

The following statements are equivalent:

(i) *S is ODS.*

(ii) $H_i \perp H_j$ *for all* $i, j \in \{1, \dots, n\}$, $i \neq j$.

Proof. (i) ⇒ (ii): Suppose that S is ODS. By Definition 2.8, the map $\varphi : (x_1, \dots, x_n) \to x_1 + \dots + x_n$ is an isometric isomorphism from $\prod = H_1 \times \dots \times H_n$ onto S. By Lemma 2.7 we have that

$$\langle \varphi(x), \varphi(y) \rangle_H = \langle x, y \rangle_{\prod},$$

for all $x, y \in \prod$. Let $x_1 \in H_1$ and $x_2 \in H_2$. If we put $x = (x_1, 0, \dots, 0)$, $y = (0, x_2, 0, \dots, 0)$ then we easily obtain that

$$\langle x_1, x_2 \rangle_H = \langle \varphi(x), \varphi(y) \rangle_H = \langle x, y \rangle_{\prod} = 0.$$

Therefore, $H_1 \perp H_2$. Similarly, $H_i \perp H_j$ for all $i, j \in \{1, \dots, n\}$, $i \neq j$.

(ii) ⇒ (i): Suppose that $H_i \perp H_j$ for $i \neq j$, and let us first prove that S is ADS. Let $x_1 + \dots + x_n = 0$, $x_i \in H_i$. We have

$$\begin{aligned} \|x_1\|^2 &= \langle x_1, x_1 \rangle_H = \langle x_1, x_1 \rangle_H + \langle x_1, x_2 \rangle_H + \dots + \langle x_1, x_n \rangle_H \\ &= \langle x_1, x_1 + x_2 + \dots + x_n \rangle_H = \langle x_1, 0 \rangle_H = 0. \end{aligned}$$

Hence, $x_i = 0$ for all $i = 1, 2, \dots, n$. To prove that S is ODS, it remains to prove that $\|\varphi(x)\| = \|x\|$, for all $x \in \prod$. By the orthogonality, we have

$$\|\varphi(x_1, \dots, x_n)\|^2 = \|x_1 + \dots + x_n\|^2 = \|x_1\|^2 + \dots + \|x_n\|^2 = \|(x_1, \dots, x_n)\|_2^2.$$

□

Remark 2.10. *Let* H_1 *and* H_2 *be closed subspaces of a unitary space* H *such that* $H_1 \perp H_2$. *Then the sum* $H_1 \overset{\perp}{\oplus} H_2$ *need not be closed in* H.

However, if H *is Hilbert space then this sum is closed in* H. *For example, take* $x \in \mathrm{cl}(H_1 \overset{\perp}{\oplus} H_2)$. *Then there is a sequence* $z_n = x_n + y_n \in H_1 \overset{\perp}{\oplus} H_2$, *with* $x_n \in H_1$, $y_n \in H_2$ *and* $\lim\limits_{n\to\infty} z_n = z$. *By the Pythagorean Theorem, we conclude that* $(x_n)_n$ *is a Cauchy sequence in* H_1, *and* $(y_n)_n$ *is a Cauchy sequence in* H_2. *Since* H_1 *and* H_2 *are closed, we get* $\lim\limits_{n\to\infty} x_n = x \in H_1$ *and* $\lim\limits_{n\to\infty} y_n = y \in H_2$. *Consequently,* $z = x + y \in H_1 \overset{\perp}{\oplus} H_2$, *so* $H_1 \overset{\perp}{\oplus} H_2$ *is closed.*

Inductivey, if H_i, $i = 1, 2, \dots, n$ *are mutually orthogonal subspaces of a Hilbert space* H *then, by induction on* n, *we can easily show that the sum* $H_1 \overset{\perp}{\oplus} \cdots \overset{\perp}{\oplus} H_n$ *is closed in* H.

If H is a Hilbert space then an operator $A \in \mathcal{B}(H)$ is self-adjoint if $A = A^*$.

Definition 2.11. *Let* H *be a unitary space. A bounded idempotent* $P : H \to H$ *is orthogonal if* $H = \mathrm{Im}\, P \overset{\perp}{\oplus} \mathrm{Ker}\, P$.

