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Global nonexistence for the damped wave equation with nonlinear memory on the Heisenberg group

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Abstract. Sufficient conditions are obtained for the nonexistence of global solutions to the damped wave equation

$$u_{tt} - \Delta_{\mathbb{H}} u + |u|^{m-1} u_t = \int_0^t (t-s)^{-\gamma} |u(.,s)|^p ds,$$

where $\Delta_{\mathbb{H}}$ is the Kohn-Laplace operator on the $(2N+1)$-dimensional Heisenberg group $\mathbb{H}$. Then, this result is extended to the case of 2×2 system of the same type. Our technique of proof is based on a duality argument.

1. Introduction and preliminaries

In this paper, we are first concerned with the nonexistence of global weak solutions for the following semi-linear wave equation with nonlinear mixed damping term:

$$u_{tt} - \Delta_{\mathbb{H}} u + |u|^{m-1} u_t = \int_0^t (t-s)^{-\gamma} |u(.,s)|^p ds, \tag{1}$$

where $0 < \gamma < 1$, $m > 1$ and $p > 1$. Supplemented with the initial conditions

$$u(\eta, 0) = u_0(\eta), \quad u_t(\eta, 0) = u_1(\eta), \tag{2}$$

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where $\Delta_{\mathbb{H}}$ is the Kohn-Laplace operator on the $(2N+1)$-dimensional Heisenberg group. Then we extend our analysis to the 2×2 system of the same type

$$\begin{cases} u_{tt} - \Delta_{\mathbb{H}} u + |u|^{m-1} u_t = \displaystyle\int_0^t (t-s)^{-\gamma_1} |v(.,s)|^q ds, \\ v_{tt} - \Delta_{\mathbb{H}} v + |v|^{m-1} v_t = \displaystyle\int_0^t (t-s)^{-\gamma_2} |u(.,s)|^p ds, \\ u(\eta,0) = u_0(\eta) \quad , \quad u_t(\eta,0) = u_1(\eta), \\ v(\eta,0) = v_0(\eta) \quad , \quad v_t(\eta,0) = v_1(\eta), \end{cases} \tag{3}$$

where $p > 1$, $q > 1$ and $0 < \gamma < 1$. Our article is motivated by the paper by A. Hakem et al. [4] which deals with the blow-up of solutions for the following Cauchy problem

$$u_{tt} - \Delta u + |u|^{m-1} u_t = \int_0^t (t-s)^{-\gamma} |u(.,s)|^p ds; \quad t > 0, \quad x \in \mathbb{R}^n,$$

subjected to the initial data

$$u_0(0,x) = u_0(x), \; u_t(0,x) = u_1(x), \; x \in \mathbb{R}^n,$$

where the unknown function u, is real-valued, $n \geq 1$, $0 < \gamma < 1$, $m > 1$ and $p > 1$. More precisely, they proved that if

$$n \leq min\left\{ \frac{2(m + (1-\gamma)p)}{(p - 1 + (1-\gamma)(m-1))}, \quad \frac{2(1 + (2-\gamma)p)}{(\frac{(p-1)(2-\gamma)}{p-m} + \gamma - 1)(p-1)} \right\},$$

or $p \leq \frac{1}{\gamma}$, then the solution of the above problem does not exist globally in time.
The problem of nonexistence of global solutions in the Heisenberg group has received specific attention in recent years. See for instance ([1], [8]). For more details on Heisenberg groups and partial differential equations in Heisenberg groups, we refer the reader to ([1], [8], [9], [15]) and the references therein. For the reader convenience, some background facts used in the sequel are recalled.

