Property \((Z_{E_a})\) for direct sums

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Abstract. We show that generally the properties \((Z_{E_a})\) and \((Z_{I_n})\) introduced by the author are not preserved under direct sum of operators. Moreover, If \(S\) and \(T\) are Banach spaces operators satisfying property \((Z_{E_a})\) or \((Z_{I_n})\), we give conditions on \(S\) and \(T\) to ensure the preservation of these properties by the direct sum \(S \oplus T\). Some crucial applications are also given.

1. Introduction

For \(T\) in the Banach algebra \(L(X)\) of bounded linear operators acting on an infinite dimensional complex Banach space \(X\), we will denote by \(\sigma(T)\) the spectrum of \(T\), by \(\sigma_a(T)\) the approximate point spectrum of \(T\), by \(\sigma_d(T)\) the null space of \(T\), by \(\sigma_r(T)\) the range of \(T\) and by \(\beta(T)\) its defect. If \(a(T) < \infty\) and \(\beta(T) < \infty\), then \(T\) is called a Fredholm operator and its index is defined by \(\text{ind}(T) = a(T) - \beta(T)\).

A Weyl operator is a Fredholm operator of index 0 and the Weyl spectrum is defined by \(\sigma_w(T) = \{\lambda \in \mathbb{C} \mid \lambda I - T\text{ is not a Weyl operator}\}\). \(T \in L(X)\) is called a semi-Fredholm if \(\sigma(T)\) is closed and \(a(T) < \infty\) (resp., \(\beta(T) < \infty\)).

For a bounded linear operator \(T\) and \(n \in \mathbb{N}\), let \(T^{[n]} : \mathcal{R}(T^n) \to \mathcal{R}(T^n)\) be the restriction of \(T\) to \(\mathcal{R}(T^n)\). \(T \in L(X)\) is said to be b-Weyl if for some integer \(n \geq 0\), the range \(\mathcal{R}(T^n)\) is closed and \(T^{[n]}\) is Weyl; its index is defined as the index of the Weyl operator \(T^{[n]}\). The respective b-Weyl spectrum is defined by \(\sigma_{bw}(T) = \{\lambda \in \mathbb{C} \mid \lambda I - T\text{ is not a b-Weyl operator}\}\). \(T \in L(X)\) is called a semi-b-Fredholm if for some integer \(n \geq 0\), the range \(\mathcal{R}(T^n)\) is closed and \(T^{[n]}\) is semi-Fredholm; and the semi-b-Fredholm spectrum is defined by \(\sigma_{sbw}(T) = \{\lambda \in \mathbb{C} \mid \lambda I - T\text{ is not a semi-b-Fredholm operator}\}\), see [4].

The ascent of an operator \(T\) is defined by \(a(T) = \inf\{n \in \mathbb{N} \mid \mathcal{N}(T^n) = \mathcal{N}(T^{n+1})\}\), and the descent of \(T\) is defined by \(d(T) = \inf\{n \in \mathbb{N} \mid \mathcal{R}(T^n) = \mathcal{R}(T^{n+1})\}\), with \(\inf = 0\). According to [10], a complex number \(\lambda \in \sigma(T)\) is a pole of the resolvent of \(T\) if \(T - \lambda I\) has finite ascent and finite descent, and in this case they are equal. We recall that \(T \in L(X)\) is said to be left Drazin invertible if \(a(T) < \infty\) and \(\mathcal{R}(T^{a(T)+1})\) is closed; and the left-Drazin spectrum of \(T\) is defined by \(\sigma_{ld}(T) = \{\lambda \in \mathbb{C} \mid T - \lambda I\text{ is not left Drazin invertible}\}\). A complex number \(\lambda \in \sigma_{ld}(T)\) is a left pole of \(T\) if \(T - \lambda I\) is left Drazin invertible.

In the following, we recall the definition of a property which has a relevant role in local spectral theory.

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In particular, $\sigma_E$ at every point

Definition 1.1. [11] An operator $T \in L(X)$ is said to have the single valued extension property (SVEP) at $\lambda_0 \in \mathbb{C}$, if for every open neighborhood $\mathcal{U}$ of $\lambda_0$, the only analytic function $f : \mathcal{U} \rightarrow X$ which satisfies the equation $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in \mathcal{U}$ is the function $f \equiv 0$. An operator $T \in L(X)$ is said to have the SVEP if $T$ has the SVEP at every point $\lambda \in \mathbb{C}$.