Remark 2.12. *Let* $P \in \mathcal{B}(H)$, $P \neq 0$, *be a bounded idempotent on a unitary space* H. *The following characterizations are well-known:*

$$P \text{ is orthogonal} \quad \Leftrightarrow \quad \mathrm{Im}\, P \perp \mathrm{Ker}\, P \quad \Leftrightarrow \quad \|P\| = 1. \tag{5}$$

If H *is a Hilbert space then an idempotent* $P \in \mathcal{B}(H)$ *is orthogonal if and only if* P *is self-adjoint.*

Theorem 2.13. *Let* $H_1, \dots, H_n$ *be subspaces of a unitary space* H *such that the sum* $S = H_1 + \dots + H_n$ *is ADS. The following statements are equivalent:*

(i) *S is ODS.*

(ii) *The map* $E_i : S \to S$ *defined by*

$$E_i(x_1 + x_2 + \dots + x_n) = x_i, \qquad x_j \in H_j,\ j = 1, 2, \dots, n$$

is an orthogonal bounded idempotent for every $i = 1, 2, \dots, n$.

Proof. The proof follows by Theorem 2.5, its proof, Theorem 2.9 and property (5). □

3. Decomposition of the identity

Let R be a ring with identity 1.

Definition 3.1. *For idempotents $e, f \in R$ we say that they are mutually orthogonal if $ef = fe = 0$. Idempotents $e_1, e_2, \dots, e_n \in R$ are mutually orthogonal if they are mutually orthogonal in pairs. The equality (in the case when it holds)*

$$1 = e_1 + e_2 + \cdots + e_n,$$

where $e_1, e_2, \dots, e_n \in R$ are mutually orthogonal idempotents, is called the decomposition of the identity in the ring R.

It should be distinguished the notion of mutually orthogonal idempotents e and f given by Definition 3.1 and the notion of orthogonal idempotent e in a ring with involution R (this is an element which satisfies $e = e^2 = e^*$ and we will call such element as self-adjoint idempotent rather then orthogonal idempotent). Because of the above reason some authors say that mutually orthogonal idempotents $e_1, \dots, e_n$ are pairwise disjoint.

Let $1 = e_1 + \cdots + e_m$ and $1 = f_1 + \cdots + f_n$ be two decompositions of the identity in the ring R. An arbitrary element $x \in R$ can be written in the form

$$x = 1 \cdot x \cdot 1 = (e_1 + \cdots + e_m) \cdot x \cdot (f_1 + \cdots + f_n) = \sum_{i=1}^{m} \sum_{j=1}^{n} e_i x f_j = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}, \tag{6}$$

where $x_{ij} = e_i x f_j \in e_i R f_j$. Note that for all $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$ the sets $e_i R e_i$ i $f_j R f_j$ are rings, and the set $e_i R f_j$ is at the same time left $e_i R e_i$-module and right $f_j R f_j$-module. Moreover, for all $r \in e_i R e_i$, $s \in f_j R f_j$ and $x \in e_i R f_j$, we have $(rx)s = r(xs)$. Therefore, $e_i R f_j$ is $e_i R e_i$-$f_j R f_j$-bimodule. It is not difficult to check that the sum (6) defines decomposition of the ring R in the direct sum of these bimodules:

$$R = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} e_i R f_j. \tag{7}$$

It is suitable to write x in the matrix form

$$x = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{bmatrix}_{e \times f}. \tag{8}$$

It is important to emphasize that every element $x \in R$ has the unique matrix representation with respect to decompositions $1 = e_1 + \cdots + e_m$ and $1 = f_1 + \cdots + f_n$.
If $y = [y_{ij}]_{e \times f}$ then it is clear that $x + y = [x_{ij} + y_{ij}]_{e \times f}$. Let $1 = g_1 + \cdots + g_k$ be another decomposition of the identity in the ring R. As above,

$$R = \bigoplus_{j=1}^{n} \bigoplus_{l=1}^{k} f_j R g_l,$$

and for every $z \in R$ there is the unique matrix representation

$$z = [z_{jl}]_{f \times g},$$

where $z_{jl} = f_j z g_l \in f_j R g_l$, $j = 1, \dots, n$, $l = 1, \dots, k$. By the mutual orthogonality of involved idempotents, it is easy to show that for the multiplication of elements x and z we can use the ordinary matrix rule:

$$xz = [w_{il}]_{e \times g}, \quad \text{where} \quad w_{il} = \sum_{j=1}^{n} x_{ij} z_{jl}.$$

When $m = n$ and $e_i = f_i$, $i = 1, 2, \ldots, n$, the decomposition (7) is known as the two-sided Peirce decomposition of the ring R, [3].

