Functional Analysis,
Approximation and
Computation
10 (2) (2018), 23–31

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/faac

ON THE CLASS OF n-REAL POWER POSITIVE OPERATORS ON A HILBERT SPACE

Abdelkader Benali[a]

[a] *Faculty of The Exact Science And Computer Science, Mathematics Department,University of Hassiba Benbouali, Chlef Algeria. B.P. 151 Hay Essalem, chlef 02000, Algeria*

Abstract. In this paper, we introduce a new class of operators acting on a complex Hilbert space $\mathcal{H}$ which is called n-real power positive operators, denoted by $[n\mathcal{RP}]$. An operator $T \in \mathcal{B}(\mathcal{H})$ is called n-real power positive operator if $T^n + T^{*n} \geq 0$ or equivalently $Re\langle T^n x \mid x\rangle \geq 0$ for all $x \in \mathcal{H}$, where n is positive integer number greater than 1.

1. Introduction and terminologies

Let $\mathcal{H}$ be a complex Hilbert space. Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators defined in $\mathcal{H}$. Let T be an operator in $\mathcal{B}(\mathcal{H})$. The operator T is called normal if it satisfies the following condition $T^*T = TT^*$, i.e., T commutes with T^*. The class of quasi-normal operators denoted by $[QN]$, was first introduced and studied by A. Brown ([1]) in 1953. The operator T is quasi-normal if T commutes with T^*T, i.e.; $T(T^*T) = (T^*T)T$. A. A. S. Jibril (see [2, 3]), in 2008 introduced the class of n-power normal operators as a generalization of normal operators and its denoted by $[nN]$. The operator T is called n-power normal if T^n commutes with T^* , i.e.; $T^nT^* = T^*T^n$. In the year 2011, O. A. Mahmoud Sid Ahmed introduced the class of n-power quasi-normal operators denoted by $[nQN]$ (see [6, 7]), as a generalization of quasi-normal operators. An operator T is called n-power quasi-normal if T^n commutes with T^*T , i.e.; $T^n(T^*T) = (T^*T)T^n$.

Recently in [5], the authors introduced and studied the operator T satisfying $T^2 \geq -T^{*2}$. In this search, we introduce a new class of operators namely n-real power positive operator denoted by $[n\mathcal{RP}]$. An operator $T \in [n\mathcal{RP}]$ if and only $T^n + T^{*n} \geq 0$, for some integer $n = 1, 2, 3, ...$ Let $T \in \mathcal{B}(\mathcal{H})$. We can write

$$T = A + iB \tag{1}$$

where A, B are Hermitian. Such a decomposition is unique, and we have

$$A = \frac{1}{2}\left(T + T^*\right), \quad B = \frac{1}{2i}\left(T - T^*\right). \tag{2}$$

The operators A and B are called the real and imaginary parts of T, and the decomposition (1) is called the Cartesian decomposition of T.

2010 *Mathematics Subject Classification*. Primary 47B20; Secondary 47B99.
Keywords. Operator; Normal, Quasi-normal; Hilbert space.
Received: 210 April 2017; Accepted: 14 May 2017
Communicated by Dijana Mosić
Email address: benali4848@gmail.com , a.benali@univhb-chlef.dz (Abdelkader Benali)

2. Some basic properties of $[n\mathcal{RP}]$

In section two we study some of the basic properties of operators in $[n\mathcal{RP}]$.

Definition 2.1. *For $n \in \mathbb{N}$, an operator $T \in \mathcal{L}(H)$ is said to be n-real power positive operator if*

$$T^n + T^{*n} \geq 0.$$

We denote the set of n-real power positive operators by $[n\mathcal{RP}]$.