It follows easily that $T \in L(X)$ has the SVEP at every point of the boundary $\partial \sigma(T)$ of the spectrum $\sigma(T)$. In particular, $T$ has the SVEP at every point of $\text{iso}(T)$.

Evidently, $T \in L(X)$ has SVEP at every isolated point of the spectrum. We summarize in the following list the usual notations and symbols needed later.

Notations and symbols:
iso $A$: isolated points of a subset $A \subset \mathbb{C}$,
acc $A$: accumulations points of a subset $A \subset \mathbb{C}$,
$D(0,1)$: the closed unit disc in $\mathbb{C}$,
$C(0,1)$: the unit circle of $\mathbb{C}$,
$\Pi(T)$: poles of $T$,
$\Pi^0(T)$: poles of $T$ of finite rank,
$\Pi_0(T)$: left poles of $T$,
$\sigma_0(T)$: eigenvalues of $T$,
$\sigma_{ab}(T)$: eigenvalues of $T$ of finite multiplicity,
$E_0(T) := \sigma_0(T) \cap \sigma_0^0(T)$,
$E(T) := \sigma_0(T) \cap \sigma_0(T)$,
$E_0(T) := \sigma_0(T) \cap \sigma_0(T)$,
$(\sigma_0^0(T) := \sigma_0(T) \cap \sigma_0(T)$,
$\sigma_0(T) = \sigma(T) \cap \Pi_0(T)$: Browder spectrum of $T$,
$\sigma_{aw}(T) = \sigma_0(T) \cap \Pi_0^0(T)$: upper-Browder spectrum of $T$,
$\sigma_w(T)$: Weyl spectrum of $T$,
$\sigma_{bwf}(T)$: b-Weyl spectrum of $T$; the symbol $\sqcup$ stands for the disjoint union.

Definition 1.2. [3], [5], [13], [14] Let $T \in L(X)$. We say that $T$ satisfies:
i) Property (ab) if $\sigma_0(T) \cap \sigma_0(T) = \sigma_0^0(T)$.
ii) Property (gb) if $\sigma_0(T) \cap \sigma_0(T) = \Pi_0(T)$.
iii) Property (Bab) if $\sigma_0(T) \cap \sigma_0(T) = \Pi_0^0(T)$.
iv) Browder’s theorem if $\sigma_0(T) \cap \sigma_0(T) = \Pi_0^0(T)$.
v) Property $(Z_k)$ if $\sigma_0(T) \cap \sigma_0(T) = \sigma_0(T)$.
vi) Property $(Z_{\Pi_0})$ if $\sigma_0(T) \cap \sigma_0(T) = \sigma_0(T)$.

Definition 1.3. Let $T \in L(X)$ and $S \in L(X)$. We will say that $T$ and $S$ have a shared stable sign index if for each $\lambda \notin \sigma_{ab}(T)$ and $\mu \notin \sigma_{ab}(S)$, ind$(T - \lambda I)$ and ind$(S - \mu I)$ have the same sign.

For examples we have:

1. Here and elsewhere, $\mathcal{H}$ denotes a Hilbert space. Two hyponormal operators $T$ and $S$ acting on $\mathcal{H}$ have a shared stable sign index, since $\text{ind}(S - \lambda I) \leq 0$ and $\text{ind}(T - \mu I) \leq 0$ for every $\lambda \notin \sigma_{ab}(S)$ and $\mu \notin \sigma_{ab}(T)$. Recall that $T \in L(\mathcal{H})$, is said to be hyponormal if $TT^* - TT^* \geq 0$ (or equivalently $\|Tx\| \leq \|Tx\|$ for all $x \in \mathcal{H}$). The class of hyponormal operators includes also subnormal operators and quasinormal operators, see [7].
2. It is easily verified that if $T \in L(X)$ has SVEP then ind$(T - \mu I) \leq 0$ for every $\mu \notin \sigma_{ab}(T)$. So if $S$ and $T$ have SVEP, then they have a shared stable sign index.