The Heisenberg group $\mathbb{H}$ whose points will be denoted by $\eta = (x, y, \tau)$, is the Lie group $(\mathbb{R}^{2N+1}; \circ)$ with the non-commutative group operation $\circ$ defined by

$$\eta \circ \eta' = (x + x', y + y', \tau + \tau' + 2(x.y' - x'.y))$$

for all $\eta = (x, y, \tau), \eta' = (x', y', \tau') \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$, where . denotes the standard scalar product in $\mathbb{R}^N$. This group operation endows $\mathbb{H}$ with the structure of a Lie group.
The Laplacian $\Delta_{\mathbb{H}}$ over $\mathbb{H}$ is obtained from the vector fields $X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial \tau}$ and $Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial \tau}$, by

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{N} \left(X_i^2 + Y_i^2 \right).$$

Observe that the vector field $T = \frac{\partial}{\partial \tau}$ does not appear in the equality above. This fact makes us presume a "loss of derivative" in the variable τ. The compensation comes from the relation

$$[X_i, Y_j] = -4T, \quad i, j \in 1, 2, 3, ..., N.$$

The relation above proves that $\mathbb{H}$ is a nilpotent Lie group of order 2. Explicit computation gives the expression

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{N} \left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i \frac{\partial^2}{\partial x_i \partial \tau} - 4x_i \frac{\partial^2}{\partial y_i \partial \tau} + 4(x_i^2 + y_i^2) \frac{\partial^2}{\partial \tau^2} \right).$$

A natural group of dilatations on $\mathbb{H}$ is given by

$$\delta_\lambda(\eta) = (\lambda x, \lambda y, \lambda^2 \tau), \qquad \lambda > 0,$$

whose Jacobian determinant is λ^Q, where $Q = 2N + 2$ is the homogeneous dimension of $\mathbb{H}$.
The operator $\Delta_{\mathbb{H}}$ is a degenerate elliptic operator. It is invariant with respect to the left translation of $\mathbb{H}$ and homogeneous with respect to the dilations δ_λ. More precisely, we have

$$\Delta_{\mathbb{H}}(u(\eta \circ \eta')) = (\Delta_{\mathbb{H}} u)(\eta \circ \eta'), \Delta_{\mathbb{H}}(u \circ \delta_\lambda) = \lambda^2 (\Delta_{\mathbb{H}} u) \circ \delta_\lambda, \eta, \eta' \in \mathbb{H}.$$

The natural distance from η to the origin is introduced by Folland and Stein, see [11]

$$|\eta|_{\mathbb{H}} = \left(\tau^2 + \left(\sum_{i=1}^{N} (x_i^2 + y_i^2) \right)^2 \right)^{\frac{1}{4}}.$$

Lemma 1.1. *([1])*
Let $f \in \mathcal{L}^1\left(\mathbb{R}^{2N+1}\right)$ *and* $\int_{\mathbb{R}^{2N+1}} f d\eta > 0$. *Then there exists a test function* $0 \le \varphi \le 1$ *such that*

$$\int_{\mathbb{R}^{2N+1}} f \varphi d\eta \ge 0.$$

For the reader's convenience, let us briefly recall the definition and the basic properties of the fractional derivatives.

If $AC[0;T]$ is the space of all functions which are absolutely continuous on $[0;T]$ with $0 < T < \infty$, then, for $f \in AC[0;T]$ the left-handed and right-handed Riemann-Liouville fractional derivatives $D^\alpha_{0|t} f(t)$ and $D^\alpha_{t|T} f(t)$ of order $\alpha \in (0;1)$ are defined by

$$D^\alpha_{0|t} f(t) = \partial_t J^{1-\alpha}_{0|t} f(t) \quad \textit{where} \quad J^\alpha_{0|t} g(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds, \tag{4}$$

and

$$D^\alpha_{t|T} g(t) = \frac{1}{\Gamma(1-\alpha)} \partial_t \int_t^T (t-s)^{\alpha-1} f(s) ds, \tag{5}$$

is the Riemann−Liouville fractional integral, for all $g \in L^q(0;T), 1 \le q \le \infty$. We refer the reader to [10] for the definitions above.
Furthermore, for every $f, g \in C[0;T]$ such that $D^\alpha_{0|t} f(t), D^\alpha_{t|T} f(t)$ exist and are continues, for all $t \in [0;T]$, $0 < \alpha < 1$, we have the formula of integration par parts (see (2.64) pp.46 in [10])

$$\int_0^T D^\alpha_{0|t} f(t) g(t) dt = \int_0^T f(t) D^\alpha_{t|T} g(t) dt. \tag{6}$$