Example 2.2. *Let $T = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{pmatrix} \in \mathcal{B}(\mathbb{C}^n)$. A simple computation shows that for all $z = (z_1, ...z_n) \in \mathbb{C}^n$ we have*

$$\Big\langle \Big(T^n + T^{*n}\Big) z \mid z \Big\rangle = \sum_{1 \leq k \leq n} \Big(\lambda_k^n + \overline{\lambda_k}^n\Big) |z_k|^2 = 2 \sum_{1 \leq k \leq n} Re\Big(\lambda_k^n\Big) |z_k|^2 .$$

We deduce that if $Re\Big(\lambda_k^n\Big) \geq 0$ for all $k = 1, 2, ..., n$, then $T \in [n\mathcal{RP}]$ and if $Re\Big(\lambda_k^n\Big) < 0$ for all $k = 1, 2, ..., n$, then $T \notin [n\mathcal{RP}]$.

Proposition 2.3. *Let $T \in \mathcal{B}(\mathcal{H})$ and $n \in \mathbb{N}$ the following properties hold*

(1) *if $T \in [n\mathcal{RP}]$ then so T^*.*

(2) *$T \in [n\mathcal{RP}]$ if and only if $Re \langle T^n x \mid x \rangle \geq 0$, for all $x \in \mathcal{H}$.*

(3) *If T is invertible, then $T \in [n\mathcal{RP}]$ if and only if $T^{-1} \in [n\mathcal{RP}]$.*

Proof. (1) Obvious from the Definition 2.1.

(2) In fact, it is well know that

$$\begin{aligned} T \in [n\mathcal{RP}] \Longleftrightarrow T^n + T^{*n} \geq 0 &\Longleftrightarrow \Big\langle \Big(T^n + T^{*n}\Big) x \mid x \Big\rangle \geq 0, \ \ \forall \ x \in \mathcal{H} \\ &\Longleftrightarrow \langle T^n x \mid x \rangle + \langle T^{*n} x \mid x \rangle \geq 0, \ \ \forall \ x \in \mathcal{H} \\ &\Longleftrightarrow \langle T^n x \mid x \rangle + \langle x \mid T^n x \rangle \geq 0, \ \ \forall \ x \in \mathcal{H} \\ &\Longleftrightarrow \langle T^n x \mid x \rangle + \overline{\langle T^n x \mid x \rangle} \geq 0, \ \ \forall \ x \in \mathcal{H} \\ &\Longleftrightarrow 2Re \langle T^n x \mid x \rangle \geq 0. \end{aligned}$$

(3) Assume that T is invertible and $T \in [n\mathcal{RP}]$. We have $Re \langle T^n x \mid x \rangle \geq 0, \ \ \forall \ x \in \mathcal{H}$.

It follows that for all $x \in \mathcal{H}$

$$0 \leq Re \langle T^n T^{-n} x \mid T^{-n} x \rangle = Re \langle x \mid T^{-n} x \rangle = Re \overline{\langle T^{-n} x \mid x \rangle} = Re \langle T^{-n} x \mid x \rangle .$$

Hence $T^{-1} \in [n\mathcal{RP}]$. The converse is obvious. □

The following examples show that the two classes $[n\mathcal{RP}]$ and $[(n+1)\mathcal{RP}]$ are not the same.

Example 2.4. *Let $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^2)$. A simple computation shows that*

$$T^2 + T^{*2} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \ \ \textit{and} \ \ T^3 + T^{*3} = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}.$$

For all $(u, v) \in \mathbb{C}^2$ *we have*

$$\left\langle \left(T^2 + T^{*2}\right)\begin{pmatrix} u \\ v \end{pmatrix} \mid \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle = 2|u|^2 + 4Re(u\overline{v}) + 2|v|^2$$
$$= 2\left(Re(u) + Re(v)\right)^2 + 2\left(Im(u) + Im(v)\right)^2 \geq 0.$$

Hence $T \in [2\mathcal{RP}]$.

On the other hand

$$\left\langle \left(T^3 + T^{*3}\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix} \mid \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\rangle = -2 < 0.$$

So $T \notin [3\mathcal{RP}]$.

Example 2.5. *Let* $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^2)$. *A simple computation shows that*

$$T^2 + T^{*2} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \text{ and } T^3 + T^{*3} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

It follows that $T \notin [2\mathcal{RP}]$ *and* $T \in [3\mathcal{RP}]$.