In this paper, we focus on the problem of giving conditions on the direct summands to ensure that the Fredholm-type spectral properties introduced very recently by the author in [13], hold for the direct sum. More recently, several authors have worked in this direction, see for examples [6], [8], [9], [12]. The paper
is organized as follows: after giving an introduction and some definitions in the first section, we prove in the second section that property \((Z_{II})\) is not preserved under direct sum of operators and we prove that if \(S\) and \(T\) satisfy property \((Z_{II})\) with the supplementary condition \(\Pi_\lambda(S) \cap \rho_\lambda(T) = \Pi_\lambda(T) \cap \rho_\lambda(S) = \emptyset\), then \(S \oplus T\) satisfies property \((Z_{II})\) if and only if \(\sigma_\lambda(S \oplus T) = \sigma_\lambda(S) \cup \sigma_\lambda(T)\). We obtain an analogous preservation result for property \((Z_{E_{1}})\). Some applications to quasisimilar hyponormal operators are given.

2. Properties \((Z_{E_{1}})\) and \((Z_{II})\) for direct sum of operators

We start this section by citing the following two results (see also [13]) which will be used in the proof of the main results of this paper. And in order to give a global overview of the subject, we also include their proofs.

Lemma 2.1. Let \(T \in L(X)\). The following assertions hold:
i) If \(T\) satisfies property \((Z_{E_{1}})\), then
\[E_0(T) = E^0(T) = \Pi_0(T) = \Pi(T) = E(T) = E^0(T) = E(T).\]
ii) If \(T\) satisfies property \((Z_{II})\), then \(\Pi_0(T) = \Pi(T) = \Pi(T)\).

Proof. i) Suppose that \(T\) satisfies property \((Z_{E_{1}})\), then \(\sigma(T) = \sigma_w(T) \cup E_0(T)\). Thus \(\mu \in E_0(T) \iff \mu \in \text{iso } \sigma(T) \cap \sigma_0(T)\), where \(\sigma_0(T)\) is the complement of the Weyl spectrum of \(T\). Hence \(E_0(T) = E^0(T) = \Pi_0(T) = \Pi(T) = \Pi(T)\) and \(E(T) = E^0(T)\). Consequently, \(\sigma(T) = \sigma_w(T) \cup E^0(T)\). This implies that \(E(T) = \Pi(T)\). Hence \(E_0(T) = E^0(T) = \Pi_0(T) = \Pi(T)\) and \(\Pi(T) = \Pi(T) = E(T)\). Since the inclusion \(\Pi(T) \subset \Pi(T)\) is always true, it suffices to show its opposite. If \(\mu \in \Pi(T)\), then \(a(T - \lambda I)\) is finite and since \(T\) satisfies property \((Z_{E_{1}})\), it follows that \(\mu \in \Pi(T)\) and hence the equality desired.

ii) Goes similarly with the proof of the first assertion. \(\square\)

Theorem 2.2. Let \(T \in L(X)\). The following statements are equivalent:
i) \(T\) satisfies property \((Z_{II})\);
ii) \(T\) satisfies property \((gab)\) and \(\sigma_{\text{w.r.}}(T) = \sigma_w(T)\);
iii) \(T\) satisfies property \((ab)\) and \(\Pi_\lambda(T) = \Pi_0(T)\);
iv) \(T\) satisfies property \((Bab)\) and \(\Pi_\lambda(T) = \Pi_0(T)\);
v) \(T\) satisfies Browder’s theorem and \(\Pi_\lambda(T) = \Pi_0(T)\).

Proof. (i) \iff (ii) Suppose that \(T\) satisfies property \((Z_{II})\), that’s \(\sigma(T) = \sigma_w(T) \cup \Pi_\lambda(T)\). From Lemma 2.1, \(\sigma_\lambda(T) = \sigma_\lambda(T) \cup \Pi_0(T)\). So \(T\) satisfies property \((ab)\). As \(\Pi_\lambda(T) = \Pi_\lambda(T)\), then from [5, Theorem 2.8], \(T\) satisfies property \((gab)\). Moreover, \(\sigma_w(T) = \sigma(T) \setminus \Pi_\lambda(T) = \sigma_\text{w.r.}(T)\). The reverse implication is obvious.

