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Pseudospectra of elements of reduced Banach algebras II

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Abstract. Let A be a Banach algebra with identity 1 and $p \in A$ be a non-trivial idempotent. Then $q = 1 - p$ is also an idempotent. The subalgebras pAp and qAq are Banach algebras, called reduced Banach algebras, with identities p and q respectively. Let $x \in A$ be such that $pxp = xp$, and $\varepsilon > 0$. We examine the relationship between the spectrum of $x \in A$, $\sigma(A, x)$, and the spectra of $pxp \in pAp$, $\sigma(pAp, pxp)$ and $qxq \in qAq$, $\sigma(qAq, qxq)$. Similarly, we examine the relationship betweeen the ε-pseudospectrum of $x \in A$, $\Lambda_\varepsilon(A, x)$ and ε-pseudospectra of $pxp \in pAp$, $\Lambda_\varepsilon(pAp, pxp)$ and of $qxq \in qAq$, $\Lambda_\varepsilon(qAq, qxq)$.

1. Introduction

Consider a Hilbert space H that can be expressed as the direct sum of two closed subspaces H_1 and H_2. Let P and $Q = I - P$ be the bounded linear projections onto H_1 and H_2 respectively. Suppose T is an operator such that H_1 is invariant under T, but H_2 is not. This is equivalent to $PTP = TP \neq PT$. In this case, the operator T has an upper-triangular form with respect to the decomposition of $H = H_1 \oplus H_2$, i.e.

$$T = \begin{bmatrix} PTP & PTQ \\ 0 & QTQ \end{bmatrix}_P .$$

Suppose T commutes with P (and hence with Q), then $PTQ = 0$, and it is easily seen that

$$\sigma(B(H), T) = \sigma(B(H_1), PTP) \cup \sigma(B(H_2), QTQ). \tag{1}$$

This is not necessarily true when P and T do not commute. Several authors have studied different conditions on T and P under which (1) holds in the upper triangular case. See [1], [2], [3] and [4].

We would like to extend this study to the decomposition of the spectrum of an element in an arbitrary Banach algebra. If $p \in A$ is an idempotent in a Banach algebra, then $pAp = \{pxp : x \in A\}$ is a closed subalgebra of A with identity element p, called a reduced Banach algebra. If $q = 1 - p$, then q is an idempotent too, and

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thus qAq is also a closed subalgebra of A, with identity element q. One can ask the following question: For $x \in A$ such that $pxp = xp$, when do we have

$$\sigma(A, x) = \sigma(pAp, pxp) \cup \sigma(qAq, qxq)?$$

In general, this is not true. We always have

$$\sigma(A, x) \subseteq \sigma(pAp, pxp) \cup \sigma(qAq, qxq).$$

Equality holds for certain kinds of elements. Some relations including the left and right spectrum, and the boundary of the spectrum hold. This is discussed in Section 3.

We next examine if a decomposition holds for the ε-pseudospectrum $\Lambda_\varepsilon(A, x)$ of $x \in A$ in terms of $\Lambda_\varepsilon(pAp, pxp)$ and $\Lambda_\varepsilon(qAq, qxq)$. Several results similar to the results for the spectrum hold, involving the left and right ε-pseudospectrum as well as the boundary of the pseudospectrum. This is discussed in Section 4.

In an earlier paper [6], the authors have discussed similar questions when x and p commute.

The primary objective of this note is to show that the pseudospectra of certain elements of Banach algebras can be decomposed into the pseudospectra of simpler elements of certain reduced subalgebras. This could make it easier to compute the pseudospectrum of certain 'triangular' operators or elements.

2. Definitions

Let A be a complex Banach algebra with unit 1. For $\lambda \in \mathbb{C}$, $\lambda.1$ is identified with λ. Let $\mathrm{Inv}(A) = \{x \in A : x \text{ is invertible in } A\}$ and $\mathrm{Sing}(A) = \{x \in A : x \text{ is not invertible in } A\}$.

Definition 2.1. *The spectrum of an element $x \in A$ is defined as:*

$$\sigma(A, x) := \{\lambda \in \mathbb{C} : \lambda - x \in Sing(A)\}.$$

The complement of the spectrum of x is known as the resolvent and is denoted by $\rho(A, x)$ or simply by $\rho(x)$.

Definition 2.2. *The spectral radius of an element $x \in A$ is defined as:*

$$r(A, x) := sup\{|\lambda| : \lambda \in \sigma(A, x)\}.$$

Definition 2.3. *The left spectrum of an element $x \in A$ is defined as:*

$$\sigma^l(A, x) := \{\lambda \in \mathbb{C} : \lambda - x \text{ is not left invertible}\}.$$

Definition 2.4. *The right spectrum of an element $x \in A$ is defined as:*

$$\sigma^r(A, x) := \{\lambda \in \mathbb{C} : \lambda - x \text{ is not right invertible}\}.$$

Remark 2.5. *Note that $\sigma(A, x) = \sigma^l(A, x) \cup \sigma^r(A, x)$. Also, the left and right spectra of an element of a Banach algebra can be shown to be non-empty (in fact, both contain the boundary of the spectrum). Also note that if an element of a Banach algebra is left (right) invertible, its left (respectively, right) inverse need not be unique. For example consider the operator T on $l^1(\mathbb{N})$ given by $T(x_1, x_2, \cdots) = (x_1 + x_2, x_3, x_4, \cdots)$. Let $R(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots)$ and $S(x_1, x_2, \cdots) = (x_1, 0, x_2, \cdots)$. Then R and S are both right inverses of T.*

Definition 2.6. *Let $\varepsilon > 0$. The ε-pseudospectrum $\Lambda_\varepsilon(x)$ of $x \in A$ is defined by*

$$\Lambda_\varepsilon(A, x) := \{\lambda \in \mathbb{C} : \|(\lambda - x)^{-1}\| \geq \varepsilon^{-1}\}$$

with the convention that $\|(\lambda - x)^{-1}\| = \infty$ if $\lambda - x$ is not invertible.

If the algebra A is fixed or is clear from the context, then we use simplified notations $\sigma(x), r(x), \sigma^l(x)$, $\sigma^r(x)$ and $\Lambda_\varepsilon(x)$ in place of $\sigma(A,x), r(A,x)$, $\sigma^l(A,x), \sigma^r(A,x)$ and $\Lambda_\varepsilon(A,x)$ respectively. The basic reference for pseudospectrum, especially for matrices, is the book [10]. The ε-pseudospectrum of an element of an arbitrary Banach algebra has been studied in [5].