Suppose now that the ring R possesses an involution $*$. From (6) we have

$$x^* = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^* = \sum_{j=1}^{n} \sum_{i=1}^{m} f_j^* x^* e_i^*,$$

i.e.

$$x^* = \begin{bmatrix} x_{11}^* & \cdots & x_{m1}^* \\ \vdots & \ddots & \vdots \\ x_{1n}^* & \cdots & x_{mn}^* \end{bmatrix}_{f^* \times e^*}, \tag{9}$$

where the representation of x^* is with respect to decompositions $1 = f_1^* + \cdots + f_n^*$ and $1 = e_1^* + \cdots + e_m^*$.

The decomposition of the identity $1 = e_1 + \cdots + e_n$ is orthogonal if e_i $(i = 1, 2, \ldots, n)$ are self-adjoint idempotents.

If X is a normed space, then the set $\mathcal{B}(X)$ can be considered as the ring with identity $I = I_X \in \mathcal{B}(X)$. Because of that, the notion of the decomposition of the identity I in the ring $\mathcal{B}(X)$ makes sense.

Remark 3.2. *When $e \in R$ is an idempotent, then it is clear that $e + (1 - e) = 1$ is a decomposition of the identity in the ring R.*

Suppose now that we have three idempotents e_1, e_2 and e_3, such that $e_1 + e_2 + e_3 = 1$. By squaring the equality $e_1 + e_2 = 1 - e_3$ we obtain $e_1 + e_2 + e_1e_2 + e_2e_1 = 1 - e_3$, that is $e_1e_2 = -e_2e_1$. Pre-multiplying this equation by e_1 we obtain $e_1e_2 = -e_1e_2e_1$. Post-multiplying $e_1e_2 = -e_2e_1$ by e_1 we obtain $e_1e_2e_1 = -e_2e_1$. It follows that $e_1e_2 = e_2e_1$, so $e_1e_2 = -e_1e_2$. Now, it is easy to see that $(e_1e_2)^2 = e_1e_2$. Thus, we have idempotent $p = e_1e_2$ such that $p + p = 0$. If we suppose that $R = \mathcal{B}(X)$ then it is clear that $p = 0$.

Now we can obtain the result of Stampfli [7]. If E_1, E_2, E_3 are idempotents from $\mathcal{B}(X)$ such that $E_1 + E_2 + E_3 = I$ then $E_iE_j = 0$ for $i \neq j$. But, Stampfli showed that there exists a Hilbert space H and idempotents $E_1, \ldots, E_4 \in \mathcal{B}(H)$ such that $E_1 + \cdots + E_4 = I$ and $E_1E_2 \neq 0$. In the same paper it is proved that if H is Hilbert space and if $E_1, \ldots, E_n$ are self-adjoint idempotents from $\mathcal{B}(H)$ with $E_1 + \cdots + E_n = I$ then $E_iE_j = 0$ for $i \neq j$.

Suppose now that H is a finite dimensional space, and let $E_1, \ldots, E_n$ be idempotents from $\mathcal{B}(H)$ (that is E_i are idempotent complex matrices of the same order) with $E_1 + \cdots + E_n = I$. It is well-known that then $E_iE_j = 0$ for $i \neq j$ (for the proof see, for example, Theorem 2.49 in [1]). For the sake of completeness we give the proof. Recall that the rank of an idempotent matrix is equal to its trace. Hence

$$n = trace(I) = trace(E_1) + \cdots + trace(E_n) = \operatorname{rank}(E_1) + \cdots + \operatorname{rank}(E_n).$$

It follows that $H = \operatorname{Im} E_1 \oplus \cdots \oplus \operatorname{Im} E_n$. Using $E_1 + \cdots + E_n = I$ we obtain

$$\left(\sum_{i=1}^{n} E_i \right) E_j x = E_j x$$

for all $x \in H$ and $j = 1, \ldots, n$. Since $E_iE_jx \in \operatorname{Im} E_i$, $E_jx \in \operatorname{Im} E_j$ and the sum of $\operatorname{Im} E_i$'s is direct, we must have $E_iE_jx = 0$, $i \neq j$. Hence, $E_iE_j = 0$ for $i \neq j$.