Note also that , for all $f \in AC^{n+1}[0;T]$ and all integers $n \ge 0$, we have (see (2.2.30) in [10])

$$(-1)^n \partial_t^n D^\alpha_{t|T} f(t) = D^{n+\alpha}_{t|T} f(t), \tag{7}$$

where $AC^{n+1}[0;T] = \left\{ f : [0;T] \longrightarrow \mathbb{R}, \textit{and} \quad \partial_t^n f \in AC[0;T] \right\}$ and ∂_t^n is the usual n times derivative. Moreover, for all $1 \le q \le \infty$, the following formula (see [10], lemma 2.4 pp.74).

$$D^\alpha_{0|t} J^\alpha_{0|t} := id_{L^q(0;T)}, \tag{8}$$

holds almost everywhere on $[0;T]$.

2. Main results

Let us set $\mathcal{H}_T = \mathbb{H} \times (0, T), \mathcal{H} = \mathbb{H} \times (0, \infty)$ and for $R > 0$ we denote

$$\mathcal{T}_R = \Big\{(n, t) \in \mathcal{H} : 0 \leq \tau^2 + |x|^4 + |y|^4 + t^4 \leq 2R^4\Big\}.$$

2.1. Case of a single equation

Definition 2.1. *We say that u is a local weak solution to (1)-(2) on $\mathcal{H}_T$ with initial data $u_0(\eta) \in L^m_{loc}(\mathbb{H}) \cap L^1_{loc}(\mathbb{H}), u_1(\eta) \in L^1_{loc}(\mathbb{H})$ if $u \in L^{max\{p,m\}}_{loc}(\mathcal{H}_T)$ and satisfies*

$$\int_{\mathcal{H}_T} u\varphi_{tt} d\eta dt - \int_{\mathcal{H}_T} u\Delta_{\mathbb{H}}\varphi d\eta dt - \frac{1}{m}\int_{\mathcal{H}_T} |u|^{m-1}u\varphi_t d\eta dt = \int_{\mathbb{H}} \Big(u_1(\eta) + \frac{1}{m}|u_0(\eta)|^{m-1}u_0\Big)\varphi(\eta, 0)d\eta$$
$$+ \int_{\mathbb{H}} u_0(\eta)\varphi_t(\eta, 0)d\eta + \int_{\mathcal{H}_T}\int_0^t (t - s)^{-\gamma}|u(\eta, s)|^p\varphi(\eta, t)dsd\eta dt \tag{9}$$

for any regular test function φ with $\varphi(., T) = 0$ and $\varphi \geq 0$. The solution is called global if $T = +\infty$.

Our first main result is given by the following theorem.

Theorem 2.2. *Let $0 < \gamma < 1$ and p, m such that $0 < 1 < m < p$. Assume that the initial data u_0, u_1 satisfy*

$$\int_{\mathbb{H}} |u_0|^{m-1}u_0(\eta)d\eta > 0, \qquad \int_{\mathbb{H}} u_1(\eta)d\eta > 0 \quad \text{and} \quad \int_{\mathbb{H}} u_0(\eta)d\eta > 0. \tag{10}$$

If

$$Q < min\left\{\frac{p+1}{p-1}; \quad \frac{(1-\gamma)p+m}{p-m}\right\},$$

then there exists no global nontrivial weak solution to (1)-(2).

Proof. Assume that u is a global weak solution to (1)-(2). Then for any regular test function φ, we have

$$\int_{\mathcal{H}}\int_0^t (t - s)^{-\gamma}|u(\eta, s)|^p\varphi(\eta, t)dsd\eta dt + \int_{\mathbb{H}} \Big(u_1(\eta) + \frac{1}{m}|u_0(\eta)|^{m-1}u_0\Big)\varphi(\eta, 0)d\eta$$
$$+ \int_{\mathbb{H}} u_0(\eta)\varphi_t(\eta, 0)d\eta \leq \int_{\mathcal{H}} |u||\varphi_{tt}|d\eta dt + \int_{\mathcal{H}} |u||\Delta_{\mathbb{H}}\varphi|d\eta dt + \frac{1}{m}\int_{\mathcal{H}} |u|^m|\varphi_t|d\eta dt. \tag{11}$$