Definition 2.7. *An element $x \in A$ is said to be a G_1-element if it satisfies the following equality:*

$$\|(z-x)^{-1}\| = \frac{1}{d(z,\sigma(x))} = r((z-x)^{-1}) \quad \forall z \in \mathbb{C} \setminus \sigma(x).$$

See [7].

Definition 2.8. *Let $p \in A$ be an idempotent. Then $pAp = \{pxp : x \in A\}$ is a closed subalgebra of A called a reduced Banach algebra. The identity element of this algebra is p.*

3. Spectra in reduced Banach algebras

Let A be a Banach algebra and $p \in A$ be an idempotent element, that is, $p = p^2$. We shall always assume that p is non-trivial, that is, $p \neq 0$ and $p \neq 1$. Let $q = 1 - p$. Then q is also an idempotent. Note that each $x \in A$ can be expressed uniquely as

$$x = pxp + pxq + qxp + qxq.$$

This is expressed in the matrix form as

$$x = \begin{bmatrix} pxp & pxq \\ qxp & qxq \end{bmatrix}_p.$$

Now, suppose $x \in A$ is such that $pxp = xp$, that is, $qxp = 0$. Then observe that $qxq = qx$. We will assume also that $pxp \neq px$, for if $pxp = px$, then x commutes with p, and the decomposition of the spectra and pseudospectra of such elements has been dealt with earlier in [6]. We observe that $\|p\|, \|q\| \geq 1$. We examine the relationships between the spectrum and pseudospectrum of $x \in A$ and the spectra and pseudospectra of $xp = pxp \in pAp$ and $qx = qxq \in qAq$.

The question of decomposition of the spectrum of an upper triangular operator matrix has been discussed in [3], [4] and [1]. For Z, Z' Banach spaces, let $B(Z)$ be the algebra of bounded linear operators from Z into itself and $B(Z, Z')$ be the space of bounded linear operators from Z into Z'.

Let $A \in B(X), B \in B(Y)$ and $C \in B(Y, X)$, where X and Y are Banach spaces. Let

$$M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}.$$

Various conditions on A, B and C for which $\sigma(M_C) = \sigma(A) \cup \sigma(B)$ are discussed in the above mentioned papers. We discuss some of these in the Banach algebra case, and also give some additional results.

We prove that

$$\sigma^l(pAp, pxp) \cup \sigma^r(qAq, qxq) \subseteq \sigma(A, x) \subseteq \sigma(pAp, pxp) \cup \sigma(qAq, qxq).$$

This has been shown in the operator case in [3], [4] and [1].

Lemma 3.1. *Let A be a unital Banach algebra, $p \in A$ be an idempotent, and $q = 1 - p$. Let $x \in A$ be such that $pxp = xp$, so x has the form*

$$x = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}_p$$

where $a = pxp, b = pxq$ and $d = qxq$. Here, $c = qxp = 0$. Then

1. *If x is invertible, then a is left invertible and d is right invertible.*
2. *If any two of x, a, d are invertible, then so is the third.*
3. *If x is invertible, then a is right invertible iff d is left invertible.*

Proof. Suppose x is invertible. Then $px^{-1}p$ is a left inverse of $a = pxp = xp$, since

$$\begin{aligned} px^{-1}ppxp &= px^{-1}pxp \\ &= px^{-1}xp \\ &= p. \end{aligned}$$

Similarly, it can be seen that $qx^{-1}q$ is a right inverse of $qxq = qx = d$. This proves (1).

Next, it is easy to check that if a and d are invertible, then so is x with inverse given by:

$$x^{-1} = \begin{bmatrix} a^{-1} & -a^{-1}bd^{-1} \\ 0 & d^{-1} \end{bmatrix}_p .$$

Next, suppose x is invertible and a is invertible. Then the inverse of a must be equal to its left inverse $px^{-1}p$. Hence $(xp)(px^{-1}p) = p$, which on simplifying gives $x^{-1}px = px^{-1}px$. By (1) we know $qx^{-1}q$ is a right inverse of $d = qx$. Then

$$\begin{aligned} qx^{-1}qqx &= (1-p)x^{-1}(1-p)x \\ &= x^{-1}(1-p)x - px^{-1}(1-p)x \\ &= 1 - x^{-1}px - p + px^{-1}px \\ &= 1 - p \\ &= q. \end{aligned}$$

Hence d is invertible with inverse $qx^{-1}q$.

If instead x and d are known to be invertible, it can be proved similarly that a is invertible. This proves (2).

(3) follows from (1) and (2).

□

Throughout the remaining part of this paper, we use $\sigma(a)$ and $\sigma(pAp, pxp)$ interchangeably, and similarly, $\sigma(d)$ and $\sigma(qAq, qxq)$ interchangeably.

Theorem 3.2. *Let A be a unital Banach algebra, $p \in A$ be an idempotent, and $q = 1 - p$. Let $x \in A$ be such that $pxp = xp$, so that as earlier, x has the form*

$$x = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}_p$$

where $a = pxp, b = pxq$ and $d = qxq$. Then

1. $\sigma^l(a) \cup \sigma^r(d) \subseteq \sigma(x)$
2. $\sigma(a) \cup \sigma(d) \setminus (\sigma(a) \cap \sigma(d)) \subseteq \sigma(x) \subseteq \sigma(a) \cup \sigma(d)$. *In particular, if $\sigma(a) \cap \sigma(d) = \emptyset$, then $\sigma(x) = \sigma(a) \cup \sigma(d)$.*

Proof. Since

$$\lambda - x = \begin{bmatrix} \lambda - a & -b \\ 0 & \lambda - d \end{bmatrix}_p$$

the proof follows from Lemma 3.1.

Hence, if $\sigma(a) \cap \sigma(d) = \emptyset$, then $\sigma(x) = \sigma(a) \cup \sigma(d)$. In fact, as is shown in [4], if $\sigma(a) \cap \sigma(d)$ has empty interior, then $\sigma(x) = \sigma(a) \cup \sigma(d)$. See Corollary 3.6. □

In general, it is not true that $\sigma(x) = \sigma(a) \cup \sigma(d)$.

Example 3.3. *Let $X = l^2(\mathbb{Z})$ and $T \in B(X)$ be the bilateral right shift operator. Observe that X can be expressed as*

$$X = l^2(-\mathbb{N} \cup \{0\}) \oplus l^2(\mathbb{N}).$$

Let P be the projection onto $l^2(\mathbb{N})$. Then it can verified that $PTP = TP$, and that TP is equal to the unilateral right shift operator R on $l^2(\mathbb{N})$. Then $Q = I - P$ is the projection onto $l^2(-\mathbb{N} \cup \{0\})$, $QT = QTQ$ and QT is equivalent to the unilateral left shift operator L on $l^2(\mathbb{N} \cup \{0\})$, since $QT(e_0) = 0$ and $QT(e_{-n}) = e_{-(n-1)}, n \in \mathbb{N}$.