The following Theorem 3.4 connects the notion of decomposition of the identity in the ring $\mathcal{B}(X)$ with the notion of topological direct sum.

Lemma 3.3. *If X is a normed space, and if $E_1, E_2 \in \mathcal{B}(X)$ are two idempotents such that $E_1E_2 = E_2E_1 = 0$. Then $E_1 + E_2$ is an idempotent and*

$$\operatorname{Im}(E_1 + E_2) = \operatorname{Im} E_1 \oplus \operatorname{Im} E_2,$$

where the above sum is TDS.

Proof. It is evident that $E_1 + E_2$ is an idempotent and $\operatorname{Im}(E_1 + E_2) \subseteq \operatorname{Im} E_1 + \operatorname{Im} E_2$. If $x \in \operatorname{Im} E_1$ then $0 = E_2E_1x = E_2x$, so $x = (E_1 + E_2)x \in \operatorname{Im}(E_1 + E_2)$. Similarly, $\operatorname{Im} E_2 \subseteq \operatorname{Im}(E_1 + E_2)$. If $x \in \operatorname{Im} E_1 \cap \operatorname{Im} E_2$ then $x = E_1x = E_2x$, so $x = E_1^2x = E_1E_2x = 0$. By Theorem 2.5, it is easy to prove that sum is TDS. □

Theorem 3.4. *(See also [2].)*

(i) *Let $X_1, \dots, X_n$ be subspaces of a normed space X, and let*

$$X = X_1 \oplus X_2 \oplus \cdots \oplus X_n,$$

where the sum is TDS. Then there exist idempotents $E_i \in \mathcal{B}(X)$, $i = 1, 2, \dots, n$, such that

$$I = E_1 + E_2 + \cdots + E_n$$

is a decomposition of the identity I in the ring $\mathcal{B}(X)$ and $\operatorname{Im} E_i = X_i$ for all $i = 1, 2, \dots, n$.

(ii) *Let*

$$I = E_1 + E_2 + \cdots + E_n$$

be a decomposition of the identity in the ring $\mathcal{B}(X)$. Then

$$X = \operatorname{Im} E_1 \oplus \operatorname{Im} E_2 \oplus \cdots \oplus \operatorname{Im} E_n, \tag{10}$$

where the above sum is TDS. Moreover, if $J \subseteq \{1, 2, \dots, n\}$ then $\sum_{i \in J} E_i \in \mathcal{B}(X)$ is an idempotent and

$$\operatorname{Im}\left(\sum_{i \in J} E_i\right) = \bigoplus_{i \in J} \operatorname{Im} E_i, \tag{11}$$

where the above sum is TDS.

Proof. (i): For $i = 1, 2, \dots, n$ define the map $E_i : X \to X$ with

$$E_i(x_1 + \cdots + x_n) = x_i, \qquad x_j \in X_j,\ j = 1, 2, \dots, n.$$

Since $X = X_1 \oplus X_2 \oplus \cdots \oplus X_n$ is TDS, by Theorem 2.5 and its proof it follows that E_i is a bounded idempotent and $\operatorname{Im} E_i = X_i$ for all $i = 1, 2, \dots, n$. It is evident that

$$(E_1 + \cdots + E_n)(x_1 + \cdots + x_n) = x_1 + \cdots + x_n,$$

so $E_1 + \cdots + E_n = I$. Also, for $i \neq j$ we have

$$E_i(E_j(x_1 + \cdots + x_n)) = E_i(x_j) = 0,$$

so $E_iE_j = 0$. Therefore, $I = E_1 + E_2 + \cdots + E_n$ is a decomposition of the identity in the ring $\mathcal{B}(X)$.