(i) \iff (iii) Follows directly from Lemma 2.1.

(i) \iff (iv) If \(T\) satisfies property \((Z_{II})\), then \(\sigma(T) \setminus \sigma_{\text{w.r.}}(T) = \sigma(T) \setminus \sigma_w(T) = \Pi_0(T) = \Pi_\lambda(T)\). So \(T\) satisfies property \((Bab)\). Conversely, the property \((Bab)\) for \(T\) implies from [14, Theorem 3.6] that \(\sigma_w(T) = \sigma_w(T)\). So \(\sigma_w(T) = \sigma(T) \setminus \Pi_0(T) = \sigma(T) \setminus \Pi_\lambda(T)\) and this means that \(T\) satisfies property \((Z_{II})\). The equivalence between assertions (i) and (v) is clear. \(\square\)

Now, we give the following proposition which will play an important role in this paper. Hereafter, \(Y\) denotes an infinite dimensional complex Banach space.

Proposition 2.3. (See also [12, Lemma 3]) Let \(S \in L(X)\) and \(T \in L(Y)\). Then
\[\sigma_w(S \oplus T) \subseteq \sigma_w(S) \cup \sigma_w(T).\]

Proof. If \(\lambda \notin \sigma_w(S) \cup \sigma_w(T)\) be arbitrary, then \(S - \lambda I\) and \(T - \lambda I\) are Fredholm operators of index zero. Hence \((S \oplus T) - \lambda I\) is a Fredholm operator and \(\text{ind}((S \oplus T) - \lambda I) = \text{ind}(S - \lambda I) + \text{ind}(T - \lambda I) = 0\). So \(\lambda \notin \sigma_w(S \oplus T)\) and then \(\sigma_w(S \oplus T) \subseteq \sigma_w(S) \cup \sigma_w(T)\). \(\square\)
Generally, the inclusion showed in Proposition 2.3 is proper. To see this, here and elsewhere the operators $R$ and $U$ are defined on the Hilbert space $\ell^2(\mathbb{N})$ by

$$R(x_1, x_2, \ldots) = (0, x_1, x_2, x_3, \ldots) \text{ and } U(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots).$$

Then $\sigma_w(R) = \sigma_w(U) = D(0, 1)$. Since $\alpha(R \oplus U) = \beta(R \oplus U) = 1$, then $0 \notin \sigma_w(R \oplus U)$ and hence $\sigma_w(R \oplus U) \neq \sigma_w(U) \cup \sigma_w(R)$. Observe that this example shows also that $\sigma_{bw}(R \oplus U) \neq \sigma_{bw}(R) \cup \sigma_{bw}(U)$.

However, we have the following corollary:

**Corollary 2.4.** Let $S \in L(X)$ and $T \in L(Y)$. The following assertions hold:

1. If $\sigma_{bw}(S \oplus T) = \sigma_{bw}(S) \cup \sigma_{bw}(T)$, then $\sigma_w(S \oplus T) = \sigma_w(S) \cup \sigma_w(T)$.
2. If $S$ and $T$ have a shared stable sign index, then $\sigma_w(S \oplus T) = \sigma_w(S) \cup \sigma_w(T)$.
3. If $S \oplus T$ satisfies Browder’s theorem, then $\sigma_w(S \oplus T) = \sigma_w(S) \cup \sigma_w(T)$.

**Proof.** i) Let $\lambda \notin \sigma_w(S \oplus T)$ be arbitrary and without loss of generality we can assume that $\lambda = 0$. Then $S \oplus T$ is a Weyl operator and so is B-Weyl operators. Thus $S$ and $T$ are B-Weyl operators. Since $\alpha(S) \leq \alpha(S \oplus T) < \infty$ and $\alpha(T) \leq \alpha(S \oplus T) < \infty$, then $S$ and $T$ are Weyl operators. Hence $\sigma_w(S \oplus T) \subset \sigma_w(S) \cup \sigma_w(T)$, and by Proposition 2.3, we conclude that $\sigma_w(S \oplus T) = \sigma_w(S) \cup \sigma_w(T)$.