Then we know that $\sigma(T) = \mathbb{T}$, the unit circle, whereas $\sigma(R)$ and $\sigma(L)$ are both equal to the closed unit disc $\bar{\mathbb{D}}$. It can also be seen that $\sigma^l(R) = \sigma_a(R) = \mathbb{T}$, where $\sigma_a(S)$ is the approximate point spectrum of an operator S. Also, $\sigma^r(L) = \sigma_c(L) = \mathbb{T}$, where $\sigma_c(S)$ is the continuous spectrum of an operator S.

It is not always true that $\sigma(a) \cup \sigma(d) \subseteq \sigma(x)$ as seen in Example 3.3. However, it is true that the boundary of $\sigma(a)$, denoted by $\delta\sigma(a)$, as well as $\delta\sigma(d)$ are contained in $\sigma(x)$.

Theorem 3.4. *Let A be a unital Banach algebra, $p \in A$ be an idempotent, and $q = 1 - p$. Let $x \in A$ be such that $pxp = xp$. Let a and d be as before. Then*

$$\delta\sigma(a) \cup \delta\sigma(d) \subseteq \sigma(x). \tag{2}$$

Thus

$$\sigma(a) \cup \sigma(d) \subseteq \sigma(x) \cup \text{ bounded components of } \rho(x)$$

and

$$r(x) = max\{r(a), r(d)\}.$$

Proof. Let $\lambda \in \delta\sigma(a)$. Then there exists $\{\lambda_n\}$, a sequence lying outside of $\sigma(a)$ such that $\lambda_n \to \lambda$. Suppose the sequence can be chosen to lie outside of $\sigma(x)$. Since $\|(\lambda_n - x)^{-1}\| \geq \frac{1}{\|p\|^2}\|(\lambda_n - a)^{-1}\| \to \infty$, this implies that $\lambda \in \sigma(x)$. If infinitely many λ_n are in $\sigma(x)$, then $\lambda \in \sigma(x)$. Hence $\delta\sigma(a) \subseteq \sigma(x)$. Similarly, $\delta\sigma(d) \subseteq \sigma(x)$. Hence,

$$\sigma(a) \cup \sigma(d) \subseteq \sigma(x) \cup \text{ bounded components of } \rho(x).$$

Since $\delta\sigma(a) \cup \delta\sigma(d) \subseteq \sigma(x) \subseteq \sigma(a) \cup \sigma(d)$, $r(x) = \max\{r(a), r(d)\}$, i.e.,

$$r(A, x) = \max\{r(pAp, pxp), r(qAq, qxq)\}.$$

□

Remark 3.5. *Let $B = \{y \in A : pyp = yp\}$. Then B is a closed subalgebra of A and for $y \in B$, $\sigma(A, y) \subseteq \sigma(B, y)$. Further, by (b) of Theorem 10.18 in [9], $\sigma(B, y) \subseteq \sigma(A, y) \cup$ certain bounded components of $\rho(A, y)$. It is also easy to see using Lemma 3.1 that for $x \in B$, $\sigma(B, x) = \sigma(pAp, pxp) \cup \sigma(qAq, qxq)$. Hence we obtain Theorem 3.4.*

Corollary 3.6. *Let A be a unital Banach algebra, $p \in A$ be an idempotent, and $q = 1 - p$. Let $x \in A$ be such that $pxp = xp$. Let a and d be as before. Then*

$$\sigma(a) \cup \sigma(d) \setminus (\sigma(a) \cap \sigma(d))^{o} \subseteq \sigma(x).$$

Proof.

$$\begin{aligned}\sigma(a) \cup \sigma(d) \setminus (\sigma(a) \cap \sigma(d))^{o} &= (\sigma(a) \cup \sigma(d)) \cap (\delta(\sigma(a) \cap \sigma(d)))\\ &\subseteq (\sigma(a) \cup \sigma(d)) \cap (\delta(\sigma(a)) \cup \delta(\sigma(d)))\\ &\subseteq \sigma(x). \quad \text{(by (2))}\end{aligned}$$

Hence, if $\sigma(a) \cap \sigma(d)$ has empty interior, then $\sigma(x) = \sigma(a) \cup \sigma(d)$. □

Remark 3.7. *In general,*

$$\sigma^l(a) \cup \sigma^r(d) \subseteq \sigma(x) \subseteq \sigma(a) \cup \sigma(d) \subseteq \sigma(x) \cup \text{bounded components of } \rho(x).$$

Thus if $x \in A$ is an element of the Banach algebra of the above form such that its resolvent contains no bounded component, then it is true that

$$\sigma(x) = \sigma(a) \cup \sigma(d).$$

In particular, the resolvent has no bounded component if it is connected.

We now look at some cases in which $\sigma(x) = \sigma(a) \cup \sigma(d)$. The case of compact operators is shown in Corollary 8 of [4] and written below.

Theorem 3.8. *Suppose that X is a Banach space and $P \in B(X) = A$ is a projection. Let $T \in B(X)$ be an operator such that $PTP = TP$, that is, $R(P)$ is invariant under T. Let $Q = I - P$. If PTP or QTQ is a compact operator, then*

$$\sigma(A, T) = \sigma(PAP, PTP) \cup \sigma(QAQ, QTQ).$$

In particular, this happens when P or Q or T is a compact operator.

Proof. If, say PTP is compact, then its spectrum has empty interior. Hence $\sigma(PAP, PTP) \cap \sigma(QAQ, QTQ)$ has empty interior and the result follows by Corollary 3.6. If T is a compact operator, another proof is the following: T has a countable spectrum, hence $\rho(T)$ has no bounded component. □

4. Pseudospectra in reduced Banach algebras

In [2], it has been shown that for an upper triangular operator matrix $M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$, it is true for certain C that $\Sigma(M_C) = \Sigma(A) \cup \Sigma(B)$, where for an operator T, $\Sigma(T)$ varies over a large class of spectra including the usual spectrum, the essential spectrum and the approximate point spectrum.

In this section, we discuss the decomposition of the pseudospectrum of an element of a Banach algebra that has the upper triangular form with respect to an idempotent, as discussed above.