(ii): Suppose that $I = E_1 + E_2 + \cdots + E_n$ is a decomposition of the identity in $\mathcal{B}(X)$. Without any loss of generality, suppose that $J = \{1, 2, \dots, k\}$, $1 \leq k \leq n$. Using Lemma 3.3, by induction on k one can prove that the sum $\sum_{i=1}^{k} E_i$ is a bounded idempotent,

$$\operatorname{Im}\left(\sum_{i=1}^{k} E_i\right) = \sum_{i=1}^{k} \operatorname{Im} E_i,$$

and the sum $S_J = \sum_{i=1}^{k} \operatorname{Im} E_i$ is ADS. To prove that the sum S_J is TDS, by Theorem 2.5, it is sufficient to prove that the map $P_i : S_J \to S_J$, $i = 1, 2, \dots, k$, defined by $P_i(x_1 + \cdots + x_k) = x_i$, $x_j \in \operatorname{Im} E_j$, $j = 1, 2, \dots, k$, is continuous for all $i = 1, 2, \dots, k$. Since $x_j \in \operatorname{Im} E_j$ and $E_iE_j = 0$ for $i \neq j$, we have that $x_j = E_jx_j$ and

$$E_i(x_1 + \cdots + x_k) = x_i = P_i(x_1 + \cdots + x_k).$$

Therefore, for $x = x_1 + \cdots + x_k$, we have $\|P_i x\| = \|E_i x\| \leq \|E_i\|\|x\|$, so P_i is bounded. The equality (10) is obtained in the special case when $J = \{1, 2, \ldots, n\}$. This completes the proof. Note that

$$\operatorname{Ker} E_j = \bigoplus_{i=1, i\neq j}^{n} \operatorname{Im} E_i. \tag{12}$$

□

Corollary 3.5. *Let $X_1, X_2 \ldots, X_n$ be subspaces of a normed space X, and let*

$$S = X_1 \oplus X_2 \oplus \cdots \oplus X_n,$$

be TDS. Let $J \subseteq \{1, 2, \ldots, n\}$. Then the sum

$$S_J = \bigoplus_{i\in J} X_i$$

is TDS and S_J is a closed subspace in S. Thus, if S is closed in X then S_J is closed in X.

Proof. Since $S = X_1 \oplus \cdots \oplus X_n$ is TDS, by Theorem 3.4 (i) it follows that there exists the decomposition of the identity in the ring $\mathcal{B}(S)$, $I_S = E_1 + \cdots + E_n$ where $\operatorname{Im} E_i = X_i$. Now, by part (ii) of Theorem 3.4 we obtain that

$$S_J = \bigoplus_{i\in J} X_i = \bigoplus_{i\in J} \operatorname{Im} E_i = \operatorname{Im}\left(\sum_{i\in J} E_i\right)$$

is TDS. Since $\sum_{i\in J} E_i$ is a bounded idempotent on S, it follows that S_J is closed subspace in S. □

When X is a Banach space, we can characterize the TDS of subspaces in a much easier way than in the case when X is a general normed space.

As a corollary of the Bounded inverse theorem, we obtain the following result.

Theorem 3.6. *Let X be a Banach space, and let $X_1, X_2, \ldots, X_n$ be subspaces of X such that the sum $S = X_1 + X_2 + \cdots + X_n$ is ADS. If the subspaces $X_1, X_2, \ldots, X_n$ and S are closed in X, then S is TDS and the set $\sum_{i\in J} X_i$ is closed subspace in X for all $J \subseteq \{1, 2, \ldots, n\}$.*

Proof. Let $\prod = X_1 \times X_2 \times \cdots \times X_n$ and consider the norm $\|\cdot\|_1$ on $\prod$. Let $\varphi : \prod \to S$ be the map defined by $\varphi(x_1, \ldots, x_n) = x_1 + \cdots + x_n$. In the proof of Theorem 2.5 we proved that φ is bounded bijective linear operator. Subspaces X_i, $i = 1, 2, \ldots, n$ and S are closed in X so they are Banach spaces. By Theorem 2.1, the space $\prod$ is Banach. From the Bounded inverse theorem, it follows that $\varphi^{-1} : S \to \prod$ is a bounded operator. Thus, φ is a topological isomorphism, so, by Definition 2.4, S is TDS. By Corollary 3.5 it follows that the set $\sum_{i\in J} X_i$ is a closed subspace of X. □