ii) If $S$ and $T$ have a shared stable sign index, then from [6, Lemma 2.2] we have $\sigma_{bw}(S \oplus T) = \sigma_{bw}(S) \cup \sigma_{bw}(T)$. So $\sigma_{bw}(S \oplus T) = \sigma_{bw}(S) \cup \sigma_{bw}(T)$.

iii) If $S \oplus T$ satisfies Browder’s theorem, then $\sigma_w(S \oplus T) = \sigma_w(S) \cup \sigma_w(T)$. As $\alpha(S) \leq \alpha(S \oplus T)$ and $\alpha(T) \leq \alpha(S \oplus T)$, then $\sigma_w(S \oplus T) = \sigma_w(S) \cup \sigma_w(T)$. Since the inclusion $\sigma_{bw}(S) \cup \sigma_{bw}(T) \subset \sigma_w(S) \cup \sigma_w(T)$ is always true, we then have $\sigma_{bw}(S \oplus T) = \sigma_{bw}(S) \cup \sigma_{bw}(T)$. Hence $\sigma_{bw}(S \oplus T) = \sigma_w(S) \cup \sigma_w(T)$. □

The following example shows that, in general the property $(Z_{\Pi_1})$ is not preserved under direct sum of operators.

**Example 2.5.** Let $T \in L(C^n)$ be a nilpotent operator and let $R \in L(\ell^2(\mathbb{N})$ be the operator defined above. Then $\sigma(T) = \{0\}, \sigma_w(T) = \emptyset, \Pi_{4}(T) = \{0\}$. Thus $\sigma(T) \setminus \sigma_w(T) = \Pi_{4}(T)$ and the property $(Z_{\Pi_1})$ is satisfied by $T$. Moreover, $\sigma(R) = D(0, 1), \sigma_w(R) = D(0, 1), \Pi_4(R) = \emptyset$. So $\sigma(R) \setminus \sigma_w(R) = \Pi_4(R)$ and $R$ satisfies property $(Z_{\Pi_1})$. But their direct sum $T \oplus R$ defined on the Banach space $C^n \oplus \ell^2(\mathbb{N})$ does not satisfy property $(Z_{\Pi_1})$, because $\sigma(T \oplus R) = D(0, 1), \sigma_w(T \oplus R) = D(0, 1)$ and $\Pi_4(T \oplus R) = \{0\}$. Here $\Pi_4(T) \cap \rho_w(T) = \{0\}$ and $\sigma_w(T \oplus R) = \sigma_w(T) \cup \sigma_w(R)$, where $\rho_w(\cdot) = \mathbb{C} \setminus \sigma_w(\cdot)$.

Nonetheless, in the next theorem we explore certain sufficient conditions which ensure the preservation of property $(Z_{\Pi_1})$ under direct sum of operators.

**Theorem 2.6.** Suppose that $S \in L(X)$ and $T \in L(Y)$ are such that $\Pi_{4}(S) \cap \rho_w(T) = \Pi_{4}(T) \cap \rho_w(S) = \emptyset$. If $S$ and $T$ satisfy property $(Z_{\Pi_1})$, then the following assertions are equivalent:

1. $S \oplus T$ satisfies property $(Z_{\Pi_1})$;
2. $\sigma_w(S \oplus T) = \sigma_w(S) \cup \sigma_w(T)$.

**Proof.** (ii) $\Rightarrow$ (i) Since $S$ and $T$ satisfy property $(Z_{\Pi_1})$, we then have

$$[(\sigma(S) \cup \sigma(T)) \setminus [\sigma_w(S) \cup \sigma_w(T)] = [(\sigma(S) \setminus \sigma_w(S)) \cup (\sigma(T) \setminus \sigma_w(T))] \cup [(\sigma(T) \setminus \sigma_w(T)) \cap (\sigma(S) \setminus \sigma_w(S))] = [\Pi_{4}(S) \cap \rho(T)] \cup [\Pi_{4}(T) \cap \rho(S)] \cup [\Pi_{4}(S) \cap \Pi_{4}(T)]$$