Definition 4.1. *The left ε-pseudospectrum of an element $x \in A$ is defined as:*

$$\Lambda^l_\varepsilon(A, x) := \sigma^l(A, x) \cup \{\lambda \notin \sigma^l(A, x) : \inf\{\|b\| : b \text{ a left inverse of } (\lambda - x)\} \geq \frac{1}{\varepsilon}\}.$$

Definition 4.2. *The right ε-pseudospectrum of an element $x \in A$ is defined as:*

$$\Lambda^r_\varepsilon(A, x) := \sigma^r(A, x) \cup \{\lambda \notin \sigma^r(A, x) : \inf\{\|b\| : b \text{ a right inverse of } (\lambda - x)\} \geq \frac{1}{\varepsilon}\}.$$

See [8] for the above definitions.

We give some basic properties of the left and right ε-pseudospectra, that parallel the properties of the pseudospectrum (compare with Theorem 2.3 of [5]). We observe some simple facts first.

Remark 4.3. *Suppose $a \in A$ has a left inverse. Define*

$$m^l(a) := \inf\{\|x\| : x \in A, xa = 1\} > 0.$$

Suppose $b \in A$ such that $\|a - b\| < \frac{1}{m^l(a)}$. Then b has a left inverse. To prove this, consider η such that

$$\|a - b\| < \eta < \frac{1}{m^l(a)}.$$

Then $m^l(a) < \frac{1}{\eta}$. *Hence there exists* $x \in A$ *such that* $xa = 1$ *and* $m^l(a) \leq \|x\| < \frac{1}{\eta}$. *Thus*

$$\begin{aligned}\|1 - xb\| &= \|xa - xb\| \\ &\leq \|x\|\|a - b\| \\ &< \frac{1}{\eta}\eta \\ &= 1.\end{aligned}$$

Hence xb *has an inverse, say* y. *Then* $yxb = 1$. *Thus* b *has a left inverse. Consequently,* $Inv^l(A)$, *the set of all left invertible elements of* A *is an open set. Similarly, it can be shown that* $Inv^r(A)$, *the set of all right invertible elements of* A *is an open set.*

For a right invertible element a, *define*

$$m^r(a) := \inf \{\|x\| : x \in A, ax = 1\} > 0.$$

Let

$$m(a) = \max \{m^l(a), m^r(a)\}. \tag{3}$$

Then, clearly

$$\begin{aligned}\Lambda^l_\varepsilon(a) &= \{\lambda \in \mathbb{C} : m^l(\lambda - a) \geq \frac{1}{\varepsilon}\}, \\ \Lambda^r_\varepsilon(a) &= \{\lambda \in \mathbb{C} : m^r(\lambda - a) \geq \frac{1}{\varepsilon}\}, \text{ and} \\ \Lambda_\varepsilon(a) &= \{\lambda \in \mathbb{C} : m(\lambda - a) \geq \frac{1}{\varepsilon}\}.\end{aligned} \tag{4}$$

The convention followed is that $m^l(a) = \infty$ *(respectively* $m^r(a) = \infty$*) if* a *does not have a left (respectively, right) inverse. Similarly, for every* $r > 0$, *the set* $\{a \in A : m^l(a) < r\}$ *is open. Let* $m^l(a) < \eta < r$, *as before. Then there exists* $x \in A$ *such that* $xa = 1$ *and* $m^l(a) < \|x\| < \eta < r$. *If* $\|b - a\| < \frac{1}{m^l(a)}$, *we have seen that* b *is left invertible with a left inverse equal to* $(xb)^{-1}x = (1 - x(a - b))^{-1}x$. *Let* $\varepsilon' > 0$ *be such that* $1 + \varepsilon' < \frac{r}{\eta}$. *Then there exists* $\delta > 0$ *such that*

$$\|(1 - x(a - b))^{-1}\| < 1 + \varepsilon' < \frac{r}{\eta} \text{ whenever } \|a - b\| < \delta.$$

Hence $\|(1 - x(a - b))^{-1}x\| < r$. *Thus for* $\|a - b\| < \min \{\frac{1}{m^l(a)}, \delta\}, m^l(b) < r$.

Theorem 4.4. *Let* A *be a Banach algebra,* $a \in A$ *and* $\varepsilon > 0$. *Then*

1. $\Lambda^l_\varepsilon(a) \cup \Lambda^r_\varepsilon(a) = \Lambda_\varepsilon(a)$.
2. $\sigma^l(a) = \bigcap\limits_{\varepsilon>0} \Lambda^l_\varepsilon(a)$.
3. $\Lambda^l_{\varepsilon_1}(a) \subseteq \Lambda^l_{\varepsilon_2}(a)\,(0 < \varepsilon_1 < \varepsilon_2)$.
4. $\Lambda^l_\varepsilon(a + \lambda) = \Lambda^l_\varepsilon(a) + \lambda\,(\lambda \in \mathbb{C})$.
5. $\Lambda^l_\varepsilon(\lambda a) = \lambda\Lambda^l_{\frac{\varepsilon}{|\lambda|}}(a)\,(\lambda \in \mathbb{C} \setminus \{0\})$.
6. $\Lambda^l_\varepsilon(a) \subseteq D(0; \|a\| + \varepsilon)$. *Further, if* a *is left invertible, and* $0 < \varepsilon < \frac{1}{m^l(a)}$, *then* $\Lambda^l_\varepsilon(a) \subseteq \{z \in \mathbb{C} : \frac{1}{m^l(a)} - \varepsilon \leq |z| \leq \|a\| + \varepsilon\}$.
7. $\Lambda^l_\varepsilon(a)$ *is a non-empty compact subset of* $\mathbb{C}$.
8. $\Lambda^l_\varepsilon(a + b) \subseteq \Lambda^l_{\varepsilon+\|b\|}(a)\,(b \in A)$.
9. $\sigma^l(a + b) \subseteq \Lambda^l_\varepsilon(a)\,(b \in A, \|b\| \leq \varepsilon)$, *that is,* $\bigcup\limits_{\|b\|\leq\varepsilon} \sigma^l(a + b) \subseteq \Lambda^l_\varepsilon(a)$.

10. $\Lambda^l_\varepsilon(a) + D(0;\delta) \subseteq \Lambda^l_{\varepsilon+\delta}(a)$ $(\delta > 0)$.
11. $\sigma^l(a) + D(0;\varepsilon) \subseteq \Lambda^l_\varepsilon(a)$.

Proof. 1. The proof is clear by Equations (3) and (4).