In the proof of Theorem 3.6, we also could use the following argument to prove that the space $\sum_{i\in J} X_i$ is closed in X. Namely, it is easy to show that the set

$$S_k = X_1 \times \cdots \times X_k \times \{0\} \times \cdots \times \{0\}$$

is closed in $\prod$. Since φ^{-1} is continuous, we have that the set $(\varphi^{-1})^{-1}(S_k) = \varphi(S_k) = X_1 + \cdots + X_k$ is closed subspace in S. Therefore, it is closed in X, since S is closed in X.

Corollary 3.7. *Let $H_1, \ldots, H_n$ be subspaces (not necessarily closed) of a unitary space H such that $X_i \perp X_j$ for $i \neq j$, $i, j = 1, 2, \ldots, n$. Let $J \subseteq \{1, \ldots, n\}$. If the sum (which is orthogonal by Theorem 2.9) $S = X_1 + \cdots + X_n$ is closed in X then the sum*

$$S_J = \sum_{i\in J} X_i$$

is closed subspace in X.

Proof. The proof follows by Theorem 2.9 and Corollary 3.5. □

Let H be a unitary space. Decomposition of the identity in the ring $\mathcal{B}(H)$, $I = E_1 + \cdots + E_n$ is orthogonal if E_i, $i = 1, \ldots, n$ are orthogonal idempotents. When H is a Hilbert space then this definition agrees with previously introduced definition of orthogonal decomposition of the identity in the ring R.

Theorem 3.8. (i) *Let $X_1, X_2, \ldots, X_n$ be subspaces of a unitary space H and let*

$$X = X_1 \overset{\perp}{\oplus} X_2 \overset{\perp}{\oplus} \cdots \overset{\perp}{\oplus} X_n.$$

Then there exist orthogonal idempotents $E_i \in \mathcal{B}(X)$, $i = 1, 2, \ldots, n$, such that

$$I = E_1 + E_2 + \cdots + E_n$$

is an orthogonal decomposition of the identity in the ring $\mathcal{B}(H)$ and $\operatorname{Im} E_i = X_i$ for all $i = 1, 2, \ldots, n$.

(ii) *Let*

$$I = E_1 + E_2 + \cdots + E_n$$

be an orthogonal decomposition of the identity in the ring $\mathcal{B}(H)$. Then

$$X = \operatorname{Im} E_1 \overset{\perp}{\oplus} \operatorname{Im} E_2 \overset{\perp}{\oplus} \cdots \overset{\perp}{\oplus} \operatorname{Im} E_n. \tag{13}$$

Moreover, if $J \subseteq \{1, 2, \ldots, n\}$ then $\sum_{i \in J} E_i \in \mathcal{B}(X)$ is orthogonal idempotent and

$$\operatorname{Im}\left(\sum_{i \in J} E_i\right) = \overset{\perp}{\bigoplus_{i \in J}} \operatorname{Im}(E_i). \tag{14}$$

Proof. The proof follows from Theorem 3.4, its proof, the identity (12), Theorem 2.9 and property (5). □

Remark 3.9. *Let X be a normed space and let X_1 and X_2 be closed subspaces such that $X = X_1 + X_2$ and $X_1 \cap X_2 = \{0\}$, i.e. $X = X_1 \oplus X_2$ is ADS. Then the map $P : X \to X$ defined by $P(x_1 + x_2) = x_1$, $x_1 \in X_1$, $x_2 \in X_2$ is linear and $P^2 = P$. But, P is not necessarily a bounded operator. If X is Banach space then P is bounded, which can be proved using the Closed graph theorem. Thus, if X is only normed space, then the closeness of subspaces X_1 and X_2 is not sufficient condition for the ADS $X_1 \oplus X_2$ to be TDS.*

It should be noted that TDS is defined by some authors as an ADS $X = X_1 \oplus X_2$ for which X_1 and X_2 are closed subspaces in X. Because of that, we give the following definition.