The assumption $\Pi_{4}(S) \cap \rho_w(T) = \Pi_{4}(T) \cap \rho_w(S) = \emptyset$ implies that $\Pi_{4}(S) \cap \rho(T) = \Pi_{4}(T) \cap \rho(S) = \emptyset$; where $\rho_w(\cdot) = \mathbb{C} \setminus \sigma_w(\cdot)$. Thus

$$[(\sigma(S) \cup \sigma(T)) \setminus [\sigma_w(S) \cup \sigma_w(T)] = \Pi_{4}(S) \cap \Pi_{4}(T).$$
On the other hand, as we know that \( \sigma_{id}(S \oplus T) = \sigma_{id}(S) \cup \sigma_{id}(T) \), we then have

\[
\Pi_{s}(S \oplus T) = \sigma_{s}(S \oplus T) \setminus \sigma_{id}(S \oplus T) = [\sigma_{s}(S) \cup \sigma_{s}(T)] \setminus [\sigma_{id}(S) \cup \sigma_{id}(T)] = [\sigma_{s}(S) \setminus \sigma_{id}(S)] \cup [\sigma_{s}(T) \setminus \sigma_{id}(T)] \cup [\sigma_{id}(S) \setminus \sigma_{id}(S)] \cup [\sigma_{id}(T) \setminus \sigma_{id}(T)] = [\sigma_{s}(S) \setminus \sigma_{id}(S)] \cup [\sigma_{s}(T) \setminus \sigma_{id}(T)] = \Pi_{s}(S) \cap \Pi_{s}(T).
\]

Hence \( \Pi_{s}(S \oplus T) = [\sigma(S) \cup \sigma(T)] \setminus [\sigma_{w}(S) \cup \sigma_{w}(T)] \). As by hypothesis \( \sigma_{w}(S \oplus T) = \sigma_{w}(S) \cup \sigma_{w}(T) \), then \( \Pi_{s}(S \oplus T) = \sigma(S \oplus T) \setminus \sigma_{w}(S \oplus T) \) and this shows that \( S \oplus T \) satisfies property \((Z_{1},)\). (i) \( \Rightarrow \) (ii) If \( S \oplus T \) satisfies property \((Z_{1},)\) then from Theorem 2.2, \( S \oplus T \) satisfies Browder’s theorem. Thus by Corollary 2.4, \( \sigma_{w}(S \oplus T) = \sigma_{w}(S) \cup \sigma_{w}(T) \). \( \square \)

**Remark 2.7.** Generally, we cannot ensure the transmission of the property \((Z_{1},)\) from two operators \( S \) and \( T \) to their direct sum even if \( \Pi_{s}(S) \cap \rho_{s}(T) = \Pi_{s}(T) \cap \rho_{s}(S) = \emptyset \). For this, the operators \( R \) and \( U \) defined above satisfy property \((Z_{1},)\), because \( \sigma(U) = \sigma_{w}(U) = D(0,1) \) and \( \rho(U) = \emptyset \). But this property is not satisfied by their direct sum, since \( \Pi_{s}(R \oplus U) = \emptyset \), \( \sigma(R \oplus U) = D(0,1) \) and \( \sigma_{w}(R \oplus U) \leq D(0,1) \). Remark that \( \Pi_{s}(R) \cap \rho_{s}(U) = \Pi_{s}(U) \cap \rho_{s}(R) = \emptyset \).

A bounded linear operator \( A \in L(X, Y) \) is said to be **quasi-invertible** if it is injective and has dense range. Two bounded linear operators \( T \in L(X) \) and \( S \in L(Y) \) on complex Banach spaces \( X \) and \( Y \) are **quasisimilar** provided there exist quasi-invertible operators \( A \in L(X, Y) \) and \( B \in L(Y, X) \) such that \( AT = SA \) and \( BS = TB \).

**Corollary 2.8.** If \( S \in L(H) \) and \( T \in L(H) \) are quasisimilar hyponormal operators and satisfy property \((Z_{1},)\), then \( S \oplus T \) satisfies property \((Z_{1},)\).