Example 2.6. *Let* $T = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^3)$. *It is easily to see that* $T \notin [n\mathcal{RP}]$ *for all* $n = 1, 2, ...$

Proposition 2.7. ([5]) *Let* $T = A + iB \in \mathcal{B}(\mathcal{H})$,*then* $T^2 \geq T^{*2}$ *if and only if* $A^2 \geq B^2$.

In the following proposition, we generalize Proposition 2.2.

Proposition 2.8. *Let* $T \in \mathcal{B}(\mathcal{H})$, $T = A + iB$ *such that* $AB + BA = 0$ *and* $n \in \mathbb{N}$. *Then the following properties hold*

(1) $T \in [2n\mathcal{RP}]$ *if and only if* $\left(A^2 - B^2\right)^n \geq 0$.

(2) $T \in [(2n + 1)\mathcal{RP}]$ *if and only if* $A\left(A^2 - B^2\right)^n \geq 0$.

Proof. (1) A simple computation shows that

$$T^{2n} = \left(A + iB\right)^{2n} = \left(A^2 - B^2\right)^n \text{ and } T^{*(2n)} = \left(A - iB\right)^{2n} = \left(A^2 - B^2\right)^n$$

and so

$$T^{2n} + T^{*2n} = 2\left(A^2 - B^2\right)^n.$$

Hence

$$T \in [2n\mathcal{RP}] \Longleftrightarrow \left(A^2 - B^2\right)^n \geq 0$$

as required.

(2) A similar argument gives

$$T^{(2n+1)} = \left(A + iB\right)^{2n+1} = \left(A + iB\right)\left(A^2 - B^2\right)^n \text{ and } T^{*(2n+1)} = \left(A + iB\right)^{*(2n+1)} = \left(A - iB\right)\left(A^2 - B^2\right)^n$$

and so

$$T^{2n+1} + T^{*2n+1} = 2A\left(A^2 - B^2\right)^n.$$

Hence,

$$T \in [(2n + 1)\mathcal{RP}] \Longleftrightarrow A\left(A^2 - B^2\right)^n \geq 0$$

as required. □

Proposition 2.9. *Let $T, S \in \mathcal{B}(\mathcal{H})$ and $n \in \mathbb{N}$. If $T \in [n\mathcal{RP}]$ and S is unitary equivalent to T, then $S \in [n\mathcal{RP}]$.*

Proof. By assumption, there is a unitary equivalent operator $U \in \mathcal{B}(\mathcal{H})$ such that $S = U^*TU$, which implies that

$$S^* = U^*T^*U.$$

Thus we have

$$S^n = U^*T^nU \quad \text{and } S^{*n} = U^*T^{*n}U.$$

Since U is unitary and using the fact that $T^n \geq -T^{*n}$ we conclude that

$$U^*T^nU \geq -U^*T^{*n}U.$$

Thus $S^n \geq -S^{*n}$. □

Remark 2.10. *The following example shows that in general the class $[n\mathcal{RP}]$ is note closed under translation.*

Example 2.11. *Consider* $T = \begin{pmatrix} 1+i & 0 & 0 \\ 0 & 1+i & 0 \\ 0 & 0 & 1+i \end{pmatrix} \in \mathcal{B}(\mathbb{C}^3)$. *From Example 2.1, it is easy to see that $T \in [3\mathcal{RP}]$ and* $T - 2I = \begin{pmatrix} -1+i & 0 & 0 \\ 0 & -1+i & 0 \\ 0 & 0 & -1+i \end{pmatrix} \notin [3\mathcal{RP}]$.

Proposition 2.12. *Let $T \in \mathcal{B}(\mathcal{H})$. If $T \in \bigcap_{1 \leq k \leq n} [k\mathcal{RP}]$ for some $n = 1, 2, ...$, then $T + \lambda I \in [n\mathcal{RP}]$ for all $\lambda \geq 0$.*

Proof. Since

$$\left(T + \lambda I\right)^n = \sum_{0 \leq k \leq n} \binom{n}{k} \lambda^{n-k} T^k = \lambda^n + \sum_{1 \leq k \leq n} \binom{n}{k} \lambda^{n-k} T^k$$

we have for all $x \in \mathcal{H}$

$$Re\left\langle \left(T + \lambda I\right)^n x \mid x \right\rangle = \lambda^n \|x\|^2 + \sum_{1 \leq k \leq n} \binom{n}{k} \lambda^{n-k} \underbrace{Re\left\langle T^k x \mid x \right\rangle}_{\geq 0} \geq 0.$$

□

In the following theorem we give a sufficient conditions under which the class $[n\mathcal{RP}]$ is closed under sum of two operators.