Definition 3.10. *A subspace Z of a normed space X is topologically complemented (complemented for short) in X if there exists a subspace $Z_1 \subseteq X$ such that $X = Z \oplus Z_1$ where the sum is TDS.*

Lemma 3.11. *Let Z be a subspace of a normed space X.*

(i) *Z is complemented in X if and only if there exists an idempotent $P \in \mathcal{B}(X)$ such that $Z = \operatorname{Im} P$.*

(ii) *If X is a Banach space, then Z is complemented in X if and only if Z is closed in X and there exists a closed subspace $Z_1 \subseteq X$ such that $X = Z + Z_1$ and $Z \cap Z_1 = \{0\}$.*

(iii) *If X is a Hilbert space then Z is complemented in X if and only if Z is closed in X.*

Proof. The statemet (i) follows by Theorem 2.5. The statement (ii) follows by Theorem 3.6. The statement (iii) follows by the well-known property that for every closed subspace Z of a Hilbert space X holds $X = Z \overset{\perp}{\oplus} Z^{\perp}$. □

4. Operator matrix induced by a direct sum

Let X and Y be normed spaces. In previous section we proved that if we have two decompositions of the identity in the ring R then any element $x \in R$ can be represented in an appropriate matrix form (8). In this section we will consider the case when $R = \mathcal{B}(X)$. Moreover, it turns out that the analogy holds even on the set $\mathcal{B}(X, Y)$, which is not a ring.

Let $I_X = F_1 + \cdots + F_n$ and $I_Y = E_1 + \cdots + E_m$ be decompositions of the identities in rings $\mathcal{B}(X)$ and $\mathcal{B}(Y)$, respectively. Suppose that $A \in \mathcal{B}(X, Y)$. Then

$$A = I_Y \cdot A \cdot I_X = (E_1 + \cdots + E_m)A(F_1 + \cdots + F_n) = \sum_{i=1}^{m} \sum_{j=1}^{n} E_i A F_j, \tag{15}$$

and thus, for $x = x_1 + \cdots + x_n \in \operatorname{Im} F_1 \oplus \cdots \oplus \operatorname{Im} F_n = X$, we have

$$Ax = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} x_j, \tag{16}$$

where the operator $A_{ij} : \operatorname{Im} F_j \to \operatorname{Im} E_i$ is defined by $A_{ij}x_j := E_i A F_j x_j = E_i A F_j x$. Therefore, $\|A_{ij}x_j\| \leq \|E_i A F_j\| \|x_j\|$, so $A_{ij} \in \mathcal{B}(\operatorname{Im} F_j, \operatorname{Im} E_i)$.

Suppose now that $A_{ij} \in \mathcal{B}(\operatorname{Im} F_j, \operatorname{Im} E_i)$ and let the operator $A : X \to Y$ be defined by (16). It follows that

$$\begin{aligned} \|Ax\| &\leq mM(\|x_1\| + \cdots + \|x_n\|) = mM(\|F_1 x\| + \cdots + \|F_n x\|) \\ &\leq mM(\|F_1\| + \cdots + \|F_n\|)\|x\|, \end{aligned}$$

where $M = \max\left\{\|A_{ij}\| : i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\right\}$. Thus, $A \in \mathcal{B}(X, Y)$. Therefore, $A \in \mathcal{B}(X, Y)$ if and only if $A_{ij} \in \mathcal{B}(\operatorname{Im} F_j, \operatorname{Im} E_i)$. In that case, the equation (16), i.e. (15) can be written in the following matrix form

$$A = \begin{bmatrix} A_{11} & \ldots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \ldots & A_{mn} \end{bmatrix} : \begin{bmatrix} \operatorname{Im} F_1 \\ \vdots \\ \operatorname{Im} F_n \end{bmatrix} \to \begin{bmatrix} \operatorname{Im} E_1 \\ \vdots \\ \operatorname{Im} E_m \end{bmatrix},$$

which is analogous with representation (8) for elements $x \in R$.