2. By the definition of left pseudospectrum, $\sigma^l(a) \subseteq \bigcap_{\varepsilon>0} \Lambda^l_\varepsilon(a)$. Suppose $\lambda - a$ is left invertible and $(\lambda - a)^l$ is a left inverse. Then there exists ε_0 such that $\|(\lambda - a)^l\| < \frac{1}{\varepsilon_0}$. Hence $m^l(\lambda - a) < \frac{1}{\varepsilon_0}$.
3. Let $\lambda \in \Lambda^l_{\varepsilon_1}(a)$. If $\lambda \in \sigma^l(a)$, we are done. Otherwise, $m^l(\lambda - a) \geq \frac{1}{\varepsilon_1} > \frac{1}{\varepsilon_2}$.
4. Let $z \in \Lambda^l_\varepsilon(a + \lambda)$. If $z \in \sigma^l(a + \lambda)$, then $(z - \lambda) \in \sigma^l(a)$, hence $z \in \sigma^l(a) + \lambda \subseteq \Lambda^l_\varepsilon(a) + \lambda$. Otherwise, $((z - \lambda) - a) = z - a - \lambda$ is left invertible and $m^l((z - \lambda) - a) \geq \frac{1}{\varepsilon}$. Hence $z - \lambda \in \Lambda^l_\varepsilon(a)$ and $z \in \Lambda^l_\varepsilon(a) + \lambda$. Hence

$$\Lambda^l_\varepsilon(a + \lambda) \subseteq \Lambda^l_\varepsilon(a) + \lambda.$$

The other inclusion is similar. Suppose $z \in \Lambda^l_\varepsilon(a) + \lambda$. Then $z - \lambda \in \Lambda^l_\varepsilon(a)$. If $z - \lambda \in \sigma^l(a)$, then $z - (a + \lambda) = z - \lambda - a$ is not left invertible, hence $z \in \sigma^l(a + \lambda) \subseteq \Lambda^l_\varepsilon(a + \lambda)$. If $z - \lambda - a$ is left invertible, then $z - (a + \lambda)$ is left invertible and $m^l(z - (a + \lambda)) = m^l(z - \lambda - a) \geq \frac{1}{\varepsilon}$, hence $z \in \Lambda^l_\varepsilon(a + \lambda)$.

5. Let $z \in \Lambda^l_\varepsilon(\lambda a)$, where $\lambda \neq 0$. If $z \in \sigma^l(\lambda a)$, then $z - \lambda a$ is not left invertible, hence $\frac{z}{\lambda} - a$ is not left invertible, thus

$$z \in \lambda\sigma^l(a) \subseteq \lambda\Lambda^l_{\frac{\varepsilon}{|\lambda|}}(a).$$

Otherwise, if $z - \lambda a$ is left invertible, then $\frac{z}{\lambda} - a$ is left invertible and for any left inverse $(z - \lambda a)^l$, $\lambda(z - \lambda a)^l$ is a left inverse of $\frac{z}{\lambda} - a$. Further, $|\lambda|\|(z - \lambda a)^l\| \geq \frac{|\lambda|}{\varepsilon}$. Hence $z \in \lambda\Lambda^l_{\frac{\varepsilon}{|\lambda|}}(a)$.

6. $\Lambda^l_\varepsilon(a) \subseteq \Lambda_\varepsilon(a) \subseteq D(0; \|a\| + \varepsilon)$. We next observe a simple fact. If a is left invertible and a^l is any left inverse, then

$$\left\{\frac{1}{\lambda} : \lambda \in \sigma^l(a)\right\} \subseteq \sigma^l(a^l). \tag{5}$$

This follows because $a^l - \frac{1}{\lambda} = a^l(1 - \frac{1}{\lambda}a) = \frac{1}{\lambda}a^l(\lambda - a)$. Hence if $a^l - \frac{1}{\lambda}$ is left invertible, so is $\lambda - a$.
Now, suppose a is left invertible and $z \in \mathbb{C}$ is such that $|z| < \frac{1}{m^l(a)} - \varepsilon$. Then, by the definition of $m^l(a)$, there exists $a^l \in A$ such that $a^l a = 1$ and $\|a^l\| < \frac{1}{|z|+\varepsilon}$. Hence $|z| < \frac{1}{\|a^l\|} - \varepsilon < \frac{1}{\|a^l\|}$. Thus, $\frac{1}{z} - a^l$ is invertible and $\|(\frac{1}{z} - a^l)^{-1}\| \leq \frac{1}{\frac{1}{|z|} - \|a^l\|}$. By the fact observed in (5), $z - a$ is left invertible and there exists a left inverse $(z - a)^l$ such that

$$\begin{aligned}
\|(z - a)^l\| &= \left\|(\frac{1}{z} - a^l)^{-1}\frac{1}{z}a^l\right\| \\
&\leq \frac{\|a^l\|\|(\frac{1}{z} - a^l)^{-1}\|}{|z|} \\
&\leq \frac{\|a^l\|}{|z|}\frac{1}{\frac{1}{|z|} - \|a^l\|} \\
&= \frac{1}{\frac{1}{\|a^l\|} - |z|} \\
&< \frac{1}{\varepsilon}.
\end{aligned}$$

Hence $z \notin \Lambda^l_\varepsilon(a)$. This shows that $\Lambda^l_\varepsilon(a) \subseteq \{z \in \mathbb{C} : \frac{1}{m^l(a)} - \varepsilon \leq |z|\}$.

7. $\Lambda^l_\varepsilon(a)$ is bounded by (6). It is closed by Remark 4.3.

   Alternately, we observe that the infimum of a family of continuous functions is upper semi-continuous and that if f is an upper semi-continuous function on a topological space X, then $\{x \in X : f(x) < \alpha\}$ is open for every $\alpha \in \mathbb{R}$. This shows that $(\Lambda^l_\varepsilon(a))^c = \{\lambda \notin \sigma^l(A, x) : \inf\{\|b\| : b \text{ a left inverse of } (\lambda - a)\} < \frac{1}{\varepsilon}\}$ is open and hence the left pseudospectrum is closed.

8. Let $a, b \in A$ and $\varepsilon > 0$. Suppose $\lambda \notin \Lambda^l_{\varepsilon+\|b\|}(a)$. Then $\lambda - a$ is left invertible and there exists a left inverse $(\lambda - a)^l$ such that $\|(\lambda - a)^l\| < \frac{1}{\varepsilon+\|b\|}$. Hence

   $$\|(\lambda - a - b) - (\lambda - a)\| = \|b\| < \varepsilon + \|b\| < \left\|(\lambda - a)^l\right\|^{-1}.$$

   Hence $\lambda - a - b$ is left invertible with a left inverse given by

   $$(1 - (\lambda - a)^l b)^{-1}(\lambda - a)^l.$$

   It can be verified that

   $$\begin{aligned}\|(\lambda - a - b)^l - (\lambda - a)^l\| &= \|(\lambda - a - b)^l((\lambda - a) - (\lambda - a - b))(\lambda - a)^l\| \\ &< \frac{\|b\|\|(\lambda - a - b)^l\|}{\varepsilon + \|b\|}.\end{aligned}$$

   Thus

   $$\begin{aligned}\|(\lambda - a - b)^l\| &\le \|(\lambda - a - b)^l - (\lambda - a)^l\| + \|(\lambda - a)^l\| \\ &< \frac{\|b\|\|(\lambda - a - b)^l\|}{\varepsilon + \|b\|} + \frac{1}{\varepsilon + \|b\|}.\end{aligned}$$

   Thus we get $\|(\lambda - a - b)^l\| < \frac{1}{\varepsilon}$. Hence $\lambda \notin \Lambda^l_\varepsilon(a + b)$.