Theorem 2.13. *Let $T, S \in [n\mathcal{RP}]$ such that $T^kS = -S^kT$ for $k = 1, 2, ..., n-1$ for some integer $n = 2, 3....$, then $T + S \in [n\mathcal{RP}]$.*

Proof. Form the hypothesis it is clear that $\left(T + S\right)^n = T^n + S^n$ and so that

$$(T + S)^n + (T^* + S^*)^n = \underbrace{T^n + T^{*n}}_{\geq 0} + \underbrace{S^n + S^{*n}}_{\geq 0} \geq 0.$$

□

The following lemma is well know.

Lemma 2.14. *Let $T, S \in \mathcal{B}(\mathcal{H})$ such that $T \geq S$. Then for all $A \in \mathcal{B}(\mathcal{H})$ we have $A^*TA \geq A^*SA$.*

Proposition 2.15. *Let $n \in \mathbb{N}$. If $T \in [n\mathcal{RP}]$ is such that $T^*T^2 = T^2T^*$, then $T^*T^2 \in [n\mathcal{RP}]$.*

Proof. Since $T \in [n\mathcal{RP}]$ we have by Lemma 2.1 that

$$\begin{aligned} T^n + T^{*n} \geq 0 &\Longrightarrow T^{*n}T^nT^n + T^{*2n}T^n \geq 0 \\ &\Longrightarrow \left(T^*T^2\right)^n + \left(T^{*2}T\right)^n \geq 0 \ \ (since \ \ T^*T^2 = T^2T^*) \\ &\Longrightarrow \left(T^*T^2\right)^n + \left(T^*T^2\right)^{*n} \geq 0. \end{aligned}$$

Hence $T^*T^2 \in [n\mathcal{RP}]$ as required. □

In the following proposition we give a characterization of the class $[2\mathcal{RP}]$.

Proposition 2.16. *If $T \in \mathcal{B}(\mathcal{H})$ is normal then we have*

$$T \in [2\mathcal{RP}] \text{ if and only if } \ 2\Big(Re(T)\Big)^2 \geq |T|^2$$

where $|T| = \left(T^*T\right)^{\frac{1}{2}}$.

Proof. Assume that $2\Big(Re(T)\Big)^2 \geq |T|^2$ so we have

$$\begin{aligned} 2\Big(Re(T)\Big)^2 \geq |T|^2 &\Longrightarrow 2\left(\frac{T+T^*}{2}\right)^2 \geq T^*T \\ &\Longrightarrow T^2 + 2T^*T + T^{*2} \geq 2T^*T \ (since \ \ T \ \ is \ normal\ ) \\ &\Longrightarrow T^2 + T^{*2} \geq 0. \end{aligned}$$

We deduce that $T \in [2\mathcal{RP}]$.

Conversely, assume that $T \in [2\mathcal{RP}]$. By the fact that T^*T is positive we have the following implications

$$\begin{aligned} T^2 + T^{*2} \geq 0 &\Longrightarrow T^2 + 2T^*T + T^{*2} \geq 2T^*T \\ &\Longrightarrow \left(T + T^*\right)^2 \geq 2T^*T \ \ (since \ \ T \ \ is \ normal\ ) \\ &\Longrightarrow \Big(2Re(T)\Big)^2 \geq 2|T|^2 \\ &\Longrightarrow 2\Big(Re(T)\Big)^2 \geq |T|^2. \end{aligned}$$

□

Theorem 2.17. *Let $T \in \mathcal{B}(\mathcal{H})$. Then the following properties hold:*