**Proof.** As \( S \) and \( T \) are quasisimilar hyponormal, then by [6, Lemma 2.8] we have \( \Pi(T) = \Pi(S) \). The property \((Z_{1},)\) for \( S \) and for \( T \) entails from Lemma 2.1, that \( \Pi(T) = \Pi_{s}(T) \) and \( \Pi(T) = \Pi_{w}(S) \). So \( \Pi_{s}(T) \cap \rho_{s}(T) = \Pi_{w}(S) \cap \rho_{s}(S) = \emptyset \). Moreover, since \( S \) and \( T \) are hyponormal operators, then they have a shared stable index. This implies from Corollary 2.4 that \( \sigma_{w}(S \oplus T) = \sigma_{w}(S) \cup \sigma_{w}(T) \). But this is equivalent from Theorem 2.6, to say that \( S \oplus T \) satisfies property \((Z_{1},)\). \( \square \)

Similarly to theorem 2.6, we prove a preservation result for property \((Z_{1},)\) under direct sum of operators. Firstly remark that in general, we cannot expect that property \((Z_{1},)\) will be satisfied by the direct sum \( S \oplus T \) if its components satisfy property \((Z_{1},)\). For instance, we give the following example:

**Example 2.9.** Let \( T \) and \( R \) be the operators defined in Example 2.5, then \( T \) and \( R \) satisfy property \((Z_{1},)\), because \( \sigma(T) \setminus \sigma_{s}(T) = E_{s}(T) = [0], \sigma(R) \setminus \sigma_{w}(R) = E_{w}(R) = \emptyset \). But \( T \oplus R \) does not satisfy property \((Z_{1},)\), because \( \sigma(T \oplus R) \setminus \sigma_{s}(T \oplus R) = \emptyset \neq E_{s}(T \oplus R) = [0] \). Here, observe that \( \sigma_{w}(R) = \emptyset \), \( \sigma_{w}(T) = [0] \) and \( \sigma_{w}(T \oplus R) = \sigma_{w}(T) \cup \sigma_{w}(R) = D(0,1) \).

However, we characterize in the next theorem the stability of property \((Z_{1},)\) under direct sum via union of Weyl spectra of its summands, which in turn are supposed to have the same eigenvalues. Before this, we recall that \( \sigma_{w}(S \oplus T) = \sigma_{w}(S) \cup \sigma_{w}(T) \). Moreover, if \( A \) and \( B \) are bounded subsets of complex plane \( \mathbb{C} \) then \( \text{acc}(A \cup B) = \text{acc}(A) \cup \text{acc}(B) \).

**Theorem 2.10.** Let \( S \in L(X) \) and \( T \in L(Y) \) be such that \( \sigma_{w}(S) = \sigma_{w}(T) \). If \( S \) and \( T \) satisfy property \((Z_{1},)\), then the following assertions are equivalent:

(i) \( S \oplus T \) satisfies property \((Z_{1},)\);

(ii) \( \sigma_{w}(S \oplus T) = \sigma_{w}(S) \cup \sigma_{w}(T) \).
Proof. (ii) $\Rightarrow$ (i) Suppose that $\sigma_w(S \oplus T) = \sigma_w(S) \cup \sigma_w(T)$. As $S$ and $T$ satisfy property $(Z_{E_n})$, i.e. $\sigma(S \setminus \sigma_w(S) = E_a(S)$ and $\sigma(T) \setminus \sigma_w(T) = E_a(T)$, we then have
\[
\sigma(S \oplus T) \setminus \sigma_w(S \oplus T) = [(\sigma(S) \setminus \sigma_w(S)) \cap \rho_p(T)] \cup [(\sigma(T) \setminus \sigma_w(T)) \cap \rho_p(S)] \\
\cup[(\sigma(S) \setminus \sigma_w(S)) \cap (\sigma(T) \setminus \sigma_w(T))] \\
= [E_a(T) \cap \rho(S)] \cup [E_a(S) \cap \rho(T)] \cup [E_a(S) \cap E_a(T)].
\]
Since by hypothesis $\sigma_p(T) = \sigma_p(S)$, then $E_a(T) \cap \rho_p(S) = E_a(S) \cap \rho_p(T) = 0$ which implies that $E_a(T) \cap \rho(T) = \emptyset$. Thus
\[
\sigma(S \oplus T) \setminus \sigma_w(S \oplus T) = E_a(S) \cap E_a(T).
\]
On the other hand, $\sigma_p(S \oplus T) = \sigma_p(S) = \sigma_p(T)$. This implies that
\[
E_a(S \oplus T) = \{\text{iso} \sigma_a(S \oplus T) \cap \rho_p(S) \}
= \{\text{iso} \sigma_a(S) \cup \sigma_a(T) \cap \rho_p(S) \}
= \{\text{iso} \sigma_a(S) \cup \sigma_a(T) \cap \rho_p(S) \}
= \{\text{iso} \sigma_a(S) \cup \sigma_a(T) \cap \rho_p(S) \}
= \{\text{iso} \sigma_a(S) \cup \sigma_a(T) \cap \rho_p(S) \}
= [E_a(S) \cap \rho_p(T)] \cup [E_a(T) \cap \rho_p(S)] \cup [E_a(S) \cap E_a(T)]
= E_a(S) \cap E_a(T).
\]
Hence $\sigma(S \oplus T) \setminus \sigma_w(S \oplus T) = E_a(S \oplus T)$ and this shows that property $(Z_{E_n})$ is satisfied by $S \oplus T$. 
(i) $\Rightarrow$ (ii) If $S \oplus T$ satisfies property $(Z_{E_n})$, then by Lemma 2.1, $S \oplus T$ satisfies property $(Z_{E_n})$. Therefore we have the equality $\sigma_a(S \oplus T) = \sigma_w(S) \cup \sigma_w(T)$, as seen in the proof of Theorem 2.6. \hfill \Box