9. Let $\lambda \in \sigma^l(a + b)$, then $\lambda - a - b$ is not left invertible. If $\lambda \in \sigma^l(a)$, we are done. Otherwise, suppose $\lambda - a$ is left invertible and $(\lambda - a)^l$ is any left inverse of it. Now, $\lambda - a - b = (1 - b(\lambda - a)^l)(\lambda - a)$. Since $\lambda - a - b$ is not left invertible and $\lambda - a$ is left invertible, $1 - b(\lambda - a)^l$ is not left invertible. Hence $\|b(\lambda - a)^l\| \ge 1$. Thus $\|(\lambda - a)^l\| \ge \frac{1}{\|b\|} \ge \frac{1}{\varepsilon}$. Hence $m^l(\lambda - a) \ge \frac{1}{\varepsilon}$.

10. Let $z \in \Lambda^l_\varepsilon(a)$ and $\lambda \in \mathbb{C}$ with $|\lambda| \le \delta$. Then

   $$\begin{aligned}z + \lambda \in \Lambda^l_\varepsilon(a) + \lambda &= \Lambda^l_\varepsilon(a + \lambda) \text{ (by (4))} \\ &\subseteq \Lambda^l_{\varepsilon+|\lambda|}(a) \text{ (by (8))} \\ &\subseteq \Lambda^l_{\varepsilon+\delta}(a) \text{ (by (3))}.\end{aligned}$$

11. Let $\lambda \notin \sigma^l(a)$. Throughout, let $(\lambda - a)^l$ denote an arbitrary left inverse. Then, using (5), we obtain

$$\begin{aligned}
d(\lambda, \sigma^l(a)) &= \inf\{|\lambda - z| : z \in \sigma^l(a)\} \\
&= \inf\{|z| : z \in \sigma^l(\lambda - a)\} \\
&\geq \inf\{|\frac{1}{z}| : z \in \sigma^l((\lambda - a)^l)\} \\
&= \frac{1}{\sup\{|z| : z \in \sigma^l((\lambda - a)^l)\}} \\
&\geq \frac{1}{\sup\{|z| : z \in \sigma((\lambda - a)^l)\}} \\
&= \frac{1}{r((\lambda - a)^l)} \\
&\geq \frac{1}{\|(\lambda - a)^l\|}.
\end{aligned}$$

Thus, if $d(\lambda, \sigma^l(a)) \leq \varepsilon$, then $\|(\lambda - a)^l\| \geq \frac{1}{\varepsilon}$. Hence, $m^l(\lambda - a) \geq \frac{1}{\varepsilon}$.

Clearly, the analogous statements of (2)-(11) for the right spectrum and the right pseudospectrum are also true. □

Let us now consider the question of the decomposition of the pseudospectrum of a 'triangular' element.

Theorem 4.5. *Let A be a unital Banach algebra, $p \in A$ be an idempotent, and $q = 1 - p$. Let $x \in A$ be such that $pxp = xp$. Let $K = max\ \{\|p\|^2, \|q\|^2\}$. Then*

$$\Lambda^l_\varepsilon(pAp, pxp) \cup \Lambda^r_\varepsilon(qAq, qxq) \subseteq \Lambda_{K\varepsilon}(A, x).$$

Proof. As before, let

$$x = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}_p.$$

As observed earlier, if $\lambda - x$ is invertible, then $p(\lambda - x)^{-1}p$ is a left inverse of $\lambda p - a$, and $q(\lambda - x)^{-1}q$ is a right inverse of $\lambda q - d$. Suppose $\lambda \notin \Lambda_{K\varepsilon}(A, x)$. Then $(\lambda - x)$ is invertible and $\|(\lambda - x)^{-1}\| < \frac{1}{K\varepsilon}$. Hence $(\lambda p - pxp)$ is left invertible and

$$\|p(\lambda - x)^{-1}p\| \leq \|p\|^2\|(\lambda - x)^{-1}\| < \|p\|^2\frac{1}{K\varepsilon} \leq \frac{1}{\varepsilon}.$$

Hence $\inf\{\|b\| : b \text{ a left inverse of } (\lambda - a)\} < \frac{1}{\varepsilon}$ and $\lambda \notin \Lambda^l_\varepsilon(pAp, pxp)$. Similarly, $(\lambda q - qxq)$ is right invertible and $\inf\{\|b\| : b \text{ a right inverse of } (\lambda - a)\} < \frac{1}{\varepsilon}$, thus $\lambda \notin \Lambda^r_\varepsilon(qAq, qxq)$.
□

We discuss what happens in the case that we know $\sigma(x) = \sigma(a) \cup \sigma(d)$. In the operator matrix case, it is shown in [1] that $\sigma(M_C) = \sigma(A) \cup \sigma(B)$ for all C in the closure of the set

$$R(\delta_{(A,B)}) + N(\delta_{(A,B)}) + \cup_{\lambda\in\mathbb{C}}N(L_{A-\lambda}) + \cup_{\lambda\in\mathbb{C}}N(R_{B-\lambda}),$$

where $\delta_{(A,B)}(T) = AT - TB, L_{A-\lambda}(T) = (A - \lambda)T$ and $R_{B-\lambda}(T) = T(B - \lambda), \forall T \in B(Y, X)$ and $N(S), R(S)$ denote the null space and range space of $S \in B(B(Y, X))$. Equality also occurs if either A or B is normal, or more generally, if A is co-hyponormal or B is hyponormal. See [3] and [4]. Some other cases were discussed in Section 3.