(1) *For $n = 2, 3, ...,$ if T^n is unitary equivalent to T^{*n-1} then*

$$T \in [n\mathcal{RP}] \Longleftrightarrow T \in [(n-1)\mathcal{RP}].$$

(2) *For $n = 2, 3, ...,$ if T^k is unitary equivalent to T^{*k-1} for all $k \in \{1, 2....., n\}$, then*

$$T \in [n\mathcal{RP}] \Longleftrightarrow T \in [\mathcal{RP}].$$

Proof. (1) From the hypothesis there exists an operator $U \in \mathcal{B}(\mathcal{H})$: $U^*U = UU^* = I$ such that $T^n = U^*T^{*n-1}U$.

Firstly, assume that $T \in [n\mathcal{RP}]$, it follows that

$$T^n + T^{*n} \geq 0 \Longrightarrow U^*T^{*n-1}U + U^*T^{n-1}U \geq 0 \Longrightarrow U^*\Big(T^{n-1} + T^{*n-1}\Big)U \geq 0.$$

By Lemma 2.1, we deduce that $T^{n-1} + T^{*n-1} \geq 0$ and hence $T \in [(n-1)\mathcal{RP}]$.

Conversely, assume that $T \in [(n-1)\mathcal{RP}]$. We have by Lemma 2.1

$$T^{n-1} + T^{*n-1} \geq 0 \Longrightarrow U^*\Big(T^{n-1} + T^{*n-1}\Big)U \geq 0 \Longrightarrow T^n + T^{*n} \geq 0.$$

Hence $T \in [n\mathcal{RP}]$.

(2) From the hypothesis there exists a unitary operator U_k such that

$$T^k = U_k^* T^{*k-1} U_k \;\text{ for }\; k = 1, 2, ..., n.$$

If we assume that $T \in [n\mathcal{RP}]$ we have from (1) that $T \in [(n-1)\mathcal{RP}]$. Repeating the process with $T \in [(n-1)\mathcal{RP}]$ we obtain that $T \in [(n-2)\mathcal{RP}]$. Hence the following implications hold

$$T \in [n\mathcal{RP}] \Longrightarrow T \in [(n-1)\mathcal{RP}] \Longrightarrow T \in [(n-2)\mathcal{RP}] \Longrightarrow ....T \in [2\mathcal{RP}] \Longrightarrow T \in [\mathcal{RP}].$$

Conversely, assume that $T \in [\mathcal{RP}]$. By Lemma 2.1 we obtain

$$T^2 + T^{*2} = U_2^*\Big(T + T^*\Big)U_2 \geq 0 \Longrightarrow T \in [2\mathcal{RP}].$$

Also

$$T^3 + T^{*3} = U_3^*\Big(T^2 + T^{*2}\Big)U_3 \geq 0 \Longrightarrow T \in [3\mathcal{RP}].$$

Repeating the process we obtain

$$T^n + T^{*n} = U_n^*\Big(T^{n-1} + T^{*n-1}\Big)U_n \geq 0 \Longrightarrow T \in [n\mathcal{RP}].$$

This completes the proof. □

Proposition 2.18. *Let $T \in \mathcal{B}(\mathcal{H})$. Consider $F = T^{n-1} + T^*$ and $G = T^{n-1} - T^*$ for some $n \in \mathbb{N}$. If T is normal then the following equivalence holds*

$$T \in [n\mathcal{RP}] \quad \textit{if and only if} \quad FF^* \geq GG^*.$$

Proof. Since T is normal we have

$$\begin{aligned} FF^* - GG^* &= \Big(T^{n-1} + T^*\Big)\Big(T^{*n-1} + T\Big) - \Big(T^{n-1} - T^*\Big)\Big(T^{*n-1} - T\Big) \\ &= 2(T^n + T^{*n}). \end{aligned}$$

From which it follows that

$$T \in [n\mathcal{RP}] \Longleftrightarrow T^n + T^{*n} \geq 0 \Longleftrightarrow FF^* - GG^* \geq 0.$$