Let

$$\varphi(\eta, t) = D^{1-\gamma}_{t|T}\xi(\eta, t).$$

Hence, from (4), we obtain

$$\Gamma(\gamma)\int_{\mathcal{H}} J^{1-\gamma}_{0|t}|u(\eta, t)|^p D^{1-\gamma}_{t|T}\xi(\eta, t)d\eta dt + \int_{\mathbb{H}} \Big(u_1(\eta) + \frac{1}{m}|u_0(\eta)|^{m-1}u_0\Big)D^{1-\gamma}_{t|T}\xi(\eta, 0)d\eta$$
$$+ \int_{\mathbb{H}} u_0(\eta)\partial_t\left(D^{1-\gamma}_{t|T}\xi(\eta, 0)\right)d\eta \leq \int_{\mathcal{H}} |u||\partial^2_{tt}D^{1-\gamma}_{t|T}\xi|d\eta dt + \int_{\mathcal{H}} |u||\Delta_{\mathbb{H}}D^{1-\gamma}_{t|T}\xi|d\eta dt \tag{12}$$
$$+\frac{1}{m}\int_{\mathcal{H}_T} |u|^m|\partial_t D^{1-\gamma}_{t|T}\xi|d\eta dt.$$

From (6), (7) and (8), we infer that

$$\begin{aligned}&\Gamma(\gamma)\int_{\mathcal{H}}|u|^p\xi(\eta,t)d\eta dt+\int_{\mathbb{H}}\left(u_1(\eta)+\frac{1}{m}|u_0(\eta)|^{m-1}u_0\right)D^{1-\gamma}_{t|T}\xi(\eta,0)d\eta+\int_{\mathbb{H}}u_0(\eta)D^{2-\gamma}_{t|T}\xi(\eta,0)d\eta\\&\leq\int_{\mathcal{H}}|u||D^{3-\gamma}_{t|T}\xi|d\eta dt+\int_{\mathcal{H}}|u||\Delta_{\mathbb{H}}D^{1-\gamma}_{t|T}\xi|d\eta dt+\frac{1}{m}\int_{\mathcal{H}_T}|u|^m|D^{2-\gamma}_{t|T}\xi|d\eta dt.\end{aligned}\tag{13}$$

Using $\varepsilon-$ Young's inequality, it is obvious that

$$\begin{aligned}&\int_{\mathcal{H}}|u|^p\xi(\eta,t)d\eta dt+\int_{\mathbb{H}}\left(u_1(\eta)+\frac{1}{m}|u_0(\eta)|^{m-1}u_0\right)D^{1-\gamma}_{t|T}\xi(\eta,0)d\eta+\int_{\mathbb{H}}u_0(\eta)D^{2-\gamma}_{t|T}\xi(\eta,0)d\eta\\&\leq C\Big\{\mathcal{A}(p,\gamma,\xi)+\mathcal{B}(p,\gamma,\xi)+\mathcal{C}(p,\gamma,\xi)\Big\},\end{aligned}\tag{14}$$

where

$$\mathcal{A}(p,\gamma,\xi)=\int_{\mathcal{H}}|\xi|^{\frac{-1}{p-1}}|D^{3-\gamma}_{t|T}\xi(\eta,t)|^{\frac{p}{p-1}}d\eta dt,\tag{15}$$

$$\mathcal{B}(p,\gamma,\xi)=\int_{\mathcal{H}}|\xi|^{\frac{-1}{p-1}}|\Delta_{\mathbb{H}}\left(D^{1-\gamma}_{t|T}\xi(\eta,t)\right)|^{\frac{p}{p-1}}d\eta dt,\tag{16}$$

and

$$\mathcal{C}(p,\gamma,\xi)=\int_{\mathcal{H}}|\xi|^{\frac{-m}{p-m}}|D^{2-\gamma}_{t|T}\xi(\eta,t)|^{\frac{p}{p-m}}d\eta dt.\tag{17}$$