Theorem 4.6. *Let A be a unital Banach algebra, $p \in A$ be an idempotent, and $q = 1 - p$. Let $x \in A$ be such that $pxp = xp$. Let $K = max \{\|p\|^2, \|q\|^2\}$. Suppose that $\sigma(A, x) = \sigma(pAp, pxp) \cup \sigma(qAq, qxq)$. Then*

$$\Lambda_\varepsilon(pAp, pxp) \cup \Lambda_\varepsilon(qAq, qxq) \subseteq \Lambda_{K\varepsilon}(A, x).$$

Proof. Suppose $\lambda \notin \Lambda_{K\varepsilon}(A, x)$. Then $(\lambda - x)$ is invertible and $\|(\lambda - x)^{-1}\| < \frac{1}{K\varepsilon}$. Then by the assumption, $\lambda p - pxp$ is invertible and

$$\|(\lambda p - pxp)^{-1}\| \leq \|p\|^2 \|(\lambda - x)^{-1}\| < \|p\|^2 \frac{1}{K\varepsilon} \leq \frac{1}{\varepsilon}.$$

Similarly, $(\lambda q - qxq)$ is invertible and $\|(\lambda q - qxq)^{-1}\| < \frac{1}{\varepsilon}$. □

Theorem 4.7. *Let $A = B(X)$ for a Banach space X, and suppose P is an idempotent bounded operator on X. Let $Q = I - P$, and T be a bounded operator on X such that $TP = PTP$. Suppose either QTQ or PTP is compact. Then*

$$\Lambda_\varepsilon(PAP, PTP) \cup \Lambda_\varepsilon(QAQ, QTQ) \subseteq \Lambda_{K\varepsilon}(A, T),$$

where $K = max \{\|P\|^2, \|Q\|^2\}$.

Proof. Suppose $\lambda \notin \Lambda_{K\varepsilon}(A, T)$. Then $\lambda - T$ is invertible and $\|(\lambda - T)^{-1}\| < \frac{1}{K\varepsilon}$. By Theorem 3.8, $\lambda P - PTP$ and $\lambda Q - QTQ$ are invertible. As in the proof of Theorem 4.6, $\|(\lambda P - PTP)^{-1}\| < \frac{1}{\varepsilon}$ and $\|(\lambda Q - QTQ)^{-1}\| < \frac{1}{\varepsilon}$. □

Theorem 4.8. *Let A be a unital Banach algebra, $p \in A$ be an idempotent, and $q = 1 - p$. Let $x \in A$ be such that $pxp = xp$. Suppose x is a G_1-element. Then with a and d as earlier,*

$$\Lambda_\varepsilon(x) \subseteq \Lambda_\varepsilon(a) \cup \Lambda_\varepsilon(d).$$

Proof. Let $\lambda \in \Lambda_\varepsilon(x)$. If $\lambda \in \sigma(a) \cup \sigma(d)$, then $\lambda \in \Lambda_\varepsilon(a) \cup \Lambda_\varepsilon(d)$. If instead, $\lambda - a$ and $\lambda - d$ are invertible, then so is $\lambda - x$ by Theorem 3.2. Further,

$$\begin{aligned}
\|(\lambda - x)^{-1}\| &= r((\lambda - x)^{-1}) \\
&= \max \{r((\lambda - a)^{-1}), r((\lambda - d)^{-1})\} \text{ (by Theorem 3.4)} \\
&\leq \max \{\|(\lambda - a)^{-1}\|, \|(\lambda - d)^{-1}\|\}.
\end{aligned}$$

Hence, $\|(\lambda - x)^{-1}\| \geq \frac{1}{\varepsilon}$ implies that either $\|(\lambda - a)^{-1}\|$ or $\|(\lambda - d)^{-1}\|$ is greater than or equal to $\frac{1}{\varepsilon}$. □

Hereafter we assume that $K = 1$, that is, $\|p\| = 1$ and $\|q\| = 1$.

Theorem 4.9. *Let A be a unital Banach algebra, $p \in A$ be an idempotent, and $q = 1 - p$. Let $x \in A$ be such that $pxp = xp$ and suppose $\|p\| = \|q\| = 1$. Then*

$$\Lambda_{\frac{\varepsilon^2}{2\varepsilon + \|x\|}}(x) \subseteq \Lambda_\varepsilon(a) \cup \Lambda_\varepsilon(d).$$

Proof. Suppose $\lambda \notin \Lambda_\varepsilon(a) \cup \Lambda_\varepsilon(d)$. Then $\lambda p - a$ and $\lambda q - d$ are invertible and their inverses have norm less than $\frac{1}{\varepsilon}$. Then

$$(\lambda - x)^{-1} = \begin{bmatrix} (\lambda - a)^{-1} & (\lambda - a)^{-1} b (\lambda - d)^{-1} \\ 0 & (\lambda - d)^{-1} \end{bmatrix}_p$$

and

$$\begin{aligned}
\|(\lambda - x)^{-1}\| &\leq \|(\lambda - a)^{-1}\| + \|(\lambda - a)^{-1} b (\lambda - d)^{-1}\| + \|(\lambda - d)^{-1}\| \\
&< \frac{1}{\varepsilon} + \frac{1}{\varepsilon}\|b\|\frac{1}{\varepsilon} + \frac{1}{\varepsilon} \\
&= \frac{2}{\varepsilon} + \frac{1}{\varepsilon^2}\|pxq\| \\
&\leq \frac{2}{\varepsilon} + \frac{1}{\varepsilon^2}\|x\|.
\end{aligned}$$

□

Theorem 4.10. *Let A be a unital Banach algebra, $p \in A$ be an idempotent, and $q = 1 - p$. Let $x \in A$ be such that $pxp = xp$ and suppose $\|p\| = \|q\| = 1$. Suppose p has the property that for every*

$$y = \begin{bmatrix} e & f \\ 0 & g \end{bmatrix}_p$$

$\|y\| = max \{\|e\| + \|f\|, \|g\|\}$ or $max \{\|f\| + \|g\|, \|e\|\}$. Then

$$\Lambda_{\frac{\varepsilon^2}{\varepsilon+\|x\|}}(x) \subseteq \Lambda_\varepsilon(a) \cup \Lambda_\varepsilon(d).$$

Proof. Suppose $\lambda \notin \Lambda_\varepsilon(a) \cup \Lambda_\varepsilon(d)$. Then $\lambda p - a$ and $\lambda q - d$ are invertible and their inverses have norm less than $\frac{1}{\varepsilon}$. Now,

$$(\lambda - x)^{-1} = \begin{bmatrix} (\lambda - a)^{-1} & (\lambda - a)^{-1} b(\lambda - d)^{-1} \\ 0 & (\lambda - d)^{-1} \end{bmatrix}_p.$$

Suppose the norm is given by the maximum absolute column sum. Then

$$\|(\lambda - x)^{-1}\| = \max \{\|(\lambda - a)^{-1}\|, \|(\lambda - a)^{-1} b(\lambda - d)^{-1}\| + \|(\lambda - d)^{-1}\|\}.$$

Since $b = pxq$, and $\|p\| = \|q\| = 1$, $\|(\lambda - x)^{-1}\| < \frac{\varepsilon+\|x\|}{\varepsilon^2}$. The same inequality holds if the norm is given by the maximum absolute row sum. $\square$

As in the case of the spectrum, we can show that the boundaries of the pseudospectra of a and d lie in the pseudospectrum of x, if $\|p\| = \|q\| = 1$.