□

Proposition 2.19. *Let $T \in \mathcal{B}(\mathcal{H})$.*

(1) If T is almost subprojection, then

$$T \in [2\mathcal{RP}] \quad \textit{if and only if} \;\; T \in [4\mathcal{RP}].$$

(2) If T is idempotent, then

$$T \in [\mathcal{RP}] \quad \textit{if and only if} \;\; T \in [n\mathcal{RP}] \;\; \textit{for } n = 2, 3....$$

Proof. (1) Since T is almost subprojection, $T^4 = T^{*2}$ (see [4]) we have for all $x \in \mathcal{H}$

$$Re\left\langle T^2x \mid x\right\rangle = Re\left\langle T^{*4}x \mid x\right\rangle = Re\left\langle x \mid T^4x\right\rangle = Re\overline{\langle T^4x \mid x\rangle} = Re\left\langle T^4x \mid x\right\rangle$$

So

$$T \in [2\mathcal{RP}] \Longleftrightarrow T \in [4\mathcal{RP}].$$

(2) Since T is idempotent we have $T = T^2 = ... = T^n$ and so that

$$T^n + T^{*n} = T + T^*.$$

Hence the desired result.

□

The following examples show that a operator $T \in [n\mathcal{RP}]$ need not be almost subprojection and vice versa.

Example 2.20. *Let* $T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ *be an operator acting in two- dimensional complex Hilbert space. then* $T \in [n\mathcal{RP}]$ *for all* $n \in \mathbb{N}$*. Now, by direct calculation* $T^4 = \begin{pmatrix} 1 & 0 \\ 0 & 16 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} = T^{*2}$

Theorem 2.21. *Let* $T, S \in \mathcal{B}(\mathcal{H})$*. Assume that* $T, S \in \bigcap\limits_{1\leq k\leq n} [k\mathcal{RP}]$ *for some integer* $n = 1, 2, ...$ *If* $TS = ST = T + S$ *, then* $TS \in [n\mathcal{RP}]$*.*

Proof. For $n = 1$. Assume that T and S are in $[\mathcal{RP}]$. We have

$$TS + (TS)^* = T + T^* + S + S^* \geq 0$$

and so $TS \in [\mathcal{RP}]$.

For $n = 2$. Assume that T and S are in $[k\mathcal{RP}]$ for $k = 1, 2$. We have

$$\begin{aligned} (TS)^2 + (TS)^{*2} &= (T + S)^2 + (T^* + S^*)^2 \\ &= T^2 + 2TS + S^2 + T^{*2} + 2T^*S^* + S^{*2} \\ &= \underbrace{T^2 + T^{*2}}_{\geq 0} + 2\underbrace{\left(TS + (TS)^*\right)}_{\geq 0} + \underbrace{S^2 + S^{*2}}_{\geq 0} \end{aligned}$$

and so $TS \in [2\mathcal{RP}]$. Assume that this result is true for $n - 1$ and we prove it for n. Let T and S are in $[k\mathcal{RP}]$ for $k = 1, 2, ..., n$.

Since $TS = ST = T + S$ we have

$$\begin{aligned} \left(TS\right)^n + \left(TS\right)^{*n} &= \left(T + S\right)^n + \left(T^* + S^*\right)^n \\ &= T^n + T^{*n} + \sum_{1\leq p\leq n-1} \binom{n}{p}\left(T^pS^{n-p} + T^{*p}S^{*n-p}\right) + S^n + S^{*n}. \end{aligned}$$

It suffice to prove under the assumptions that $T^pS^{n-p} + T^{*p}S^{*n-p} \geq 0$ for $p = 1, 2, ..., n - 1$.