Corollary 2.11. Let $S \in L(X)$ and $T \in L(Y)$ be quasisimilar operators satisfying property $(Z_{E_n})$. If $S$ or $T$ has SVEP, then $S \oplus T$ satisfies property $(Z_{E_n})$.

Proof. The quasisimilarity of $S$ and $T$ implies that $\sigma_p(S) = \sigma_p(T)$. It implies also from [1, Theorem 2.15] that $S$ and $T$ have SVEP. So they have a shared stable sign index and hence $\sigma_w(S \oplus T) = \sigma_w(S) \cup \sigma_w(T)$. But this is equivalent from Theorem 2.10, to say that $S \oplus T$ satisfies property $(Z_{E_n})$. \hfill \Box

Examples 2.12.

1. A bounded linear operator $T \in L(H)$ is said to be $p$-hyponormal, with $0 < p \leq 1$, if $(T^*T)^p \geq (TT^*)^p$ and is said to be log-hyponormal if $T$ is invertible and satisfies $\log(T^*T) \geq \log(TT^*)$. According to [2], if $T \in L(H)$ is invertible and $p$-hyponormal, there exists $S \in L(H)$ log-hyponormal quasisimilar to $T$. Then $\sigma_p(S) = \sigma_p(T)$. Since $S$ has SVEP, then $S$ and $T$ have a shared stable sign index and so $\sigma_w(S \oplus T) = \sigma_w(S) \cup \sigma_w(T)$. Moreover, if $S$ and $T$ satisfy property $(Z_{E_n})$, then $S \oplus T$ satisfies property $(Z_{E_n})$.

2. Let $V$ denote the Volterra operator on the Banach space $C[0, 1]$ defined by $V(f)(x) = \int_0^x f(t)dt$ for all $f \in C[0, 1]$. $V$ is injective and quasinilpotent. $\sigma(V) = \sigma_p(V) = \{0\}$ and $\Pi_{E_n}(V) = \emptyset$. So $V$ satisfies property $(Z_{1L})$. It is already mentioned that $R$ satisfies property $(Z_{1L})$. As $R$ and $V$ have SVEP, then they have a shared stable sign index. On the other hand, $\Pi_{E_n}(R) \cap \rho_w(V) = \Pi_{E_n}(V) \cap \rho_w(R) = \emptyset$. Hence $V \oplus R$ satisfies property $(Z_{1L})$.

We finish this paper by posing the following two questions arising from Corollary 2.4.

Let $S \in L(X)$ and $T \in L(Y)$. Is it true that?

1. If $\sigma_w(S \oplus T) = \sigma_w(S) \cup \sigma_w(T)$, then $\sigma_{\text{wim}}(S \oplus T) = \sigma_{\text{wim}}(S) \cup \sigma_{\text{wim}}(T)$.

2. If $\sigma_w(S \oplus T) = \sigma_w(S) \cup \sigma_w(T)$, then $S \oplus T$ satisfies Browder’s theorem.

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References