Similarly to the case of a ring, the addition and the multiplication of operators written in matrix forms are performed using the known matrix rules. More precisely, if the operator $B \in \mathcal{B}(X, Y)$ has the matrix form $[B_{ij}]_{m \times n}$ with respect to decompositions $I_X = F_1 + \cdots + F_n$ and $I_Y = E_1 + \cdots + E_m$ then the matrix form of $A + B$ is $[A_{ij} + B_{ij}]_{m \times n}$. Further, let $I_Z = G_1 + \cdots + G_k$ be the decomposition of the identity of $\mathcal{B}(Z)$, where Z is another normed space. Let $[C_{jl}]_{n \times k}$ be the matrix form of an operator $C \in \mathcal{B}(Z, X)$. Then the operator $AC \in \mathcal{B}(Z, Y)$ has the matrix form $[D_{il}]_{m \times k}$, where $D_{il} = \sum_{j=1}^{n} A_{ij} C_{jl}$. The proof of these properties are easy.

The case when X and Y are complete vector spaces are the most interesting. Because of this, in the following theorem, we will summarize the previous considerations in the case of Banach spaces.

Theorem 4.1. *Let $X_1, \ldots, X_n$ be closed subspaces of Banach space X, and let $Y_1, \ldots, Y_m$ be closed subspaces of Banach space Y. Let*

$$X = X_1 \oplus \cdots \oplus X_n \quad \text{and} \quad Y = Y_1 \oplus \cdots \oplus Y_m,$$

where the above sums are ADS. Then these sums are TDS and they induce the decomposition of the identity in the ring $\mathcal{B}(X)$

$$I_X = F_1 + \cdots + F_n,$$

where $\operatorname{Im} F_j = X_j$, $j = 1, 2, \ldots, n$ and the decomposition of the identity in the ring $\mathcal{B}(Y)$,

$$I_Y = E_1 + \cdots + E_m,$$

where $\operatorname{Im} E_i = Y_i$*,* $i = 1, 2, \ldots, m$*. Moreover, for* $J \subseteq \{1, 2, \ldots, n\}$ *and* $I \subseteq \{1, 2, \ldots, m\}$*, the sums*

$$\bigoplus_{j\in J} X_j \ \text{and} \ \bigoplus_{i\in I} Y_i$$

are closed subspaces in X *and* Y *respectively. Furthermore, let* $A : X \to Y$ *be a liner operator. Then* A *can be represented in the following matrix form*

$$A = \begin{bmatrix} A_{11} & \ldots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \ldots & A_{mn} \end{bmatrix} : \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} \to \begin{bmatrix} Y_1 \\ \vdots \\ Y_m \end{bmatrix}, \tag{17}$$

where $A_{ij} : X_j \to Y_i$ *is linear operators defined by* $A_{ij}x_j = E_iAF_jx_j$*. If* A *is bounded then the operators* A_{ij} *are bounded. Conversely, let* $A_{ij} : X_j \to Y_i$ *be linear operators and let the operator* $A : X \to Y$ *be defined by* $Ax = \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}x_j$*, where* $x = x_1 + \cdots + x_n$*,* $x_j \in X_j$*. If the operators* A_{ij} *are bounded then the operator* A *is bounded. In this case, every submatrix of the matrix (17) defines appropriate bounded operator.*

Suppose now that X and Y are Hilbert spaces. For $A \in \mathcal{B}(X, Y)$ we can consider its Hilbert adjoint operator $A^* \in \mathcal{B}(Y, X)$. Let

$$X = X_1 \overset{\perp}{\oplus} \cdots \overset{\perp}{\oplus} X_n \ \text{ and } \ Y = Y_1 \overset{\perp}{\oplus} \cdots \overset{\perp}{\oplus} Y_m.$$

Then the appropriate decompositions of the identity in the rings $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ are orthogonal. Using the representation (9), we can easily obtain that A^* has the following matrix form:

$$A^* = \begin{bmatrix} A_{11}^* & \ldots & A_{m1}^* \\ \vdots & \ddots & \vdots \\ A_{1n}^* & \ldots & A_{mn}^* \end{bmatrix} : \begin{bmatrix} Y_1 \\ \vdots \\ Y_m \end{bmatrix} \to \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}.$$

Of course, the above representation is not valid in the case when observed sums are not orthogonal.

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