Let us set

$$\xi_R(\eta,t)=\phi^{\omega}\left(\frac{\tau^2+|x|^4+|y|^4+t^4}{R^4}\right),\tag{18}$$

where ϕ is the following standard cut– off function

$$\Phi(r)=\begin{cases}1, & 0\leq r\leq 1,\\ \searrow & 1\leq r\leq 2,\\ 0, & r\geq 2.\end{cases}$$

We point out that $supp(\xi_R)$ is a subset of $\mathcal{T}_R$ while $supp(\xi_R)_{tt})$; $supp(\Delta_{\mathbb{H}}\xi_R)$ and $supp(\Delta_{\mathbb{H}}(\xi_R)_t)$ are subsets of

$$\mathfrak{T}_R=\Big\{(n,t)\in\mathcal{H}:R^4\leq\tau^2+|x|^4+|y|^4+t^4\leq 2R^4\Big\}.$$

Then we have

$$\begin{aligned}\Delta_{\mathbb{H}}\Phi^{\omega}&=\tfrac{4\omega(N+4)}{R^4}\left(|x|^2+|y|^2\right)\Phi'\Phi^{\omega-1}+\tfrac{16\omega}{R^8}\left(\left(|x|^6+|y|^6\right)+2\tau\left(|x|^2-|y|^2\right)x.y+\tau^2\left(|x|^2+|y|^2\right)\right)\Phi''\Phi^{\omega-1}\\&+\tfrac{16\omega(\omega-1)}{R^8}\left(\left(|x|^6+|y|^6\right)+2\tau\left(|x|^2-|y|^2\right)x.y+\tau^2\left(|x|^2+|y|^2\right)\right)\Phi'^2\Phi^{\omega-2}.\end{aligned}$$

At this stage, we use the scaled variables

$$\tilde{\tau}=R^{-2}\tau,\quad\tilde{x}=R^{-1}x,\quad\tilde{y}=R^{-1}y,\quad\tilde{t}=R^{-1}t,\tag{19}$$

we obtain easily the estimates

$$\mathcal{A}(p,\gamma,\xi)\leq CR^{\nu_1},\quad\mathcal{B}(p,\gamma,\xi)\leq CR^{\nu_2},\quad\mathcal{C}(p,\gamma,\xi)\leq CR^{\nu_3},\tag{20}$$

where

$$\nu_1 = \frac{(\gamma-3)p}{p-1} + Q + 1, \quad \nu_2 = \frac{-2p}{p-m} + Q + 1, \quad \nu_3 = \frac{(\gamma-2)p}{p-m} + Q + 1.$$

Therefore, from the condition (10) and the above estimates, we deduce

$$\int_{\mathcal{H}} |u|^p \xi(\eta, t) d\eta dt \leq C \left(R^{\nu_1} + R^{\nu_2} + R^{\nu_3} \right). \tag{21}$$

If $\max\{\nu_1; \nu_2; \nu_3\} < 0$ then by letting $R \longrightarrow \infty$ in (21) and using the dominate convergence theorem, we get

$$\int_{\mathcal{H}} |u|^p = 0 \quad \Longrightarrow u = 0.$$

This contradicts our assumption about u. This finishes the proof. □

2.2. Case of system

Definition 2.3. *We say that the pair (u, v) is a local weak solution to (3) on $\mathcal{H}_T$ with initial data $(u_0(\eta), v_0(\eta)) \in L^m_{loc}(\mathbb{H}) \cap L^1_{loc}(\mathbb{H}) \times L^m_{loc}(\mathbb{H}) \cap L^1_{loc}(\mathbb{H})$ and $(u_1(\eta), v_1(\eta)) \in L^1_{loc}(\mathbb{H}) \times L^1_{loc}(\mathbb{H})$ if $(u, v) \in L^{max\{p,m\}}_{loc}(\mathcal{H}_T) \times L^{max\{q,m\}}_{loc}(\mathcal{H}_T)$ and satisfies*