Theorem 4.11. *Let A be a unital Banach algebra, $p \in A$ be an idempotent, and $q = 1 - p$. Let $x \in A$ be such that $pxp = xp$ and suppose $\|p\| = \|q\| = 1$. Then, with a and d as earlier,*

$$\delta\Lambda_\varepsilon(a) \cup \delta\Lambda_\varepsilon(d) \subseteq \Lambda_\varepsilon(x).$$

Proof. Suppose $\lambda \notin \Lambda_\varepsilon(x)$. Then $\lambda - x$ is invertible and $\|(\lambda - x)^{-1}\| < \frac{1}{\varepsilon}$. If $\lambda p - a$ is invertible (which is true iff $\lambda q - d$ is invertible), then $\|(\lambda p - a)^{-1}\| \leq \|(\lambda - x)^{-1}\| < \frac{1}{\varepsilon}$, and similarly, $\|(\lambda q - d)^{-1}\| \leq \|(\lambda - x)^{-1}\| < \frac{1}{\varepsilon}$. Hence $\lambda \notin \delta\Lambda_\varepsilon(a) \cup \delta\Lambda_\varepsilon(d)$, for the boundary of the pseudospectrum of an element s is contained in the level set $\{\lambda \in \mathbb{C} : \|(\lambda - s)^{-1}\| = \frac{1}{\varepsilon}\}$.

On the other hand, if $\lambda p - a$ is not invertible, then $\lambda \in \sigma(a)$ and also, $\lambda \in \sigma(d)$. However, since for any element s, $\sigma(s) + B(0; \varepsilon)$ is contained in the interior of $\Lambda_\varepsilon(s)$ it follows that $\sigma(s) \cap \delta\Lambda_\varepsilon(s) = \emptyset$. Hence $\lambda \notin \delta\Lambda_\varepsilon(a) \cup \delta\Lambda_\varepsilon(d)$. $\square$

Remark 4.12. *Hence we have*

$$\Lambda_\varepsilon(a) \cup \Lambda_\varepsilon(d) \subseteq \Lambda_\varepsilon(x) \cup \text{ bounded components of the complement of } \Lambda_\varepsilon(x).$$

Thus, if x is an element such that $\Lambda_\varepsilon(x)^c$ has no bounded component, then $\Lambda_\varepsilon(a) \cup \Lambda_\varepsilon(d) \subseteq \Lambda_\varepsilon(x)$.
As in the case of the spectrum, we claim that

$$\Lambda_\varepsilon(a) \cup \Lambda_\varepsilon(d) \setminus (\Lambda_\varepsilon(a) \cap \Lambda_\varepsilon(d)) \subseteq \Lambda_\varepsilon(x).$$

In fact, we have the following theorem.

Theorem 4.13. *Let A be a unital Banach algebra, $p \in A$ be an idempotent, and $q = 1 - p$. Let $x \in A$ be such that $pxp = xp$ and suppose $\|p\| = \|q\| = 1$. Then, with a and d as earlier,*

$$\Lambda_\varepsilon(a) \cup \Lambda_\varepsilon(d) \setminus (\sigma(a) \cap \sigma(d))^o \subseteq \Lambda_\varepsilon(x).$$

Proof. Suppose $\lambda \notin \Lambda_\varepsilon(x)$, then $\lambda - x$ is invertible and $\|(\lambda - x)^{-1}\| < \frac{1}{\varepsilon}$. If $\lambda \in \Lambda_\varepsilon(a) \cup \Lambda_\varepsilon(d)$, then we must have $\lambda \in \sigma(a) \cup \sigma(d)$ (hence $\lambda \in \sigma(a) \cap \sigma(d)$), for if $\lambda - a$ were invertible, $\|(\lambda - a)^{-1}\| = \|p(\lambda - x)^{-1}\| < \frac{1}{\varepsilon}$. Further, such a λ must belong to $(\sigma(a) \cap \sigma(d))^{\mathrm{o}}$, for if $\lambda \in \delta(\sigma(a) \cap \sigma(d))$, then it must belong to $\sigma(x)$ as observed in Corollary 3.6. □

Example 4.14. *We re-examine Example 3.3 for pseudospectra. Since T is unitary and $\sigma(T) = \mathbb{T}$, the unit circle, $\Lambda_\varepsilon(T) = \{\lambda \in \mathbb{C} : 1-\varepsilon \leq |\lambda| \leq 1+\varepsilon\}$ for $\varepsilon < 1$ and $\Lambda_\varepsilon(T) = D(0; 1+\varepsilon)$ for other $\varepsilon \geq 1$. Since R and L are G_1 operators and $\sigma(R) = \sigma(L) = \bar{\mathbb{D}}$, $\Lambda_\varepsilon(R) = \Lambda_\varepsilon(L) = D(0; 1+\varepsilon)$ (see Lemma 3.6 in [5]). Hence for $\varepsilon \geq 1$, $\Lambda_\varepsilon(T) = \Lambda_\varepsilon(R) \cup \Lambda_\varepsilon(L)$, whereas for $\varepsilon < 1$, $\Lambda_\varepsilon(R) \cup \Lambda_\varepsilon(L) = \Lambda_\varepsilon(T) \cup$ bounded component of $(\Lambda_\varepsilon(T))^c$.*

The following is a corollary of Theorem 4.6, Theorem 4.8, Remark 4.12 and Theorem 4.13.

Corollary 4.15. *Let A be a unital Banach algebra, $p \in A$ be an idempotent, and $q = 1 - p$. Let $x \in A$ be such that $pxp = xp$ and suppose $\|p\| = \|q\| = 1$. Suppose x is a G_1 element. With a and d as earlier, suppose any one of the following is true:*

1. $(\sigma(a) \cap \sigma(d))^{\mathrm{o}} = \emptyset$
2. $\sigma(x) = \sigma(a) \cup \sigma(d)$ *(Note that* $(1) \implies (2)$*)*
3. $(\Lambda_\varepsilon(x))^c$ *has no bounded component.*

Then

$$\Lambda_\varepsilon(a) \cup \Lambda_\varepsilon(d) = \Lambda_\varepsilon(x).$$

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