For $p = 1$ we have

$$\begin{aligned} TS^{n-1} + T^*S^{*n-1} &= TSS^{n-2} + T^*S^*S^{*n-2} \\ &= (T+S)S^{n-2} + (T^*+S^*)S^{*n-2} \\ &= TS^{n-2} + T^*S^{*n-2} + \underbrace{S^{n-1} + S^{*n-1}}_{\geq 0} \\ &= TSS^{n-3} + T^*S^*S^{*n-3} + \underbrace{S^{n-1} + S^{*n-1}}_{\geq 0} \\ &= TS^{n-3} + T^*S^{*n-3} + \underbrace{S^{n-2} + S^{*n-2}}_{\geq 0} + \underbrace{S^{n-1} + S^{*n-1}}_{\geq 0} \\ &= \dots \quad \dots \quad \dots \\ &= \underbrace{T + T^*}_{\geq 0} + \sum_{1\leq k\leq n-1} \Big(\underbrace{S^k + S^{*k}}_{\geq 0}\Big). \end{aligned}$$

For $p = 2$ we have

$$\begin{aligned} T^2S^{n-2} + T^{*2}S^{*n-2} &= TSTS^{n-3} + T^*S^*T^*S^{*n-3} \\ &= T^2S^{n-3} + TS^{n-2} + T^{*2}S^{*n-3} + T^*S^{*n-2} \\ &= T^2S^{n-4} + TS^{n-3} + TS^{n-2} + T^{*2}S^{*n-4} + T^*S^{*n-3} + T^*S^{*n-2} \\ &= T^2S^{n-5} + TS^{n-4} + TS^{n-3} + TS^{n-2} \\ &\quad + T^{*2}S^{*n-5} + T^*S^{*n-4} + T^*S^{*n-3} + T^*S^{*n-2} \\ &= \dots \quad \dots \quad \dots \quad \dots \\ &= T^2 + T^{*2} + \sum_{1\leq k\leq n-2} \Big(TS^k + T^*S^{*k}\Big). \end{aligned}$$

A simple calculation shows that

$$TS^k + T^*S^{*k} = T + T^* + \sum_{1\leq j\leq k} \Big(S^j + S^{*j}\Big).$$

We deduce that

$$\begin{aligned} & T^2S^{n-2} + T^{*2}S^{*n-2} \\ = \;& T^2 + T^{*2} + \sum_{1\leq k\leq n-2} \Big(T + T^* + \sum_{1\leq j\leq k} \Big(S^j + S^{*j}\Big)\Big) \geq 0. \end{aligned}$$

Same way for $p = 3, ..., n-1$. Hence $\big(TS\big)^n + \big(TS\big)^{*n} \geq 0$ as required. □

Example 2.22. *Let* $S = T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. *It is easy to see that* $T \in [k\mathcal{RP}]$ *for* $k = 1, 2, ..., n$ *and* $TS \in [n\mathcal{RP}]$.

The following example shows that Theorem 2.3 is not necessarily true if $TS \neq S + T$.

Example 2.23. *Let* $T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ *and* $S = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. *We have* T *and* S *are in* $[\mathcal{RP}]$, $TS \neq T + S$ *and* $TS \notin [2\mathcal{RP}]$.

ACKNOWLEDGMENT

The author would like to express his cordial gratitude to the referee for valuable advice and suggestions. I Would like also to thank Prof. O. A. Mahmoud Sid Ahmed for his helpful discussions and remarks.

References

[1] A. Brown, *On a class of operators*, Proc. Amer. Math. Soc, 4 (1953), 723-728.
[2] A. A. S. Jibril.*On n-Power Normal Operators*.The Journal for Science and Engenering . Volume 33, Number 2A.(2008) 247-251.
[3] A. A. S. Jibril, *On 2-Normal Operators*, Dirasat, Vol.(23) No.2.
[4] A. A. S, Jibril, *On subprojection sperators*, International Mathematical Journal, 4(3), (2003), pp. 229–238.
[5] M. Guesba and M. Nadir, *On operators for wich* $T^2 \geq -T^{*2}$. The Australian Journal of Mathematical Analysis and Applications. Volume 13, Issue 1, Article 6 (2016), pp. 1–5.
[6] O. A. M. Sid Ahmed, *On the class of n-power quasi-normal operators on Hilbert spaces*, Bull. Math. Anal. Appl., 3(2), (2011), 213–228.
[7] O. A. M. Sid Ahmed, *On Some Normality-Like properties and Bishops property* (β) *for a class of operators on Hilbert spaces*, International Journal of Mathematics and Mathematical Sciences, (2012),(20 pages).