$$\begin{aligned}\int_{\mathcal{H}_T} u\varphi_{tt} d\eta dt - \int_{\mathcal{H}_T} u \Delta_{\mathbb{H}} \varphi d\eta dt - \frac{1}{m} \int_{\mathcal{H}_T} |u|^{m-1} u \varphi_t d\eta dt = \int_{\mathbb{H}} \left(u_1(\eta) + \frac{1}{m} |u_0(\eta)|^{m-1} u_0 \right) \varphi(\eta, 0) d\eta \\ + \int_{\mathbb{H}} u_0(\eta) \varphi_t(\eta, 0) d\eta + \int_{\mathcal{H}_T} \int_0^t (t-s)^{-\gamma_1} |v(\eta, s)|^q \varphi(\eta, t) ds d\eta dt,\end{aligned} \tag{22}$$

and

$$\begin{aligned}\int_{\mathcal{H}_T} v\varphi_{tt} d\eta dt - \int_{\mathcal{H}_T} v \Delta_{\mathbb{H}} \varphi d\eta dt - \frac{1}{m} \int_{\mathcal{H}_T} |v|^{m-1} v \varphi_t d\eta dt = \int_{\mathbb{H}} \left(v_1(\eta) + \frac{1}{m} |v_0(\eta)|^{m-1} v_0 \right) \varphi(\eta, 0) d\eta \\ + \int_{\mathbb{H}} v_0(\eta) \varphi_t(\eta, 0) d\eta + \int_{\mathcal{H}_T} \int_0^t (t-s)^{-\gamma_2} |u(\eta, s)|^p \varphi(\eta, t) ds d\eta dt,\end{aligned} \tag{23}$$

for any regular test function φ with $\varphi(., T) = 0$ and $\varphi \geq 0$. The solution is called global if $T = +\infty$.

Our main result in this section is:

Theorem 2.4. *Let $0 < \gamma_i < 1$ and p, q, m such that $0 < 1 < m < p$ and $0 < 1 < m < q$. Assume that the initial data u_0, u_1, v_0, v_1 satisfy*

$$\int_{\mathbb{H}} |u_0|^{m-1} u_0(\eta) d\eta > 0, \quad \int_{\mathbb{H}} u_1(\eta) d\eta > 0 \quad and \quad \int_{\mathbb{H}} u_0(\eta) d\eta > 0, \tag{24}$$

$$\int_{\mathbb{H}} |v_0|^{m-1} v_0(\eta) d\eta > 0, \quad \int_{\mathbb{H}} v_1(\eta) d\eta > 0 \quad and \quad \int_{\mathbb{H}} v_0(\eta) d\eta > 0. \tag{25}$$

If

$$Q < max \{\lambda; \theta\},$$

where

$$\lambda = min \left\{ 1 + 2\frac{p+1}{pq-1}, \quad 1 + \frac{(2-\gamma_1)p + 2m}{pq-m}, \quad \frac{(1-\gamma_2)pq + 2mp + m}{pq-m}, \quad \frac{(1-\gamma_2)pq + (2-\gamma_1)mp + m^2}{pq-m^2} \right\},$$

and

$$\theta = min \left\{ 1 + 2\frac{q+1}{pq-1}, \quad 1 + \frac{(2-\gamma_2)q + 2m}{pq-m}, \quad \frac{(1-\gamma_1)pq + 2mq + m}{pq-m}, \quad \frac{(1-\gamma_1)pq + (2-\gamma_2)mq + m^2}{pq-m^2} \right\}.$$

Then the system (3) does not have a global nontrivial weak solution.

Proof. Suppose that the pair (u, v) is a global weak solution to (3). Then for any regular test functions φ_i, we have

$$\begin{aligned}&\int_{\mathcal{H}}\int_0^t (t-s)^{-\gamma_1}|v(\eta,s)|^q\varphi_1(\eta,t)dsd\eta dt + \int_{\mathbb{H}}\Big(u_1(\eta)+\frac{1}{m}|u_0(\eta)|^{m-1}u_0\Big)\varphi_1(\eta,0)d\eta + \int_{\mathbb{H}} u_0(\eta)(\varphi_1)_t(\eta,0)d\eta \\ &\leq \int_{\mathcal{H}}|u||(\varphi_1)_{tt}|d\eta dt + \int_{\mathcal{H}}|u||\Delta_{\mathbb{H}}\varphi_1|d\eta dt + \frac{1}{m}\int_{\mathcal{H}}|u|^m|(\varphi_1)_t|d\eta dt,\end{aligned} \tag{26}$$

and

$$\begin{aligned}&\int_{\mathcal{H}}\int_0^t (t-s)^{-\gamma_2}|u(\eta,s)|^p\varphi_2(\eta,t)dsd\eta dt + \int_{\mathbb{H}}\Big(v_1(\eta)+\frac{1}{m}|v_0(\eta)|^{m-1}v_0\Big)\varphi_2(\eta,0)d\eta + \int_{\mathbb{H}} v_0(\eta)(\varphi_2)_t(\eta,0)d\eta \\ &\leq \int_{\mathcal{H}}|v||(\varphi_2)_{tt}|d\eta dt + \int_{\mathcal{H}}|v||\Delta_{\mathbb{H}}\varphi_2|d\eta dt + \frac{1}{m}\int_{\mathcal{H}}|v|^m|(\varphi_2)_t|d\eta dt.\end{aligned} \tag{27}$$

Let us set

$$\varphi_i(\eta,t) = D_{t|T}^{1-\gamma_i}\xi(\eta,t).$$

Then proceeding in the same manner used in the proof of theorem (2.2), we obtain

$$\Gamma(\gamma_1)\int_{\mathcal{H}}|v|^q\xi(\eta,t)d\eta dt \leq \int_{\mathcal{H}}|u||D_{t|T}^{3-\gamma_1}\xi|d\eta dt + \int_{\mathcal{H}}|u||\Delta_{\mathbb{H}}D_{t|T}^{1-\gamma_1}\xi|d\eta dt + \frac{1}{m}\int_{\mathcal{H}_T}|u|^m|D_{t|T}^{2-\gamma_1}\xi|d\eta dt, \tag{28}$$

similarly

$$\Gamma(\gamma_2)\int_{\mathcal{H}}|u|^p\xi(\eta,t)d\eta dt \leq \int_{\mathcal{H}}|v||D_{t|T}^{3-\gamma_2}\xi|d\eta dt + \int_{\mathcal{H}}|v||\Delta_{\mathbb{H}}D_{t|T}^{1-\gamma_2}\xi|d\eta dt + \frac{1}{m}\int_{\mathcal{H}_T}|v|^m|D_{t|T}^{2-\gamma_2}\xi|d\eta dt. \tag{29}$$

Using the Holder inequality, we get the estimates

$$\begin{aligned}\int_{\mathcal{H}}|v|^q\xi(\eta,t)d\eta dt \leq C\Bigg\{&\Big(\int_{\mathcal{H}}|u|^p\xi d\eta dt\Big)^{\frac{1}{p}}\Big(\mathcal{A}^{\frac{p-1}{p}}(p,\gamma_1,\xi)+\mathcal{B}^{\frac{p-1}{p}}(p,\gamma_1,\xi)\Big) \\ &+\Big(\int_{\mathcal{H}}|u|^p\xi d\eta dt\Big)^{\frac{m}{p}}\mathcal{C}^{\frac{p-m}{p}}(p,\gamma_1,\xi)\Bigg\},\end{aligned} \tag{30}$$

and

$$\begin{aligned}\int_{\mathcal{H}}|u|^p\xi(\eta,t)d\eta dt \leq C\Bigg\{&\Big(\int_{\mathcal{H}}|v|^q\xi d\eta dt\Big)^{\frac{1}{q}}\Big(\mathcal{A}^{\frac{q-1}{q}}(q,\gamma_2,\xi)+\mathcal{B}^{\frac{q-1}{q}}(q,\gamma_2,\xi)\Big) \\ &+\Big(\int_{\mathcal{H}}|v|^q\xi d\eta dt\Big)^{\frac{m}{q}}\mathcal{C}^{\frac{q-m}{q}}(q,\gamma_2,\xi)\Bigg\}.\end{aligned} \tag{31}$$

For the simplicity let us set

$$\mathcal{I} = \Big(\int_{\mathcal{H}}|u|^p\xi d\eta dt\Big)^{\frac{1}{p}} \quad \text{and} \quad \mathcal{J} = \Big(\int_{\mathcal{H}}|v|^q\xi d\eta dt\Big)^{\frac{1}{q}},$$

then we can deduce easily

$$\mathcal{J}^q \leq C\Big\{\Big(\mathcal{A}^{\frac{p-1}{p}}(p,\gamma_1,\xi)+\mathcal{B}^{\frac{p-1}{p}}(p,\gamma_1,\xi)\Big)\mathcal{I} + \mathcal{C}^{\frac{p-m}{p}}(p,\gamma_1,\xi)\mathcal{I}^m\Big\}, \tag{32}$$

and

$$\mathcal{I}^p \le C\left\{\left(\mathcal{A}^{\frac{q-1}{q}}(q,\gamma_2,\xi)+\mathcal{B}^{\frac{q-1}{q}}(q,\gamma_2,\xi)\right)\mathcal{J}+C^{\frac{q-m}{q}}(q,\gamma_2,\xi)\mathcal{J}^m\right\}. \tag{33}$$

Taking into account the test function given by (18) and moreover, we use the change of variables (21), we arrive at

$$\mathcal{J}^q \le C\{R^{\alpha_1}\mathcal{I}+R^{\alpha_2}\mathcal{I}^m\}, \tag{34}$$

$$\mathcal{I}^p \le C\{R^{\alpha_3}\mathcal{J}+R^{\alpha_4}\mathcal{J}^m\}, \tag{35}$$

where

$$\alpha_1 = Q-1+\frac{Q+1}{p};\quad \alpha_2=\gamma_1+Q-1-\frac{m(Q+1)}{p}$$

$$\alpha_3 = Q-1+\frac{Q+1}{q};\quad \alpha_4=\gamma_2+Q-1-\frac{m(Q+1)}{q}.$$

From lemma 04 in [15], we obtain

$$\mathcal{I}^{pq} \le C\left\{R^{\beta_1}+R^{\beta_2}+R^{\beta_3}+R^{\beta_4}\right\},$$

$$\mathcal{J}^{pq} \le C\left\{R^{\beta_1'}+R^{\beta_2'}+R^{\beta_3'}+R^{\beta_4'}\right\},$$

with

$$\beta_1=\frac{q}{pq-1}\left(pq(Q-1)-2p-Q-1\right);\quad \beta_2=\frac{q}{pq-m}\left(pq(Q-1)+(\gamma_1-2)p-m(Q+1)\right)$$

$$\beta_3=\frac{q}{pq-m}\left(pq(\gamma_2+Q-1)-2mp-m(Q+1)\right);\quad \beta_4=\frac{q}{pq-m^2}\left(pq(\gamma_2+Q-1)+(\gamma_1-2)pm-m^2(Q+1)\right)$$

$$\beta_1'=\frac{p}{pq-1}\left(pq(Q-1)-2q-Q-1\right);\quad \beta_2'=\frac{p}{pq-m}\left(pq(Q-1)+(\gamma_2-2)q-m(Q+1)\right)$$

$$\beta_3'=\frac{p}{pq-m}\left(pq(\gamma_1+Q-1)-2mq-m(Q+1)\right);\quad \beta_4'=\frac{p}{pq-m^2}\left(pq(\gamma_1+Q-1)+(\gamma_2-2)qm-m^2(Q+1)\right).$$

Taking either $max\{\beta_1;\beta_2;\beta_3;\beta_4\}<0$ or $max\left\{\beta_1';\beta_2';\beta_3';\beta_4'\right\}<0$ and using the same arguments as in the previous proof one can show that $u=v=0$. This completes the proof. □

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